SIAM J. MATH. ANAL.
Vol. 1, No. 1, February 1970

# PSEUDOPARABOLIC PARTIAL DIFFERENTIAL EQUATIONS*

R. E. SHOWALTER† AND T. W. TING‡

**1. Introduction.** Various physical phenomena have led to a study of a mixed boundary value problem for the partial differential equation

$$\frac{\partial}{\partial t}u - \eta\Delta\frac{\partial}{\partial t}u = k\Delta u, \tag{1.1}$$

where $\Delta$ denotes the Laplacian differential operator. The initial and boundary conditions for this equation are the same as those posed for solutions of the parabolic equation

$$\frac{\partial}{\partial t}u = k\Delta u \tag{1.2}$$

which is obtained from (1.1) by setting $\eta = 0$. The class of equations which are considered here will be called *pseudoparabolic*, not only because the problems which are well-posed for the parabolic equation are also well-posed for these equations, but because the generalized solution to the parabolic equation (1.2) satisfying mixed initial and boundary conditions can be obtained as the limit of a sequence of solutions to the corresponding problem for equation (1.1) corresponding to any null sequence for the coefficient $\eta$. That is, a solution of the parabolic equation can be approximated by a solution of (1.1).

More statements on the comparison of these problems will appear in the following.

A study of nonsteady flow of second order fluids [36] leads to a mixed boundary value problem for the one-dimensional case of (1.1) for the velocity of the fluid. In [36] the role of the material constant $\eta$ was examined, for this constant distinguishes this theory of second order fluids from earlier ones. This principal result of interest here is that the mixed boundary value problem is mathematically well-posed.

Equations of the form (1.1) are satisfied by the hydrostatic excess pressure within a portion of clay during consolidation [35]. In this context the constant $\eta$ is a composite soil property with the dimensions of viscosity. If one assumes that the resistance to compression is plastic (proportional to the rate of compression), then equation (1.1) results with $\eta > 0$. However the classical Terzaghi assumption that any increment in the hydrostatic excess pressure is proportional to an increment of the ratio of pore volume to solid volume in the clay leads to the parabolic (1.2).

---

* Received by the editors October 30, 1968, and in revised form March 12, 1969.

† Department of Mathematics, University of Texas, Austin, Texas 78712. The work of this author was supported by a National Science Foundation fellowship.

‡ Department of Mathematics, University of Illinois, Urbana, Illinois 61801. The work of this author was supported in part by the National Science Foundation under Grant GP-7374.

As a final example of the physical origin of (1.1) we mention the theory of seepage of homogeneous fluids through a fissured rock [4]. A fissured rock consists of blocks of porous and permeable material separated by fissures or "cracks." The liquid then flows through the blocks and also between the blocks through the fissures. In this context an analysis of the pressure in the fissures leads to (1.1), where $\eta$ represents a characteristic of the fissured rock. A decrease in $\eta$ corresponds to a reduction in block dimensions and an increase in the degree of fissuring, and (1.1) then tends to coincide with the classical parabolic equation (1.2) of seepage of a liquid under elastic conditions.

The equation which we shall consider here is an example of the general class of equations of Sobolev type, sometimes referred to as the Sobolev–Galpern type. These are characterized by having mixed time and space derivatives appearing in the highest order terms of the equation. Such an equation was studied by Sobolev [34], and he used a Hilbert space approach to determine that both the Cauchy problem on the whole space and the mixed boundary value problem on a bounded domain are well-posed for the equation

$$\frac{\partial^2}{\partial t^2}(\Delta u) + \frac{\partial^2}{\partial x^2}(u) = 0. \tag{1.3}$$

This equation can be handled by the methods considered here.

The methods of generalized functions [11], [16] have been used on various classes of Sobolev type equations. In particular Galpern [15] investigated the Cauchy problem for a system of equations of the form

$$M\left(t, \frac{\partial}{\partial x_k}\right)\frac{\partial \mathbf{u}}{\partial t} + L\left(t, \frac{\partial}{\partial x_k}\right)\mathbf{u} = 0, \tag{1.4}$$

where $\mathbf{u}$ is a vector of functions and $M$ and $L$ are quadratic polynomial matrices depending on $t$. An analysis by Fourier transforms was used to assert existence and regularity of a solution to this system. Kostachenko and Eskin [24] discussed correctness classes of generalized functions for (1.4) with constant coefficients.

Zalenyak [41] obtained a class of solution of (1.3) satisfying a homogeneous initial condition and then [42] exhibited a class of solutions for the more general equation

$$\sum_{i=1}^{N} \frac{\partial^i}{\partial t^i}\left(a_{i1}\frac{\partial^2 u}{\partial x^2} + a_{i2}\frac{\partial^2 u}{\partial y^2} + b_i(x, y)\frac{\partial u}{\partial x} + c_i(x, y)\frac{\partial u}{\partial y} + d_i(x, y)u\right) = 0$$

in which the $a_{ij}$ are constants.

In the following we shall consider equations of the form

$$M\frac{\partial u}{\partial t} + Lu = f$$

for which $M$ and $L$ are second order differential operators in the space variable and $M$ is elliptic. These operators are independent of $t$ but contain variable coefficients.

This class of equations contains (1.1), and the original Sobolev equation (1.3) can be handled similarly. A generalized mixed boundary value problem for this equation will be solved in the Hilbert space $H_0^1$ which is the Sobolev space of functions having square integrable first order derivatives and which vanish on the boundary in a generalized sense. The Sobolev spaces are introduced in § 2 along with other information that will be used in the following development. The statement of the generalized form of the problem and of the existence and uniqueness of the solution are the content of § 3.

The proof of the existence-uniqueness theorem comprises § 4, and the regularity of the solution is demonstrated in § 5. In particular it is shown that the solution is just as smooth as the initial function and the coefficients of the equation allow it to be. These results depend on the well-developed theory of the Dirichlet problem by means of $L^2$ estimates.

The asymptotic behavior of solutions is discussed in § 6 where it is shown that the solution decays exponentially along with all first order space derivatives. Section 7 extends the existence, uniqueness and regularity results to the nonhomogeneous equation with a time-varying boundary condition.

The results contained in § 8 account for the name *pseudo-parabolic* which we have given to the equation under consideration. In particular it is shown that the solution of (1.1) depends continuously on the coefficient $\eta$, and that if $\eta$ is close to zero then the corresponding solution of (1.1) is arbitrarily close to the solution of (1.2) which satisfies the same initial and boundary data.

Finally in § 9, a similar problem is posed and solved in the Schauder space of functions with uniformly Hölder-continuous derivatives. It is shown that the problem is well-posed in this Banach space, and the same method of constructing a solution as used in the Hilbert space development is applicable here. This section is independent of the previous material, but it depends on the solution of the Dirichlet problem by means of the estimates of Schauder.

**2. Preliminary material.** In this section we shall recall some standard definitions and notations for various spaces of functions. In particular we shall discuss the domain $G$ associated with the problem we are to consider as well as the Sobolev spaces of functions defined on $G$.

$R^n$ will denote the $n$-dimensional real Euclidean space with points specified by coordinates of the form

$$x = (x_1, x_2, \cdots, x_n).$$

For any open set $\Omega$ in $R^n$ we shall denote by $C^m(\Omega)$ the set of all functions defined on $\Omega$ which have continuous derivatives of all orders up through the integer $m$. By $C^m(\hat{\Omega})$ we shall mean those elements of $C^m(\Omega)$ for which all the indicated derivatives are uniformly continuous and hence have unique continuous extensions to the boundary of $\Omega$, and we set

$$C^\infty(\hat{\Omega}) = \bigcap_{m=1}^{\infty} C^m(\hat{\Omega}).$$

The support of a function on $\Omega$ is the closure of the set of points for which the function is nonzero. The set consisting of those functions in $C^\infty(\hat{\Omega})$ with compact support contained in $\Omega$ is denoted by $C_0^\infty(\Omega)$. Each of the sets defined above is a linear space under pointwise addition and scalar multiplication of the elements. The $\alpha$th derivative of a function $\varphi$ in $C^m(\Omega)$ is denoted by

$$D^\alpha \varphi = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}} \varphi,$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ is an $n$-tuple of nonnegative integers and the order of this derivative is denoted by

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

The domain $G$ associated with the problem is a bounded open point set in $R^n$ whose boundary $\partial G$ is an $(n-1)$-dimensional manifold with $G$ all on one side of $\partial G$. With regard to the degree of smoothness of the boundary we shall say that $\partial G$ is of the class $C^m$ for a positive integer $m$ if at each point of $\partial G$ there is a neighborhood $\Omega$ in which $\partial G$ has a representation of the form

$$x_i = g(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n),$$

where $g$ is in $C^m(\Omega)$.

We shall make use of a generalization of the concept of differentiation in order to obtain a large class of differentiable functions. Let $L^2(G)$ denote the space of (equivalence classes of) square-summable functions on $G$.

DEFINITION 2.1. For each integer $k \geqq 0$, $H^k(G)$ is the set of (equivalence classes of) real-valued measurable functions $f$ on $G$ for which the $\alpha$th derivative $D^\alpha f$ belongs to $L^2(G)$ whenever $|\alpha| \leqq k$.

The linear space $H^k(G)$ has a norm and scalar product defined on it by

$$\|f\|_k = \left( \sum_{|\alpha| \leqq k} \int_G |D^\alpha f|^2 \right)^{1/2}$$

and

$$(f, g)_k = \sum_{|\alpha| \leqq k} \int_G (D^\alpha f \cdot D^\alpha g),$$

respectively. From the definition of $H^k(G)$ and the completeness of $L^2(G)$ it follows easily that $H^k(G)$ is complete with respect to the indicated norm and is hence a Hilbert space.

We shall want to distinguish those elements of $H^k(G)$ which vanish on $\partial G$ in some generalized sense. This is accomplished as follows.

DEFINITION 2.2. For each integer $k \geqq 0$, $H_0^k(G)$ is the closure of $C_0^\infty(G)$ in $H^k(G)$.

Thus $H_0^k(G)$ is a closed subspace of $H^k(G)$. It can be shown that if $\partial G$ is of the class $C^k$ and if $\varphi$ belongs to $C^{k-1}(\mathrm{cl}(G))$, then $\varphi$ is in $H_0^k(G)$ if and only if $\varphi$ is in $H^k(G)$ and $D^\alpha \varphi = 0$ on $\partial G$ whenever $|\alpha| \leqq k - 1$. Furthermore it can be shown that an element $f$ in $H^k(G)$ is in $H_0^k(G)$ if and only if $D^\alpha f$ belongs to $H_0^1(G)$ for all $\alpha$ with $|\alpha| \leqq k - 1$.

It is worthwhile to note that $C_0^\infty(G)$ is not in general a dense subset of $H^k(G)$, although it is true that $H_0^0(G) = H^0(G) = L^2(G)$ since $C_0^\infty(G)$ is dense in $L^2(G)$.

Also, we note that most rules of the calculus can be extended to generalized derivatives, [1], [12].

The following result is known as Poincare's inequality and relates the $L^2$-norm of a function to that of its derivatives.

PROPOSITION 2.1. *There is a constant $K \geqq 1$ depending only on $G$ such that for all $\varphi$ in $H_0^1(G)$*

$$\int_G \varphi^2 \leqq K \int_G \sum_{i=1}^n \left(\frac{\partial}{\partial x_i}\varphi\right)^2 .$$

The proof of this proposition [17, pp. 181–182] depends only on integration by parts.

Another useful result for domains with smooth boundaries is the Sobolev lemma. Letting $[\gamma]$ denote the greatest integer less than or equal to the real number $\gamma$, we have the following uniform bound on functions in $H^k(G)$ when $k$ is sufficiently large, [12, pp. 282–284].

PROPOSITION 2.2. *Let $\partial G$ be of class $C^1$ and $k = [n/2] + 1$. There is a constant $C_s$ (depending on $G$) such that for any $u$ in $H^k(G)$ and almost all $x$ in $G$ we have*

$$|u(x)| \leqq C_s \|u\|_k .$$

COROLLARY. *If $u$ is in $H^k(G)$, $k = [n/2] + 1$, then $u$ can be identified with a uniformly continuous function $u(x)$ on $G$ for which the above inequality is true.*

**3. The boundary value problem.** In the following we shall let $M$ and $L$ denote differential operators of second order of the form

$$M = -\sum_{i=1}^n \sum_{j=1}^n \frac{\partial}{\partial x_i} m_{ij}(x) \frac{\partial}{\partial x_j} + m(x) \tag{3.1}$$

$$L = -\sum_{i=1}^n \sum_{j=1}^n \frac{\partial}{\partial x_i} l_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^n l_i(x) \frac{\partial}{\partial x_i} + l(x). \tag{3.2}$$

The (classical) problem under consideration is that of finding a function $u(x, t)$ of the space and time variables $x$ and $t$ which satisfies the partial differential equation

$$M\left(\frac{\partial}{\partial t} u(x, t)\right) + L(u(x,t)) = 0,$$

vanishes on the boundary of the domain $G$ for all $t$ in $R$, and at $t = 0$ is equal to a given function $u_0(x)$ of the space variable $x$.

The operators $M$ and $L$ are meaningful for functions in $C^2(G)$, but we shall extend the domain of these operators in a meaningful way. This will be accomplished by using the Lax–Milgram theorem on bounded positive-definite bilinear forms in Hilbert space to obtain the corresponding Friedrichs extensions of these operators. The domains of the extended operators are dense subsets of $H_0^1(G)$, and it is in this space that the generalized boundary value problem will be formulated. We shall seek a solution $u(x, t)$ belonging to $H_0^1(G)$ for each fixed $t$ in $R$, and this will provide the generalization of the vanishing on the boundary of $G$ in view of the remarks in the previous section on the boundary behavior of functions in $H_0^k(G)$.

The following properties of the operators $M$ and $L$ will be assumed.

PROPERTY 1. ($P_1$). The coefficients occurring in (3.1) and (3.2) are bounded and measurable, and $m(x) \geqq 0$ for $x$ in $G$.

PROPERTY 2. ($P_2$). $M$ is uniformly strongly elliptic on $G$. Hence there is a constant $m_0 > 0$ for which

$$\sum_{i,j=1}^{n} m_{ij}(x)\xi_i\xi_j \geqq m_0 \sum_{i=1}^{n} (\xi_i)^2$$

whenever $\xi = (\xi_1, \cdots, \xi_n)$ is in $R^n$ and $x$ is in $G$.

PROPERTY 3. ($P_3$). For $1 \leqq i, j \leqq n$, $l_{ij}$ and $m_{ij}$ belong to $H^2(G)$.

This last assumption is used to relate the operators $M$ and $L$ to the respective bilinear forms

$$B_M(\varphi, \psi) \equiv \sum_{i,j=1}^{n} \left(m_{ij}\frac{\partial}{\partial x_j}\varphi, \frac{\partial}{\partial x_i}\psi\right)_0 \quad (m\varphi, \psi)_0$$

and

$$B_L(\varphi, \psi) \equiv \sum_{i,j=1}^{n} \left(l_{ij}\frac{\partial}{\partial x_j}\varphi, \frac{\partial}{\partial x_i}\psi\right)_0 + \sum_{i=1}^{n} \left(l_i\frac{\partial}{\partial x_i}\varphi, \psi\right) + (l\varphi, \psi)_0$$

for $\varphi, \psi$ in $C_0^\infty(G)$. It follows from an integration by parts and ($P_3$) that

$$B_M(\varphi, \psi) = (M\varphi, \psi)_0$$

and

$$B_L(\varphi, \psi) = (L\varphi, \psi)_0.$$

The generalized problem which we shall eventually formulate will be stated in terms of the bilinear forms $B_M$ and $B_L$. For this reason there is no necessity for the assumption ($P_3$), and it will be needed only when we wish to consider the linear operators $M$ and $L$ for which it is necessary to be able to differentiate the higher order coefficients.

The inequalities we derive now essentially characterize the bilinear forms $B_M$ and $B_L$. Letting $\varphi$ and $\psi$ denote arbitrary elements of $C_0^\infty(G)$, we have from the Cauchy–Schwarz inequalities

$$|B_M(\varphi, \psi)| = \left| \sum_{i,j=1}^{n} (m_{ij}\varphi_{xj}, \psi_{xi})_0 + (m\varphi, \psi)_0 \right|$$
$$\leqq \bar{m}\left(\sum_{i=1}^{n} \|\psi_{xi}\|_0^2\right)^{1/2}\left(\sum_{j=1}^{n} \|\varphi_{xj}\|_0^2\right)^{1/2} + \bar{m}\|\varphi\|_0\|\psi\|_0,$$

where $\bar{m} = \max_{1 \leqq i,j \leqq n}\{\|m_{ij}\|_\infty, \|m\|_\infty\}$. Hence there is a constant $K_m > 0$ such that

$$|B_M(\varphi, \psi)| \leqq K_m\|\varphi\|_1\|\psi\|_1 \tag{3.3}$$

for all $\varphi, \psi$ in $C_0^\infty(G)$. A similar argument will verify that for some $K_l > 0$ we have

$$|B_L(\varphi, \psi)| \leqq K_l\|\varphi\|_1\|\psi\|_1. \tag{3.4}$$

Hence $B_M$ and $B_L$ are defined by continuity for all $\varphi, \psi$ in $H_0^1(G)$.

From the ellipticity condition ($P_2$) we have for $\varphi$ in $C_0^\infty(G)$

$$B_M(\varphi, \varphi) \geqq m_0 \sum_{i=1}^{n} \|\varphi_{xi}\|_0^2.$$

Poincare's inequality then yields

$$B_M(\varphi, \varphi) \geqq \frac{m_0}{K} \|\varphi\|_0^2,$$

so we have

$$B_M(\varphi, \varphi) \geqq \frac{m_0}{2} \sum_{i=1}^{n} \|\varphi_{xi}\|_0^2 + \frac{m_0}{2K} \|\varphi\|_0^2.$$

Hence there is a constant $k_m > 0$ such that

$$B_M(\varphi, \varphi) \geqq k_m \|\varphi\|_1^2 \tag{3.5}$$

for all $\varphi$ in $C_0^\infty(G)$.

We shall demonstrate that we may assume without loss of generality that $L$ is elliptic and that

$$B_L(\varphi, \varphi) \geqq k_l \|\varphi\|_1^2 \tag{3.6}$$

for some $k_l > 0$ and all $\varphi$ in $C_0^\infty(G)$. In particular, $u(x, t)$ is a solution of the problem if and only if $v(x, t) = e^{-\alpha t}u(x, t)$ satisfies the equation

$$M\left(\frac{\partial v}{\partial t}\right) + (\alpha M + L)v = 0.$$

From (3.4) and (3.5) it follows that (3.6) is true for $\alpha M + L$ instead of $L$ if we choose $\alpha \geqq (K_l + k_l)/k_m$. That is, $(L, \varphi, \varphi)_0 \geqq -K_l\|\varphi\|_1^2$, so

$$((\alpha M + L)\varphi, \varphi)_0 \geqq (\alpha k_m - K_l)\|\varphi\|_1^2 \geqq k_l\|\varphi\|_1^2.$$

The ellipticity is verified as follows: letting $\bar{l} = \sup\{|l_{ij}(x)| : x \in G,\ 1 \leqq i, j \leqq n\}$, we have

$$\begin{aligned}\left|\sum_{j,i=1}^{n} l_{ij}(x)\xi_i\xi_j\right| &\leqq \bar{l} \sum_{i=1}^{n} |\xi_i| \sum_{j=1}^{n} |\xi_j| \\ &\leqq \bar{l}n^2\left(\sum_{i=1}^{n} (\xi_i)^2\right)^{1/2}\left(\sum_{j=1}^{n} (\xi_j)^2\right)^{1/2} \\ &= \bar{l}n^2 \sum_{i=1}^{n} (\xi_i)^2.\end{aligned}$$

Hence

$$\sum_{i=1}^{n} \sum_{j=1}^{n} l_{ij}(x)\xi_i\xi_j \geqq -\bar{l}n^2 \sum_{i=1}^{n} (\xi_i)^2,$$

so we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha m_{ij}(x)+l_{ij}(x))\xi_i\xi_j \geqq (\alpha m_0 - \bar{l}n^2)\sum_{i=1}^{n}(\xi_i)^2$$

for all $x$ in $G$, $\xi$ in $R^n$, so $\alpha M + L$ is uniformly strongly elliptic for $\alpha$ sufficiently large. As stated above, we shall hereafter assume $L$ is elliptic and that (3.6) is satisfied.

We are ready to obtain the extensions of $M$ and $L$ by means of the Lax–Milgram theorem [25, p. 171]. This asserts that there exists a linear transformation $M_0$ with domain $D(M_0)$ dense in $H_0^1(G)$ for which $B_M(\varphi, \psi) = (M_0\varphi, \psi)_0$ whenever $\varphi$ is in $D(M_0)$ and $\psi$ in $H_0^1(G)$. The range of $M_0$ is all of $H^0(G)$, and $M_0$ has an inverse which is a bounded mapping of $H^0(G)$ into $H_0^1(G)$. From ($P_3$) it follows that $(M\varphi, \psi)_0 = (M_0\varphi, \psi)_0$ for all $\varphi, \psi$ in $C_0^\infty(G)$, so $M_0$ is a (weak) extension of $M$, also known as the *minimal operator* associated with $M$, or the *Friedrichs extension.* See [25, p. 173], [31, pp. 329–335] and [21]. The discussion above can be duplicated to obtain the Friedrichs extension $L_0$ of $L$ with domain $D(L_0)$.

The generalized initial boundary value problem may now be formulated in $H_0^1(G)$ as follows: Find a strongly differentiable [18, p. 59] mapping $t \to u(t)$ of $R$ into $H_0^1(G)$ such that

$$B_M(u'(t), \varphi) + B_L(u(t), \varphi) = 0 \tag{3.7}$$

for each $t$ in $R$ and $\varphi$ in $C_0^\infty(G)$ with $u(0) = u_0$, where $u_0$ is a given "initial" function in $H_0^1(G)$.

The proof of the following existence-uniqueness theorem is the context of the next section.

THEOREM 3.1. *Assume* ($P_1$) *and* ($P_2$). *There is a unique bounded linear operator B on* $H_0^1(G)$ *which extends* $-M_0^{-1}L_0$. *If* $u_0$ *is an element of* $H_0^1(G)$, *then there is a unique strongly differentiable mapping* $t \to u(t)$ *of R into* $H_0^1(G)$ *such that*

$$u'(t) = Bu(t) \tag{3.8}$$

*for all t in R and* $u(0) = u_0$.

COROLLARY 3.1. *The vector-valued function* $u(t)$ *satisfies* (3.7).

COROLLARY 3.2. *If* $u(t)$ *belongs to* $D(L_0)$ *then* $u'(t)$ *is in* $D(M_0)$ *and*

$$M_0u'(t) + L_0u(t) = 0 \tag{3.9}$$

*for all t in R.*

**4. Existence and uniqueness.** The operators $M_0$ and $L_0$ are bijections onto $H^0(G)$ from $D(M_0)$ and $D(L_0)$ respectively. We shall show that the bijection $M_0^{-1}L_0$ from $D(L_0)$ onto $D(M_0)$ can be uniquely extended as a bounded linear operator from $H_0^1(G)$ onto itself and that the appropriate exponential of this bounded operator provides the unique solution of the problem in $H_0^1(G)$ as stated in § 3.

We shall verify that the bijection $M_0^{-1}L_0$ is bounded with respect to the norm $\|\cdot\|_1$. If $\varphi$ is in $C_0^\infty(G)$ it follows from (3.4) and (3.5) that

$$k_m\|M_0^{-1}L_0\varphi\|_1^2 \leqq (L_0\varphi, M_0^{-1}L_0\varphi)_0 \leqq K_l\|\varphi\|_1\|M_0^{-1}L_0\varphi\|_1,$$

so we have

$$\|M_0^{-1}L_0\varphi\|_1 \leqq (K_l/k_m)\|\varphi\|_1. \tag{4.1}$$

The constant $K_l/k_m$ depends only on $L$, $M$ and the domain $G$, so (4.1) is true for all $\varphi$ in $C_0^\infty(G)$. Since this set is dense in $H_0^1(G)$ it follows that $M_0^{-1}L_0$ is bounded and has a unique extension to a bounded linear operator on $H_0^1(G)$. We shall let $B$ denote the extension of $-M_0^{-1}L_0$ and remark that $L_0$ is defined only on $D(L_0)$ while $B = -M_0^{-1}L_0$ has been defined on all $H_0^1(G)$ by continuity.

By an elementary argument we can verify that the range of $B$ is all of $H_0^1(G)$ and that its inverse is bounded. Letting $\varphi$ belong to $C_0^\infty(G)$ we have from (3.6) and (3.3)

$$\begin{aligned} k_l\|L_0^{-1}M_0\varphi\|_1^2 &\leqq (M_0\varphi, L_0^{-1}M_0\varphi)_0 \\ &\leqq K_m\|\varphi\|_1\|L_0^{-1}M_0\varphi\|_1, \end{aligned}$$

so we have

$$\|L_0^{-1}M_0\varphi\|_1 \leqq (K_m/k_l)\|\varphi\|_1$$

for all $\varphi$ in $C_0^\infty(G)$; hence $B^{-1} = -L_0^{-1}M_0$ is bounded from $D(M_0)$ to $D(L_0)$. Since $D(M_0)$ is dense in $H_0^1(G)$, $B$ is onto $H_0^1(G)$. In particular if $g$ is in $H_0^1(G)$ there is a sequence $\{g_n\}$ from $D(M_0)$ which converges to $g$ in the topology of $H_0^1(G)$. The boundedness of $B^{-1}$ on $D(M_0)$ implies that the sequence $f_n = B^{-1}g_n$ is Cauchy in $D(L)$, hence converges to some element $f$ in $H_0^1(G)$. From the continuity of $B$ we conclude

$$B(f) = \lim\{B(f_n): n \to \infty\} = g.$$

The construction of $B$ is indicated by Fig. 1.

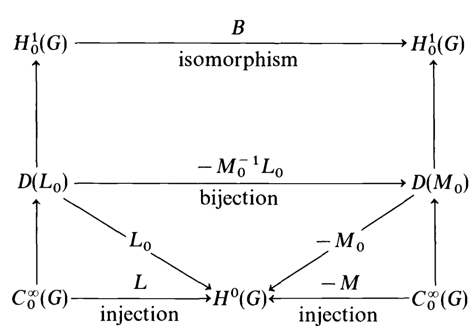


FIG. 1

From the boundedness of $B$ we are able to construct the exponential of the operator $tB$ for each real number $t$. This will yield a one-parameter group $\{E(t): t \text{ in } R\}$ of bounded operators on $H_0^1(G)$, and these will be used to construct the solution of the generalized problem. For each real number $t$, define $E(t)$ by means of the power series

$$\exp(tB) = \sum_{k=0}^{\infty} (tB)^k/k!.$$

Then $E(t)$ is the limit in the uniform operator topology of $\mathscr{L}(H_0^1(G))$ of the sequence

$$\sum_{k=0}^{n} (tB)^k/k!.$$

The convergence of this sequence follows from the completeness of the space $\mathscr{L}(H_0^1(G))$ of bounded linear operators on $H_0^1(G)$, and this is a consequence of the completeness of $H_0^1(G)$. By means of the classical arguments on the convergence of power series with absolute values replaced by the norm $\|\cdot\|_1$, we can show that the indicated power series in $tB$ is convergent for all $t$ in $R$ and that the convergence is uniform on compact subsets of $R$. In this manner we obtain for each real $t$ the bounded linear operator $E(t)$ on $H_0^1(G)$ whose norm satisfies

$$\|E(t)\|_1 \leqq \exp(|t|\,\|B\|_1).$$

For the purpose of reference we collect the properties of this group of operators on $H_0^1(G)$:

(4.2)
(a) $\{E(t): t \text{ in } R\}$ is an Abelian group, and $E(t_1 + t_2) = E(t_1)E(t_2)$, $E(0) = I$.
(b) Each $E(t)$ is a bounded linear operator on $H_0^1(G)$ and the dependence on $t$ is continuous in the uniform operator topology.
(c) $E(t)$ is differentiable in the uniform operator topology, and

$$E'(t) = B \cdot E(t).$$

The group of bounded operators $E(t)$ can now be used to construct our weak solution. Let $u_0$ be the given "initial" function in $H_0^1(G)$ and define

$$u(t) = E(t)u_0 \tag{4.3}$$

for each $t$ in $R$. From (4.2 c) it follows that

$$u'(t) = B \cdot u(t) \tag{4.4}$$

in the strong topology of $H_0^1(G)$. Furthermore we see from (4.2 a) that $u(0) = u_0$ and from (4.2 b) that $u(t)$ is a continuous function of $t$ in the strong topology of $H_0^1(G)$.

We shall verify that the solution given by (4.3) is the *only* such solution to the generalized problem. Letting $u(t)$ denote any such solution, we consider the real-valued function

$$\alpha(t) = (u(t), u(t))_1.$$

By the Cauchy–Schwarz inequality and (4.4) we have

$$|\alpha'(t)| = 2|(Bu(t), u(t))_1| \leqq 2\|B\|_1\alpha(t)$$

for all real $t$. This yields the estimate $\alpha(t) \leqq \exp(2\|B\|_1|t|)\alpha(0)$ from which we have

$$\|u(t)\|_1 \leqq \|u(0)\|_1 \exp(\|B\|_1|t|). \tag{4.5}$$

An immediate consequence of (4.5) is the uniqueness of the solution, for the difference of any two solutions is a solution which is initially zero, hence zero for all $t$ in $R$.

Finally we must verify (3.7). Since $u(t)$ belongs to $H_0^1(G)$, there is a sequence $\{\varphi_n\}$ in $C_0^\infty(G)$ converging to $u(t)$. The boundedness of $B$ on $H_0^1(G)$ implies that $\{B\varphi_n\}$ converges to $u'(t)$. But $M_0(B\varphi_n) + L_0(\varphi_n) = 0$ for all $n$, so we see

$$B_M(u'(t), \varphi) + B_L(u(t), \varphi) = \lim_{n\to\infty} B_M(B\varphi_n, \varphi) + \lim_{n\to\infty} B_L(\varphi_n, \varphi)$$
$$= \lim_{n\to\infty} [(M_0(B\varphi_n), \varphi)_0 + (L_0\varphi_n, \varphi)_0] \equiv 0.$$

Having obtained the weak solution to the generalized problem under consideration, we shall relate the extended operators $L_0$ and $M_0$ on their respective domains to the operators $L_1$ and $M_1$ which are just the extensions of $L$ and $M$ respectively to the domain $H^2(G)$ in the sense of generalized derivatives. *Hereafter we shall always assume* ($P_3$). An integration by parts shows that for all $f$ in $H_0^1(G) \cap H^2(G)$ and $g$ in $H_0^1(G)$ we have

$$(M_1 f, g)_0 = B_M(f, g),$$

and from the characterization of $D(M_0)$ in the Lax–Milgram theorem it follows that

$$H_0^1(G) \cap H^2(G) \subset D(M_0)$$

and that $M_0(f) = M_1(f)$ when $f$ belongs to $H_0^1(G) \cap H^2(G)$. Likewise we have

$$H_0^1(G) \cap H^2(G) \subset D(L_0)$$

and $L_0 = L_1$ on $H_0^1(G) \cap H^2(G)$.

**5. Regularity of the weak solution.** The group of operators $\{E(t): t \text{ in } R\}$ has enabled us to construct a solution by (4.3) of the generalized problem in the weak sense of (3.7). We shall in this section show that each of the subspaces $H_0^1(G) \cap H^p(G)$ remains invariant under the family $\{E(t)\}$, where the integer $p$ depends on the differentiability of the coefficients in $L$ and $M$ as well as the boundary of $G$. These results are based on the regularity problem for the Dirichlet problem. The invariance of these subspaces implies that the solution $u(t)$ given by (4.3) is just as smooth in the $L^2$ sense as is the initial function $u_0$. In fact the special case $L = M$ possesses the solution $u(x, t) = e^{-t}u_0(x)$, and this example shows that we may not in general expect the solution to be more smooth in the space variable than is the initial function. Thus the invariance of the subspaces is the strongest possible result. Finally we shall show that under certain smoothness conditions on the coefficients, boundary and initial function $u_0$, the solution is an analytic function of the time variable and is uniformly continuous (or differentiable) in the space variable.

In order to show that $B$ leaves invariant the spaces $H_0^1(G) \cap H^p(G)$ we shall make use of the results on the Dirichlet problem as presented in [12, pp. 270–307]. The following criterion will be used to specify the assumptions of smoothness on the generalized problem.

DEFINITION 5.1. The generalized initial boundary value problem (3.7) is *p-smooth* for the integer $p \geqq 2$, if

(i) the coefficients in (3.1) and (3.2) satisfy for $1 \leqq i; j \leqq n; l_{ij}, m_{ij} \in C^{p-1}(\mathrm{cl}(\mathbf{G})); m, l, l_i \in C^{p-2}(\mathrm{cl}(\mathbf{G}))$, with $m(x) \geqq 0$ for $x$ in $\mathrm{cl}(\mathbf{G})$;
(ii) $M$ and $L$ are uniformly strongly elliptic in $G$; and
(iii) the boundary $\partial G$ is of class $C^p$.

From [12] there is then for any $f$ in $H^{p-2}(G)$ a unique pair $u, v$ in $H_0^1(G) \cap H^p(G)$ for which $L_0 u = f$ and $M_0 v = f$.

Assume that the generalized problem is $p$-smooth and let $v$ belong to $H_0^1(G) \cap H^p(G)$. $L_0(v)$ is in $H^{p-2}(G)$, so there is a unique $u$ in $H_0^1(G) \cap H^p(G)$ for which $M_0 u = -L_0(v)$. Thus $u = -M_0^{-1} L_0 v$ is in $H_0^1(G) \cap H^p(G)$, so we see that $B$ maps $H_0^1(G) \cap H^p(G)$ into itself. Furthermore $B$ is onto $H_0^1(G) \cap H^p(G)$ from itself, since we need only solve the Dirichlet problem

$$L_0 v = -M_0 u, \qquad v \text{ in } H_0^1(G)$$

for a given $u$ in $H_0^1(G) \cap H^p(G)$ to obtain the $v$ in $H_0^1(G) \cap H^p(G)$ for which $u = -M_0^{-1} L_0 v$. We conclude that $B$ maps each of these subspaces $H_0^1(G) \cap H^q(G)$ onto itself for $p \geqq q \geqq 2$.

*Remark.* We shall hereafter assume that the problem is at least 2-smooth. It follows that if $f$ is in $H^0(G)$ there is a unique $v$ in $H_0^1(G) \cap H^2(G)$ with $M_0 v = f$; hence the domain $D(M_0)$ is contained in $H_0^1(G) \cap H^2(G)$, and by a previous remark thus equal to $H_0^1(G) \cap H^2(G)$. Similarly, $D(L_0) = H_0^1(G) \cap H^2(G)$. We collect these results in the following statement.

PROPOSITION 5.1. *Let the generalized problem be p-smooth for some integer $p \geqq 2$. Then the domains $D(L_0)$ and $D(M_0)$ of the respective Friedrich's extensions coincide with $H_0^1(G) \cap H^2(G)$ and the bounded extension $B$ of $-M_0^{-1} L_0$ on $H_0^1(G)$ leaves invariant each of the subspaces $H_0^1(G) \cap H^q(G)$, where $2 \leqq q \leqq p$.*

We shall make use of the closed graph theorem [18, p. 47] to show that

$$B : H_0^1(G) \cap H^p(G) \to H_0^1(G) \cap H^p(G)$$

is bounded with respect to the norm $\|\cdot\|_p$. The linear operator $B$ is said to be *closed* if whenever $x_n \to x_0$ and $Bx_n \to x_1$ it is necessarily true that $x_1 = Bx_0$. The closed graph theorem asserts that any such closed linear operator is necessarily bounded; its proof depends on the completeness of the space. We remark that since $H_0^1(G) \cap H^p(G)$ is a linear subset of the Hilbert space $H^p(G)$ and since $\|\cdot\|_1 \leqq \|\cdot\|_p$ on this space, $H_0^1(G) \cap H^p(G)$ is a (complete) Hilbert space with the norm $\|\cdot\|_p$.

We shall have need of similar results as this on the boundedness of a linear operator with respect to stronger topologies on subspaces, so we prove a fundamental lemma which with the above discussion implies that $B$ is bounded on $H_0^1(G) \cap H^p(G)$.

FUNDAMENTAL LEMMA. *Let $X_i$ $(i = 1, 2)$ be Banach spaces with respective norms $|\cdot|_i$. Let $Y_i$ be a subset of $X_i$ which is a Banach space with norm $\|\cdot\|_i$ and assume $|y|_i \leqq \|y\|_i$ when $y$ belongs to $Y_i$. Let $T$ be a bounded linear transformation from $X_1$ to $X_2$ such that $T$ maps $Y_1$ into $Y_2$. Then $T$ is bounded from $Y_1$ to $Y_2$.*

*Proof.* We need only show that $T$ is closed as a transformation of $Y_1$ into $Y_2$. Hence let $\{y_n : n \geqq 2\}$ be a sequence in $Y_1$ for which $\|y_n - y_0\|_1 \to 0$ and

$\|Ty_n - y_1\|_2 \to 0$ as $n \to \infty$, where $y_0 \in Y_1$ and $y_1 \in Y_2$. Since

$$\begin{aligned}|y_1 - Ty_0|_2 &\leqq |y_1 - Ty_n|_2 + |T(y_n - y_0)|_2 \\ &\leqq |y_1 - Ty_n|_2 + |T||y_n - y_0|_1 \\ &\leqq \|y_1 - Ty_n\|_2 + |T|\,\|y_n - y_0\|_1,\end{aligned}$$

we have $y_1 = Ty_0$, so $T$ is closed, hence bounded.

The significance of the boundedness of $B$ on $H_0^1(G) \cap H^p(G)$ is that the group of operators $\{E(t): t \text{ in } R\}$ is bounded on $H_0^1(G) \cap H^p(G)$. We state this as the main result of this section.

THEOREM 5.1. *If the generalized problem is p-smooth, then the group of operators* $\{E(t): t \text{ in } R\}$ *leaves invariant the subspace* $H_0^1(G) \cap H^p(G)$. *For each t in R, E(t) is a bijection of* $H_0^1(G) \cap H^p(G)$ *onto itself and is bounded with respect to the norm* $\|\cdot\|_p$.

In fact we could duplicate the discussion on the construction of the $E(t)$ but replace the norm $\|\cdot\|_1$ by $\|\cdot\|_p$ since $B$ is bounded with respect to $\|\cdot\|_p$ and thus obtain the corresponding results with $H_0^1(G)$ replaced by $H_0^1(G) \cap H^p(G)$.

Since we always assume $p \geqq 2$ it follows that $H_0^1(G) \cap H^2(G)$ is invariant under $\{E(t): t \text{ in } R\}$. Hence if $u_0$ is in $H_0^1(G) \cap H^2(G)$ the solution $u(t)$ of the equation (4.4) as given by (4.3) belongs to $H_0^1(G) \cap H^2(G)$ for each $t$ in $R$. Furthermore it follows from (4.4) and the invariance of $H_0^1(G) \cap H^2(G)$ under $B$ that $u'(t)$ belongs to $H_0^1(G) \cap H^2(G)$. But this is the domain of the extended operators $M_0$, so we may apply $M_0$ to both sides of (4.4) to obtain the equation

$$M_0u'(t) + L_0u(t) = 0. \tag{5.1}$$

That is, $M_0u'(t)$ and $L_0u(t)$ are both in $H^0(G)$, so (5.1) is equivalent to (3.7).

Since the group of operators constructed above leaves invariant the subspaces $H_0^1(G) \cap H^q(G)$ for $p \geqq q \geqq 2$ under the assumption of $p$-smoothness, it follows that this group also leaves invariant each of their (point-set) complements. That is, if $u_0$ is in $H_0^1(G) \cap H^{p-1}(G)$ but not in $H^p(G)$ then the same is true of $u(t)$ for each $t$ in $R$. Thus our transformation group preserves smoothness but does not improve it.

We can use the Sobolev lemma to obtain a sufficient condition for the solution $u(t)$ to be a continuous function of the space variable and infinitely differentiable in the time variable.

PROPOSITION 5.2. *Let the generalized problem be p-smooth and* $u_0$ *belong to* $H_0^1(G) \cap H^p(G)$, *with* $p \geqq [n/2] + 1$. *Then for each t in R, u(t) can be identified (a.e.) with a uniformly continuous function of x, denoted by u(x, t), and the mapping* $t \to u(x, t)$ *is infinitely differentiable. The function u(x, t) vanishes identically on the boundary* $\partial G$.

*Proof.* From Theorem 5.1 it follows that $u(t)$ belongs to $H_0^1(G) \cap H^p(G)$ for every $t$ in $R$, hence by Sobolev's lemma it can be identified with a uniformly continuous function $u(x, t)$ on $G$. Also from Sobolev's inequality it follows that if $\delta \neq 0$

$$\begin{aligned}&|\delta^{-1}(u(x, t + \delta) - u(x, t)) - B \cdot u(x, t)| \\ &\qquad = |(\delta^{-1}(E(\delta) - I) - B)u(x, t)| \\ &\qquad \leqq C_s\|(\delta^{-1}(E(\delta) - I) - B)u(t)\|_p,\end{aligned}$$

where the constant $C_s$ depends only on $n$ and $\partial G$. Since the group $\{E(t):t \text{ in } R\}$ is infinitely differentiable in the uniform operator topology induced by $\|\cdot\|_p$ and its $k$th derivative is $B^k \cdot E(t)$, the last term in the above inequality converges to zero as $\delta \to 0$. This establishes the differentiability of $u(x, t)$ and the equality

$$\frac{\partial}{\partial t}u(x, t) = B \cdot u(x, t)$$

for each $x$ in $G$. A repetition of this argument will show that $u(x, t)$ is infinitely differentiable with respect to $t$ and that its derivatives agree with the corresponding derivatives of $u(t)$ in $H_0^1(G) \cap H^p(G)$.

In fact we see that $u(x, t)$ is *analytic* in $t$, for the remainder term

$$R_n(x, t) = \frac{\partial^{n+1}}{\partial t^{n+1}}u(x, T)t^{n+1}/(n + 1)!$$

(where $|T| < |t|$) of the Taylor formula converges to zero as $n$ increases. That is,

$$\begin{aligned} |R_n(x, t)| &= |((tB)^{n+1}/(n + 1)!)u(x, T)| \\ &\leqq C_s\|(tB)^{n+1}/(n + 1)!\|_p\|u_0\|_p \exp(\|tB\|_p) \end{aligned}$$

by Sobolev's lemma, and the convergence of the power series for $\exp(tB)$ in $\mathscr{L}(H_0^1(G) \cap H^p(G))$ implies that its $(n + 1)$st term converges to zero in $\mathscr{L}(H_0^1(G) \cap H^p(G))$.

Finally we note that the uniform continuity of $u(x, t)$ in the space variable and its belonging to $H_0^1(G)$ imply that it vanishes on the boundary.

COROLLARY. *The solution $u(t)$ of the generalized problem can be identified with a function $u(x, t)$ in $C^m(\mathrm{cl}(\mathbf{G}))$ for each $t$ in $R$, where $m = p - [n/2] - 1$. Hence a classical solution of the problem exists if $p \geqq [n/2] + 3$.*

**6. Asymptotic behavior.** We shall investigate the asymptotic behavior of the solution of the problem under consideration. The additional assumptions of symmetry of the operators or of constant coefficients are reasonable from the standpoint of physical motivation. We shall show in this section that under the appropriate conditions the solution $u(t)$ of our problem decays exponentially along with its derivatives up through a specified order. Furthermore we shall obtain more regularity type results which will imply that if the initial function has a given number of derivatives vanishing on the boundary then the solution has this same property.

Assume throughout the remainder of this section that $M$ is symmetric and that the statements $(P_1)$ and $(P_2)$ of § 3 are valid. By letting $u_0$ in $H_0^1(G)$ be arbitrary, it follows from the strong differentiability of $u(t)$ and the symmetry of the bilinear form $B_M$ on $H_0^1(G)$ that the real-valued function

$$\gamma(t) = B_M(u(t), u(t))$$

is continuously differentiable and that

$$\tfrac{1}{2}\gamma'(t) = B_M(u'(t), u(t)).$$

From (3.7), (3.6) and (3.3), respectively, we see that

$$\tfrac{1}{2}\gamma'(t) = -B_L(u(t), u(t))$$
$$\leqq -k_l\|u(t)\|_1^2 \leqq -k_l/K_m\gamma(t).$$

Hence for all $t \geqq 0$ we have

$$\gamma(t) \leqq \gamma(0)\exp(-2k_l/K_m t).$$

Using (3.5) and (3.3) we then obtain the estimate

$$\|u(t)\|_1 \leqq (K_m/k_m)^{1/2}\|u_0\|_1 \exp(-k_l/K_m t) \tag{6.1}$$

for $t \geqq 0$. This estimate (6.1) implying the exponential decay of the solution and its first derivatives in the sense of their $L^2$-norms is true in particular whenever $M$ has constant coefficients, for then it can be written in a symmetric form.

Because of the boundedness of the operator $B$ on $H_0^1(G)$ it has made no difference whether we consider (5.1) or the equation

$$-M_0u'(t) + L_0u(t) = 0.$$

However it is apparent in the previous paragraph that the sign of $M$ is fundamental in obtaining the estimate (6.1) describing the asymptotic behavior in the norm $\|\cdot\|_1$ for the solution. Without this sign consideration we would only obtain an estimate of the form (4.5) which allows the solution to grow exponentially with the time variable. The estimate (6.1) is valid only for $t \geqq 0$, but this is the case of physical interest. The previously used estimate also implies that for $t \leqq 0$

$$\gamma(t) \geqq \gamma(0)\exp(-k_l/K_m t)$$

and by (3.3) and (3.5) would follow

$$\|u(t)\|_1 \geqq (k_m/K_m)^{1/2}\|u_0\|_1 \exp(k_l/K_m|t|) \tag{6.2}$$

whenever $t \leqq 0$. The inequalities (4.5), (6.1) and (6.2) describe the behavior of $u(t)$ in the large: the solution grows exponentially as $t \to -\infty$ and decays exponentially as $t \to \infty$ whenever $M$ is symmetric.

We should note that in order for the above results to be significant we must assume that (3.6) is true for the "original" operator $L$. That is, by replacing $L$ by $\alpha M + L$ we actually obtain the solution $e^{\alpha t}u(t)$ which is bounded by $(K_m/k_m)^{1/2}\|u_0\|_1$ $\exp((\alpha - kl/Km)t)$. But our sufficient choice for $\alpha$ given in § 3 implies that $\alpha - k_l/K_m = K_l/k_m + k_l/k_m - k_l/K_m$, and this quantity will in general be positive. In this event we would not be able to show that the solution decayed exponentially for $t \to \infty$. An example of this is the case $M = -d^2/dx^2$, $L = I$ and $u_0(x) = \{\sinh(x), 0 \leqq x \leqq \tfrac{1}{2}; \sinh(1-x), \tfrac{1}{2} \leqq x \leqq 1\}$. The solution $u(x,t) = u_0(x)e^t$ in $H_0^1(G)$ grows exponentially.

We will obtain some bounds on the higher order derivatives of the solution. To do so let us assume that the generalized problem is $(k+1)$-smooth, $k$ being an integer $\geqq 1$, and that $M$ and $L$ have constant coefficients.

Our first task is to show that the space $H_0^{1+k}(G)$ is invariant under the group of operators $\{E(t)\}$. Since $B$ has already been shown to be bounded with respect to the $(k+1)$-norm, it will suffice to show that $B$ maps $H_0^{1+k}(G)$ into itself. Hence let $\psi$ be an element of $C_0^\infty(G)$. The regularity results previously obtained imply that $B\psi$ belongs to $H^{1+k}(G)$. If $|\alpha| \leqq k$ then since $D^\alpha\psi$ belongs to $C_0^\infty(G)$ we have $BD^\alpha\psi$ belongs to $H_0^1(G) \cap H^2(G)$ and hence

$$M_0(BD^\alpha\psi) + L_0(D^\alpha\psi) = 0.$$

But $M_0$ and $L_0$ have constant coefficients, so we see

$$\begin{aligned} M_0(BD^\alpha\psi) &= -L_0(D^\alpha\psi) = -D^\alpha(L_0\psi) \\ &= D^\alpha(M_0B\psi) = M_0(D^\alpha B\psi). \end{aligned}$$

That is, we have

$$D^\alpha(B\psi) = B(D^\alpha\psi) \tag{6.3}$$

belongs to $H_0^1(G)$ whenever $|\alpha| \leqq k$, so in particular $B\psi$ must be in $H_0^{1+k}(G)$. Since $B$ maps $C_0^\infty(G)$ into $H_0^{1+k}(G)$ and is bounded with respect to the $(k + 1)$-norm, it follows that $B$ maps all of $H_0^{1+k}(G)$ into itself. Also it is easy to show that (6.3) is true for all $\psi$ in $H_0^{1+k}$; the argument is similar to that used below to verify (6.4).

We have shown that each $E(t)$ maps $H_0^{1+k}(G)$ onto itself and we shall verify that when $|\alpha| \leqq k$

$$D^\alpha E(t)\psi = E(t)D^\alpha\psi \tag{6.4}$$

for each $\psi$ in $H_0^{1+k}(G)$. Let $E_n(t)$ denote the $n$th partial sum of the series which defined $E(t)$. Since $D^\alpha$ commutes with $B$ it also commutes with each $E_n(t)$. Thus for any $\varphi$ in $C_0^\infty(G)$ we have

$$\begin{aligned} (E(t)D^\alpha\psi, \varphi)_0 &= \lim_{n\to\infty} (E_n(t)D^\alpha\psi, \varphi)_0 = \lim_{n\to\infty} (D^\alpha E_n(t)\psi, \varphi)_0 \\ &= \lim_{n\to\infty} (E_n(t)\psi, (-1)^{|\alpha|}D^\alpha\varphi)_0 = (E(t)\psi, (-1)^{|\alpha|}D^\alpha\varphi)_0 \\ &= (D^\alpha E(t)\psi, \varphi)_0. \end{aligned}$$

The desired estimates on the derivatives of a solution to the generalized problem are now easily obtained. Let $u_0$ be given in $H_0^{1+k}(G)$. Then $u(t) = E(t)u_0$ belongs to $H_0^{1+k}(G)$ and from (6.4) it follows that $D^\alpha u(t)$ is the unique solution in $H_0^1(G)$ of the generalized problem with initial condition $D^\alpha u(0) = D^\alpha u_0$. Hence we have the estimate

$$\|D^\alpha u(t)\|_1 \leqq (K_m/k_m)^{1/2}\|D^\alpha u_0\|_1 \exp\left(-\frac{kl}{Km}t\right) \tag{6.5}$$

for all $\alpha$ with $|\alpha| \leqq k$.

From the inequality (6.5) one can proceed by means of the Sobolev lemma to obtain pointwise bounds on the solution and various derivatives. The smoothness of the problem now depends only on the differentiability of the boundary $\partial G$, so the largest number $k$ for which the solution belongs to $H_0^{1+k}(G)$ and (6.5) is true when $|\alpha| \leqq k$ depends on the boundary $\partial G$ and the initial function $u_0$.

**7. The nonhomogeneous problem.** The objective in this section is to extend the previous results to the nonhomogeneous equation

$$M_1 u'(t) + L_1 u(t) = f(t) \tag{7.1}$$

with a solution in $H^2(G)$ satisfying a nonhomogeneous time-varying boundary condition. Note that for any $v$ in $H^2(G)$ the expression $M_1 v$ denotes the element of $H^0(G)$ defined as a linear combination of $v$ and its first and second order strong derivatives as specified by (3.1). It follows that the linear mapping $v \mapsto M_1 v$ is bounded from $H^2(G)$ to $H^0(G)$, and we have shown that $M_0$ is the restriction of $M_1$ to the subspace $H_0^1(G) \cap H^2(G)$. The corresponding statements hold for the operator $L_1$.

We shall first prove the following result.

LEMMA 7.1. *Assume that the (associated homogeneous) problem is 2-smooth and $f(t)$ is strongly continuous in $H^0(G)$. There is a unique mapping $t \mapsto w(t)$ of $R$ into $H_0^1(G) \cap H^2(G)$ with a strongly continuous derivative which satisfies (7.1) and the initial condition $w(0) = 0$.*

*Proof.* The operator $M_0^{-1}$ is continuous from $H^0(G)$ into $H_0^1(G)$, so it follows from the Fundamental Lemma of § 5 that it not only maps $H^0(G)$ onto $H_0^1(G) \cap H^2(G)$ but is continuous with respect to the stronger norm $\|\cdot\|_2$ on $H_0^1(G) \cap H^2(G)$. The strong continuity of $f(t)$ implies that $M_0^{-1}f(t)$ is strongly continuous with respect to $\|\cdot\|_2$. Also the continuity of the mapping $\xi \mapsto E(\xi)$ in the uniform operator topology of $\mathscr{L}(H_0^1(G) \cap H^2(G))$ implies that for each $t$ in $R$ the function

$$T \mapsto E(t - T)M_0^{-1}f(T)$$

from $R$ into $H_0^1(G) \cap H^2(G)$ is strongly continuous.

By means of the calculus of vector-valued functions [18, pp. 56–58] we have given for each real number $t$ an element of $H_0^1(G) \cap H^2(G)$ denoted by

$$w(t) = \int_0^t E(t - T)M_0^{-1}f(T)\,dT.$$

The integral is taken as a limit of Riemann sums with respect to the norm $\|\cdot\|_2$. From the differentiability of $E(t)$ it follows that $w(t)$ is differentiable with respect to $\|\cdot\|_2$ and that

$$\begin{aligned} w'(t) &= \int_0^t E'(t - T)M_0^{-1}f(T)\,dT + E(0)M_0^{-1}f(t) \\ &= \int_0^t B \cdot E(t - T)M_0^{-1}f(T)\,dT + M_0^{-1}f(t). \end{aligned}$$

The continuity and linearity of $B$ then implies that

$$w'(t) = Bw(t) + M_0^{-1}f(t).$$

Each term of this last equation belongs to $H_0^1(G) \cap H^2(G)$ so we have

$$M_0 w'(t) + L_0 w(t) = f(t),$$

where $w(t)$ has a strongly continuous derivative in $H_0^1(G) \cap H^2(G)$ and $w(0) = 0$.

The uniqueness of $w(t)$ follows from the corresponding result for the homogeneous equation by linearity.

We shall proceed by means of this lemma to the case of time-varying boundary conditions. The boundary condition is given by a function $t \mapsto \beta(t)$ from $R$ to $H^2(G)$ with a strongly continuous derivative in the $\|\cdot\|_2$-norm. The initial function $u_0$ belongs to $H^2(G)$, and these functions satisfy a compatibility condition

$$u_0 - \beta(0) \in H_0^1(G). \tag{7.2}$$

Define a function in $H^0(G)$ by

$$F(t) = f(t) - M_1\beta'(t) - L_1\beta(t)$$

for each $t$ in $R$. The continuity of $\beta$ and $\beta'$ in $H^2(G)$ implies that $F(t)$ is continuous in $H^0(G)$. From the preceding lemma we know that the function

$$v(t) = \int_0^t E(t - T)M_0^{-1}F(T)\,dT$$

in $H_0^1(G) \cap H^2(G)$ satisfies the equation

$$M_0v'(t) + L_0v(t) = F(t)$$

and the initial condition $v(0) = 0$. Now we define the function

$$u(t) = \beta(t) + E(t)(u_0 - \beta(0)) + v(t) \tag{7.3}$$

which has a strongly continuous derivative in $H^2(G)$. Furthermore we may verify directly that $u(t)$ satisfies the requirements in the following theorem which is the main result of this section.

THEOREM 7.1. *Let the (associated homogeneous) problem be 2-smooth, $f(t)$ be strongly continuous in $H^0(G)$, $\beta(t)$ have a strongly continuous derivative in $H^2(G)$, and $u_0$ be a function in $H^2(G)$ for which* (7.2) *is satisfied. There is a unique strongly differentiable function $u(t)$ in $H^2(G)$ given by* (7.3) *which satisfies* (7.1) *and for which $u(t) - \beta(t)$ is in $H_0^1(G)$ for all $t$ in $R$, and $u(0) = u_0$.*

*Remark.* In verifying (7.1) it is essential to note that $M_1M_0^{-1} = I$ on $H^0(G)$ and hence $M_1B = -L_0$ on $H_0^1(G) \cap H^2(G)$.

In the same manner we can verify the following result.

COROLLARY. *Let the problem be $p$-smooth ($p \geqq 2$), $f(t)$ be strongly continuous in $H^{p-2}(G)$, $\beta(t)$ have a strongly continuous derivative in $H^p(G)$, $u_0$ belong to $H^p(G)$ and satisfy* (7.2). *Then there is a strongly differentiable mapping $u(t)$ of $R$ into $H^p(G)$ satisfying* (7.1) *with $u(t) - \beta(t)$ belonging to $H_0^1(G)$ for all real $t$ and $u(0) = u_0$.*

**8. Remarks on parabolic equations.** In this section we shall briefly discuss an interesting relationship between the solution $u_\lambda(t)$ of the pseudoparabolic equation

$$(\lambda L_0 + I)u_\lambda'(t) + L_0u_\lambda(t) = 0 \tag{8.1}$$

and the solution $u(t)$ of the parabolic equation

$$u'(t) + L_0u(t) = 0, \tag{8.2}$$

both of which satisfy the same initial condition and a homogeneous boundary condition. From the very form of these equations one might expect that for $\lambda$

sufficiently small the solution $u_\lambda(t)$ is "close" to $u(t)$ in some generalized sense. We shall show that this is exactly the situation. This result is normally assumed in the formulation of these boundary value problems from a physical model, since one often takes $u(t)$ as an approximation for $u_\lambda(t)$ by assuming that the viscosity coefficient $\lambda$ is zero.

The generalized solution of the parabolic equation (8.2) can be constructed by means of the semigroup theory of Hille and Yosida. This method is used in [25]. The extended operator $-L_0$ is such that its resolvent set contains all of the positive real axis and furthermore

$$\|(\lambda L_0 + I)^{-1}\|_0 \leqq (\lambda l_0 + 1)^{-1}$$

for all positive numbers $\lambda$ and a constant $l_0$ depending only on $L_0$ and the domain $G$. These are precisely the conditions for which the Hille–Yosida theorem can be used to construct a strongly continuous semigroup of bounded linear operators $\{S(t): t \geqq 0\}$ with the property that if $u_0$ belongs to $D(L_0)$ then the function

$$u(t) = S(t)u_0 \tag{8.3}$$

is strongly continuous in $L^2(G)$, belongs to $D(L_0)$ and satisfies $u(0) = u_0$, $u'(t) = -L_0 u(t)$ for $t \geqq 0$.

The semigroup $\{S(t): t \geqq 0\}$ is constructed as follows. Define for each number $\lambda > 0$ an operator

$$L_\lambda = (I + \lambda L_0)^{-1} L_0$$

and show that it is a bounded operator on $L^2(G)$. Also for any $v$ in $D(L_0)$ we have

$$\lim_{\lambda \to 0} \|L_\lambda v - L_0 v\|_0 = 0.$$

Since $L_\lambda$ is bounded we can define for each number $t$ the bounded operator

$$E_\lambda(t) = \exp(-tL_\lambda).$$

It can then be shown that, for those $t \geqq 0$, $E_\lambda(t)$ converges to an operator $S(t)$ in the strong sense as $\lambda$ converges to zero, and that $\{S(t): t \geqq 0\}$ is the desired semigroup.

The relation between the solution of the parabolic problem given by (8.3) and the solution to the equation (8.1) is now clear. The operator $L_\lambda$ above can be expressed as $L_\lambda = M_0^{-1} L_0$ for the special case $M_0 = \lambda L_0 + I$ which we are considering, hence $E_\lambda(t)$ is for each $\lambda > 0$ the group of bounded operators constructed in § 4 for the equation (8.1). The solution to (8.1) is then given by

$$u_\lambda(t) = E_\lambda(t) u_0.$$

In order for the parabolic problem to be meaningful we require that $u_0$ belong to $D(L_0)$. The statement above that $E_\lambda(t)$ converges in the strong sense to $S(t)$ is exactly the result we seek. That is, for $t \geqq 0$ and $u_0$ in $D(L_0)$ we have

$$\lim_{\lambda \to 0} \|u_\lambda(t) - u(t)\|_0 = 0, \tag{8.4}$$

and this is the precise form of the statement that $u(t)$ is "close" to $u_\lambda(t)$ when $\lambda$ is small.

This result can be generalized to the equation

$$(\lambda M_0 + I)u_\lambda'(t) + L_0 u_\lambda(t) = 0, \tag{8.5}$$

for which we have the following.

THEOREM. *Assume that the generalized problem* (8.5) *is* 3-*smooth and* $u_0$ *belongs to* $H_0^1(G) \cap H^2(G)$. *Then for all* $t \geqq 0$ *the solution* $u(x, t)$ *of the parabolic equation* (8.2) *given by* (8.3) *is the* $\|\cdot\|_0$-*limit of the solutions* $u_\lambda(x, t)$ *of the pseudoparabolic equation* (8.5). (See [37].)

The proof of this result is modeled after the proof of the Hille–Yosida Theorem [39], but the details are considerably more involved since there are two different operators to consider.

**9. The Schauder estimates.** We shall begin an independent but parallel study of the problem considered previously, and this investigation is based on the solution of the Dirichlet problem by the method of Schauder. In this context the operators $M$ and $L$ are studied on the Banach space of functions with uniformly Hölder continuous second order derivatives, and we shall see that the product operator $M^{-1}L$ is bounded on this space. This will enable the construction of the solution by exponentiating this bounded operator. In proving the boundedness of $M^{-1}L$, we shall make use of the Schauder estimates (up to the boundary) and the closed graph theorem, so the completeness of the function spaces used is essential.

The existence, uniqueness and regularity results are essentially the same as those obtained previously. That is, the solution is obtained directly as the exponential of a bounded operator, and this operator leaves certain subspaces invariant. There will be no need of an analogue of Sobolev's lemma since convergence in the function space will imply pointwise convergence, hence this method always yields a pointwise solution.

A function $v(x)$ is said to belong to the class $C^{m+\alpha}(\mathrm{cl}(\mathbf{G}))$, where $m$ is a nonnegative number and $0 < \alpha < 1$, if $v$ belongs to $C^m(\mathrm{cl}(\mathbf{G}))$ and all of its $m$th order derivatives are uniformly Hölder continuous of exponent $\alpha$. By this last statement we mean

$$H_\alpha^m(v) = \sup\left\{\frac{D^j v(x) - D^j v(y)}{|x - y|^\alpha} : x, y \in G, |j| = m\right\}$$

is finite. We define on $C^{m+\alpha}(\mathrm{cl}(\mathbf{G}))$ a norm

$$|v|_{m+\alpha} = |v|_m + H_\alpha^m(v),$$

where

$$|v|_m = \sum_{i=0}^{m} \sup\{|D^j v(x)| : x \in G, |j| = i\}.$$

Furthermore one can show that $C^{m+\alpha}(\mathrm{cl}(\mathbf{G}))$ is complete with respect to the norm $|\cdot|_{m+\alpha}$, so it is a Banach space.

The boundary $\partial G$ is in the class $C^{m+\alpha}$ whenever there is at each point of $\partial G$ a neighborhood $S$ in which $\partial G$ has a parametric representation of the form

$$x_i = g(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n),$$

where $g$ belongs to $C^{m+\alpha}(\mathrm{cl}(\mathbf{S}))$.

The operators $M$ and $L$ will be assumed to have the forms

$$M = \sum_{i,j=1}^{n} m_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} m_i(x)\frac{\partial}{\partial x_i} - m(x),$$

$$L = \sum_{i,j=1}^{n} l_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} l_i(x)\frac{\partial}{\partial x_i} - l(x).$$

The following assumptions will always be made:
($A_1$): Each of the coefficients which appears above belongs to $C^\alpha(\mathrm{cl}(\mathbf{G}))$ and the coefficients $m(x)$, $l(x)$ are nonnegative.
($A_2$): $M$ and $L$ are uniformly elliptic, hence there are positive constants $m_0$ and $l_0$ for which

$$\sum_{i,j=1}^{n} m_{ij}(x)\xi_i\xi_j \geqq m_0 \sum_{i=1}^{n} (\xi_i)^2,$$

$$\sum_{i,j=1}^{n} l_{ij}(x)\xi_i\xi_j \geqq l_0 \sum_{i=1}^{n} (\xi_i)^2$$

whenever $\xi$ belongs to $R^n$ and $x$ belongs to $G$.

The technique which we shall use here is totally dependent on the existing results on the solution of the Dirichlet problem. That is, given a function $f$ in $C^\alpha(\mathrm{cl}(\mathbf{G}))$, find a function $u$ for which

$$Lu = f$$

in $G$ and $u(x) = 0$ when $x$ is on $\partial G$. In proving the existence of a solution of such a problem by the method of continuity, the following a priori estimate is essential [2], [12], [29].

THEOREM 9.1. *Assume $(A_1), (A_2)$, that $f$ belongs to $C^\alpha(\mathrm{cl}(\mathbf{G}))$ and that $\partial G$ is of class $C^{2+\alpha}$. If $u$ is a function in $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ for which $Lu = f$ in $G$ and $u = 0$ on $\partial G$, then*

(9.1) $$|u|_{2+\alpha} \leqq K_L |f|_\alpha,$$

*where $K_L$ depends only on $L$ and $G$.*

This is a very strong result and is used to prove the following existence theorem for the Dirichlet problem.

THEOREM 9.2. *Assume $(A_1), (A_2)$, that $f$ belongs to $C^\alpha(\mathrm{cl}(\mathbf{G}))$ and that $\partial G$ is of class $C^{2+\alpha}$. Then there exists a unique function $u$ in $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ for which $L(u) = f$ in $G$ and $u = 0$ on $\partial G$.*

Concerning the differentiability of solutions of the Dirichlet problem we have the following result.

THEOREM 9.3. *Let $p$ be a nonnegative integer for which $f$ and all the coefficients which appear in $L$ belong to $C^{p+\alpha}(\mathrm{cl}(\mathbf{G}))$ and for which $\partial G$ is of class $C^{p+2+\alpha}$. Then any function $u$ in $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ for which $Lu = f$ in $G$ and $u = 0$ on $\partial G$ belongs to $C^{p+2+\alpha}(\mathrm{cl}(\mathbf{G}))$.*

Corresponding results are of course valid for the operator $M$.

We are now ready to study the behavior of $L$ and $M$ on the appropriate function space. Define $C_0^{m+\alpha}(\mathrm{cl}(\mathbf{G}))$ as the set of functions in $C^{m+\alpha}(\mathrm{cl}(\mathbf{G}))$ that vanish on

$\partial G$. With the norm $|\cdot|_{m+\alpha}$, $C_0^{m+\alpha}(\mathrm{cl}(\mathbf{G}))$ is a Banach subspace of $C^{m+\alpha}(\mathrm{cl}(\mathbf{G}))$, because convergence with respect to $|\cdot|_{m+\alpha}$ implies uniform convergence of the function and hence preserves the zero condition on the boundary. From the results stated above for the Dirichlet problem it is immediate that $L$ maps $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ onto $C^{\alpha}(\mathrm{cl}(\mathbf{G}))$ in a one-to-one manner. From (9.1) it follows that $L^{-1}$ is bounded, so from the closed graph theorem it is immediate that $L$ is a linear homeomorphism of $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ onto $C^{\alpha}(\mathrm{cl}(\mathbf{G}))$. The same is true of $M$, so we may conclude that $M^{-1}L$ is a bounded linear operator on $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$.

For each real number $t$ we construct the exponential of the bounded operator $-tM^{-1}L$ by means of the power series

$$E(t) = \exp(-tM^{-1}L) = \sum_{k=0}^{\infty} (-tM^{-1}L)^k/k!$$

This power series converges with respect to the uniform operator topology induced on $\mathscr{L}(C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G})))$ by the norm $|\cdot|_{2+\alpha}$ on $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$. It is not difficult to verify that the family $\{E(t): t \text{ in } R\}$ is an infinitely differentiable group of bounded linear operators and that

$$E'(t) = -M^{-1} \cdot L \cdot E(t) \tag{9.2}$$

for all $t$ in $R$. This group of linear operators provides the existence portion of the following result.

THEOREM 9.4. *Assume that $(A_1)$ and $(A_2)$ are true, $\partial G$ is of class $C^{2+\alpha}$ and that $u_0$ is a given function in $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$. There is a unique strongly differentiable mapping*

$$t \mapsto u(t)$$

*of $R$ into $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ for which*

$$Mu'(t) + Lu(t) = 0 \tag{9.3}$$

*in $C^{\alpha}(\mathrm{cl}(\mathbf{G}))$ for all real $t$ and $u(0) = u_0$. This mapping is infinitely differentiable.*

*Proof.* Define $u(t) = E(t)u_0$. It is immediate that $u(0) = u_0$ and that $u(t)$ is infinitely differentiable. Furthermore since $M$ and $L$ are both bijections of $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ onto $C^{\alpha}(\mathrm{cl}(\mathbf{G}))$ it follows from (9.2) that (9.3) is true.

We shall verify the uniqueness of the solution. The solution must necessarily satisfy the integral equation

$$u(t) = u(0) - M^{-1}L \int_0^t u(T)\, dT$$

because of the boundedness and linearity of $M^{-1}L$ on $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$. The integral is taken as usual as the limit in the $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ topology of Riemann sums. From this equation we have

$$|u(t)|_{2+\alpha} \leqq |u(0)|_{2+\alpha} + |M^{-1}L|_{2+\alpha} \int_0^t |u(T)|_{2+\alpha}\, dT \tag{9.4}$$

for all $t$ in $R$.

LEMMA 9.1 (Gronewall). *If $\varphi$ is continuous and nonnegative on $R^+ = \{r \in R : r \geqq 0\}$ and if*

$$\varphi(\xi) \leqq c + m \int_0^\xi \varphi(T)\, dT$$

*for all $\xi \geqq 0$ then*

$$\varphi(\xi) \leqq c \exp(m\xi).$$

*Proof.* From the hypotheses we have

$$\frac{1}{m}\frac{d}{dt}\left\{\ln\left(c + m\int_0^t \varphi(T)\, dT\right)\right\} \leqq 1$$

so

$$\ln\left[\left(c + m\int_0^t \varphi(T)\, dT\right)\Big/ c\right] \leqq mt.$$

Hence

$$c + m\int_0^t \varphi(T)\, dT \leqq c \exp(mt)$$

and the result is immediate from this inequality.

This lemma together with (9.4) shows that *any* solution of the problem satisfies

$$|u(t)|_{2+\alpha} \leqq |u(0)|_{2+\alpha} \exp(|M^{-1}L|\,|t|). \tag{9.5}$$

In particular the difference between any two solutions satisfies (9.5) with $u(0) = 0$, hence the solutions are identical.

The solution thus obtained can easily be seen to be a solution in the pointwise sense. For each real number $t$, $u(t)$ belongs to $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ and is therefore a real-valued function of the space variable whose value at the point $x$ of $G$ is denoted by $u(x, t)$. Furthermore for any real $\delta \neq 0$ we have

$$\begin{aligned} |\delta^{-1}(u(x, t + \delta) - u(x, t)) + M^{-1}L[u(x, t)]| \\ &= |(\delta^{-1}(E(\delta) - I) + M^{-1}L)[u(x, t)]| \\ &\leqq |(\delta^{-1}(E(\delta) - I) + M^{-1}L)u(t)|_{2+\alpha} \\ &\leqq |\delta^{-1}(E(\delta) - I) + M^{-1}L|_{2+\alpha}|u(t)|_{2+\alpha} \end{aligned}$$

so the mapping $t \mapsto u(x, t)$, $x$ in $G$, is differentiable, in fact infinitely differentiable, since the group $\{E(t) : t \text{ in } R\}$ is infinitely differentiable. Consequently Theorem 9.4 implies that the equation (9.3) possesses a pointwise solution $u(x, t)$ which belongs to $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ for each $t$ in $R$, vanishes on the boundary $\partial G$ and is infinitely differentiable with respect to the time variable $t$.

The results on the regularity of the solution are completely analogous to those obtained previously, and the same methods may be used as before. In particular we use the results stated above on the regularity of the solution to the Dirichlet problem to prove the following.

PROPOSITION 9.1. *Let p be a nonnegative integer and assume $\partial G$ is of class $C^{p+2+\alpha}$. Let the operator L satisfy $(A_1)$ and $(A_2)$ and assume that its coefficients belong to $C^{p+\alpha}(\mathrm{cl}(\mathbf{G}))$. Then L is a linear homeomorphism of $C_0^{p+2+\alpha}(\mathrm{cl}(\mathbf{G}))$ onto $C^{p+\alpha}(\mathrm{cl}(\mathbf{G}))$.*

*Proof.* The results above for the Dirichlet problem show that $L$ is a bijection as stated, so the boundedness of $L$ and $L^{-1}$ is the only question. But this is settled by the Fundamental Lemma of § 5.

COROLLARY. *Let p be a nonnegative integer such that $\partial G$ is of class $C^{p+\alpha+2}$ and the operators M and L satisfy $(A_1)$ and $(A_2)$, and their coefficients belong to $C^{p+\alpha}(\mathrm{cl}(\mathbf{G}))$. Then $M^{-1}L$ is a linear homeomorphism of $C_0^{p+\alpha+2}(\mathrm{cl}(\mathbf{G}))$ onto itself.*

From the boundedness of $M^{-1}L$ with respect to the norm $|\cdot|_{p+\alpha+2}$ on $C_0^{p+\alpha+2}(\mathrm{cl}(\mathbf{G}))$ it follows as before that the group of operators $\{E(t):t \text{ in } R\}$ is bounded on and leaves invariant the space $C_0^{p+\alpha+2}(\mathrm{cl}(\mathbf{G}))$. This yields the following result on the regularity of solutions.

THEOREM 9.5. *Under the assumptions of the corollary above, the solution $u(t)$ of the problem* (9.3), (9.4) *belongs to $C_0^{p+\alpha+2}(\mathrm{cl}(\mathbf{G}))$ for each t in R if and only if $u_0$ belongs to $C_0^{p+\alpha+2}(\mathrm{cl}(\mathbf{G}))$.*

The nonhomogeneous problem can be handled in much the same way as was done previously. The main result in this direction is the following.

THEOREM 9.6. *Assume that $(A_1)$ and $(A_2)$ are true and the $\partial G$ is of class $C^{2+\alpha}$. Let $f(t)$ be a (strongly) continuous function of R into $C^{\alpha}(\mathrm{cl}(\mathbf{G}))$ and $\beta(t)$ a continuously differentiable function of R into $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$. Let $u_0$ belong to $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ and satisfy the "compatibility condition" $u_0 = \beta(0)$ on $\partial G$. (That is, $u_0 - \beta(0)$ is in $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$.) Then there exists a unique continuously differentiable function $u(t)$ of R into $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ such that*

(i) $Mu'(t) + Lu(t) = f(t)$,
(ii) $u(0) = u_0$, *and*
(iii) $u(t) = \beta(t)$ *on the boundary* $\partial G$.

*Proof.* Define $F(t)$ from $R$ into $C^{\alpha}(\mathrm{cl}(\mathbf{G}))$ by $F(t) = -M\beta'(t) - L\beta(t) + f(t)$. Since $M$ and $L$ are bounded (but not invertible) from $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$ into $C^{\alpha}(\mathrm{cl}(\mathbf{G}))$, we see that $F(t)$ is continuous. Since $M^{-1}$ is bounded from $C^{\alpha}(\mathrm{cl}(\mathbf{G}))$ onto $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$, we have that $M^{-1}F(t)$ is continuous in $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$, so we can define

$$v(t) = \int_0^t E(t - T)M^{-1}F(T)\,dT$$

in $C_0^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$. It follows that the continuously differentiable mapping $t \mapsto v(t)$ satisfies the equation

$$Mv'(t) + Lv(t) = F(t)$$

and initial condition $v(0) = 0$.

*Remark.* Since $M$ is not invertible (not injective on $C^{2+\alpha}(\mathrm{cl}(\mathbf{G}))$), we do *not* have $M^{-1}M =$ identity. This is of consequence if one wishes to expand $M^{-1}F(t)$ into its three terms.

Now define the continuously differentiable function

$$u(t) = v(t) + \beta(t) + E(t)[u_0 - \beta(0)].$$

This satisfies (i)–(iii) above. The uniqueness follows from Theorem 9.4 by looking at the difference between two such solutions.

## REFERENCES

[1] S. AGMON, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand, New York, 1965.

[2] S. AGMON, A. DOUGLIS AND L. NIRENBERG, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I*, Comm. Pure Appl. Math., 12 (1959), pp. 623–727.

[3] M. S. AGRONOVICH AND M. I. VISHIK, *Elliptic problems with a parameter and parabolic problems of general type*, Russian Math. Surveys, 19 (1964), pp. 53–157.

[4] G. BARENBLAT, I. ZHELTOV AND I. KOCHIVA, *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks*, J. Appl. Math. Mech., 24 (1960), pp. 1286–1303.

[5] G. BARENBLAT AND G. CHERNYI, *On moment relations on surface discontinuity in dissipative media*, J. Appl. Math. Mech., 27 (1963), pp. 1205–1218.

[6] F. E. BROWDER, *On the regularity properties of solutions of elliptic differential equations*, Comm. Pure Appl. Math., 9 (1956), pp. 351–361.

[7] ———, *A priori estimates for solutions of elliptic boundary value problems. I, II*, Nederl. Akad. Wetensch. Proc. Ser. B, 22 (1960), pp. 145–159, 160–167.

[8] ———, *On the spectral theory of elliptic differential operators. I*, Math. Ann., 142 (1961), pp. 20–130.

[9] R. CARROLL, *Partial Differential Equations*, Harper and Row, New York, to appear.

[10] R. COURANT AND D. HILBERT, *Partial Differential Equations*, Methods of Mathematical Physics, vol. 2, Interscience, New York, 1962.

[11] A. FRIEDMAN, *Generalized Functions and Partial Differential Equations*, Prentice-Hall, Englewood Cliffs, New Jersey, 1963.

[12] ———, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, New Jersey, 1964.

[13] K. O. FRIEDRICHS, *Spektraltheorie halbbeschränkter operatoren*, Math. Ann., 109 (1934), pp. 465–487, 685–713; 110 (1935), pp. 777–779.

[14] ———, *The identity of weak and strong extensions of differential operators*, Trans. Amer. Math. Soc., 55 (1944), pp. 132–151.

[15] C. A. GALPERN, Dokl. Akad. Nauk SSSR, 119 (1958), pp. 640–643.

[16] I. M. GELFAND AND G. E. SHILOV, *Generalized Functions*, vols. 1, 2, 3, Moscow, 1958.

[17] G. HELLWIG, *Partial Differential Equations*, Blaisdell, New York, 1964.

[18] E. HILLE AND R. S. PHILLIPS, *Functional Analysis and Semigroups*, Colloquium Publications, vol. 31, American Mathematical Society, New York, 1957.

[19] E. HOPF, *A remark on linear elliptic differential equations of the second order*, Proc. Amer. Math. Soc., 3 (1952), pp. 791–793.

[20] L. HORMANDER, *Linear Partial Differential Operators*, Springer-Verlag, Berlin, 1963.

[21] ———, *On the theory of general partial differential operators*, Acta Math., 94 (1955), pp. 161–248.

[22] J. HORVATH, *Topological Vector Spaces and Distributions*, Addison-Wesley, Reading, Massachusetts, 1967.

[23] O. D. KELLOG, *Foundations of Potential Theory*, Dover, New York, 1953.

[24] A. G. KOSTACHEKO AND G. I. ESKIN, *Cauchy problem for an equation of Sobolev-Galpern type*, Trudy Moscow Mat. Obshch., 10 (1961), pp. 273–284.

[25] P. D. LAX AND A. MILGRAM, *Parabolic equations*, Contributions to the Theory of Partial Differential Equations, Ann. Math. Studies, no. 33, Princeton University Press, 1954, pp. 167–190.

[26] J. L. LIONS, *Lectures on elliptic partial differential equations*, Tata Institute, Bombay, 1957.

[27] ———, *Problèmes aux limites dans les équations aux dérivées partielles*, University of Montreal, Montreal, 1962.

[28] N. MEYERS AND J. SERRIN, $H \equiv W$, Proc. Nat. Acad. Sci. U.S.A., 51 (1964), pp. 1055–1056.

[29] L. NIRENBERG, *Remarks on strongly elliptic partial differential equations*, Comm. Pure Appl. Math., 8 (1955), pp. 648–674.

[30] ———, *Existence Theorems in Partial Differential Equations*, New York University Press, New York, 1954.

[31] F. RIESZ AND B. VON SZ. NAGY, *Functional Analysis*, Frederick Ungar, New York, 1955.

[32] G. E. SHILOV, *Correct boundary value problems in a half-space for linear partial differential equations with constant coefficients*, Russian Math. Surveys, 19 (1964), pp. 1–52.

[33] S. SOBOLEV, *On a theorem of functional analysis*, Math. Sbornik, 4 (1938), pp. 471–497.

[34] ———, *Some new problems in mathematical physics*, Izv. Akad. Nauk SSSR Ser. Mat., 18 (1954), pp. 3–50.

[35] D. W. TAYLOR, *Research on Consolidation of Clays*, Massachusetts Institute of Technology Press, Cambridge, 1942.

[36] T. W. TING, *Certain non-steady flows of second order fluids*, Arch. Rational Mech. Anal., 14 (1963), pp. 1–26.

[37] ———, *Parabolic and pseudo-parabolic partial differential equations*, to appear.

[38] C. TRUESDELL AND W. NOLL, *The non-linear field theories of mechanics*, Encyclopedia of Physics, vol. III, Springer-Verlag, Berlin, 1965.

[39] K. YOSIDA, *On the differentiability and representation of one-parameter semigroups of linear operators*, J. Math. Soc. Japan, 1 (1948), pp. 15–21.

[40] ———, *Functional Analysis*, Academic Press, New York, 1965.

[41] T. ZALENYAK, *The behavior for $t \to \infty$ of solutions of a problem of S. L. Sobolev*, Soviet Math. Dokl., 2 (1961), no. 2, pp. 956–958.

[42] ———, *A problem of Sobolev*, Ibid., 3 (1962), no. 6, pp. 1756–1759.

SIAM J. MATH. ANAL.
Vol. 1, No. 1, February 1970

# ON THE BOUNDEDNESS AND THE STABILITY OF SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER*

MARTIN HARROW†

In this paper we investigate certain fourth order nonhomogeneous differential equations. Ezeilo [1] derived interesting results for the problem

$$\dddot{x} + a\ddot{x} + b\dot{x} + h(x) = p(t). \tag{1}$$

We show similar results for

$$x^{(4)} + a\dddot{x} + b\ddot{x} + c\dot{x} + h(x) = p(t), \tag{2}$$

where $a, b, c$ are given positive constants such that $ab > c$; $h(x)$ is differentiable and $p(t)$ is continuous in $x$ and $t$ respectively, and of such nature that the existence and uniqueness of the solutions, as well as their continuous dependence on the initial values is assured. Let $x(t)$ be a solution of (2) then we may write

$$\frac{dx}{dt} = \dot{x}, \quad \frac{d^2x}{dt^2} = \ddot{x}, \quad \frac{d^3x}{dt^3} = \dddot{x}, \quad \frac{d^4x}{dt^4} = x^{(4)},$$

and an equivalent system of (2) may be written as

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = w, \quad \dot{w} = -aw - bz - cy - h(x) + p(t). \tag{3}$$

THEOREM. *Suppose that* $h(0) = 0$ *and that*
(A) *$d$ is a positive constant such that*

$$h'(x) \leqq d, \qquad s \equiv abc - c^2 - a^2d > 0,$$

*and $\delta_2$ is a constant such that for all $x$*

$$d - 2asc^{-1} < \delta_2 < h'(x) \leqq d;$$

(B) *$\delta_1$ is a positive constant such that for $x \neq 0$*

$$d - \frac{ds}{s^*} \cdot \frac{c}{ab} < \delta_1 < \frac{h(x)}{x},$$

*where*

$$s^* \equiv s + 2ad(ab - c)b^{-1}.$$

*Then corresponding to any constant $\lambda$ in the interval $0 \leqq \lambda \leqq 1/2$ there always is a constant $\mu \equiv \mu(a, b, c, d, \delta_1, \delta_2, \lambda) > 0$ such that every solution $x(t)$ of (2) determined by the initial conditions*

$$x(t_0) = x_0, \quad \dot{x}(t_0) = y_0, \quad \ddot{x}(t_0) = z_0, \quad \dddot{x}(t_0) = w_0$$

* Received by the editors November 22, 1968, and in revised form April 11, 1969.

† Department of Mathematics, Sir George Williams University, Montreal, Canada, and McGill University, Montreal, Canada.

*and defined for all* $t \geqq t_0$ *satisfies*

$$\text{(4)} \quad x^2(t) + \dot{x}^2(t) + \ddot{x}^2(t) + \dddot{x}^2(t) \leqq \left\{ e^{-\mu t}\left[ D_1 + D_2 \int_{t_0}^{t} |p(\tau)|^{2(1-\lambda)} e^{\mu t}\, d\tau \right]\right\} 1/(1 - \lambda)$$

*for all* $t \geqq t_0$, *where* $D_1 \equiv D_1(a, b, c, d, \delta_1, \delta_2, t_0, x_0, y_0, z_0, w_0) > 0$ *and* $D_2 \equiv D_2(a, b, c, d, \delta_1, \delta_2) > 0$ *are constants.*

A result similar to (4) was obtained by Ezeilo [1] for (1), and all the special results which Ezeilo obtained by considering the following cases apply in the fourth order equation (2) as well.

*Case* 1. $p(t) \equiv 0$.

*Case* 2. $|p(t)|$ is bounded.

*Case* 3. $\max|p(t)| < \infty$ for all $t \geqq t_0$ and $\int_{t_0}^{t} |p(\tau)|^k d\tau < \infty$ for some $k$, $1 \leqq k \leqq 2$.

In the special linear case $h(x) = dx$ we note the following:

(i) Equation (2) becomes $x^{(4)} + a\dddot{x} + b\ddot{x} + c\dot{x} + dx = p(t)$.

(ii) The Routh-Hurwitz criteria of stability apply, namely $a > 0$, $b > 0$, $c > 0$, $d > 0$, $s = abc - c^2 - a^2 d > 0$.

(iii) The existence of $\delta_1$ and $\delta_2$ is trivially fulfilled and is unnecessary for the proof.

(iv) The conclusions of the theorem remain the same except that $D_1$ and $D_2$ are independent of $\delta_1$ and $\delta_2$.

In the nonlinear case, the difference between $h(x)$ and $dx$ is determined by $a, b, c, d$ and the hypotheses (A) and (B). A simple example illustrates the significant difference that may occur. Suppose we let

$$a = \lambda, \quad b = \lambda, \quad c = \mu\lambda^2, \quad d = \delta\lambda^2\mu(1 - \mu),$$

where $\lambda > 0, 0 < \mu < 1$ and $0 < \delta < 1$. Then $a, b, c, d$ form a Routh-Hurwitz set of numbers. It can be shown that

$$d - 2asc^{-1} = \lambda^2(1 - \mu)[\delta\mu - 2\lambda(1 - \delta)]$$

and that

$$d\left[1 - \frac{s}{s^*}\frac{c}{ab}\right] = d \cdot \frac{(1 + \delta)(1 - \mu)}{1 + \delta - 2\mu\delta}.$$

Now, for an arbitrarily small $\varepsilon$ such that $0 < \varepsilon < 1$, and a fixed $\delta$, we can choose $\mu$ near 1 so that

$$0 < \frac{(1 + \delta)(1 - \mu)}{1 + \delta - 2\mu\delta} < \varepsilon$$

and we can also choose $\lambda$ large enough so that $\lambda > \delta\mu/2(1 - \delta)$. This implies that $\eta = d - 2asc^{-1} < 0$ and hence by hypotheses (A) and (B)

$$\eta < \delta_2 < h'(x) \leqq d \quad \text{and} \quad \varepsilon d < \delta_1 < \frac{h(x)}{x}.$$

Thus, $h'(x)$ may take on negative values, and $h(x)/x$ may be considerably less than $d$.

In what follows we shall define two functions $M(x, y, z, w)$ and $N(x, y, z, w)$.

We shall prove that for certain positive constants $D_3$ and $D_4$

$$D_3(x^2 + y^2 + z^2 + w^2) \leqq W = M + N \leqq D_4(x^2 + y^2 + z^2 + w^2)$$

and that for certain positive constants, $D_5$ and $D_6$,

$$\frac{dW}{dt} \leqq -D_5(x^2 + y^2 + z^2 + w^2) + D_6(x^2 + y^2 + z^2 + w^2)^{1/2}|p(t)|,$$

where $dW/dt$ is defined in the usual way with respect to (3). The remainder of the proof follows the one given by Ezeilo [1].

*Proof of the theorem.* Consider the function

$$2M(x, y, z, w) = m_1(w + m_1 z + m_1 dc^{-1}y)^2 + c[z + m_1 y + m_1 c^{-1}h(x)]^2 + m_1 d\sigma c^{-2}(y + cm_1\alpha\sigma^{-1}z)^2 + m_2 z^2,$$

where $\alpha$ is a constant such that (as is possible in view of the condition $bc > ad$)

$$0 < \alpha < s(bc - ad)^{-1}$$

and

$$m_1 = a - \alpha, \qquad m_2 = sc^{-1} + (ad - bc)\alpha c^{-1} + \alpha m_1^2 m_3 \sigma^{-1} + \alpha dm_1 c^{-1},$$

$$\sigma = m_1 bc - m_1^2 d - c^2, \qquad m_3 = \sigma - d\alpha m_1.$$

It is easy to prove that $m_1, m_3, \sigma$ are positive, and hence that

$$m_2 > \alpha m_1^2 m_3 \sigma^{-1} + \alpha dm_1 c^{-1} > 0.$$

Thus $M(x, y, z, w)$ is positive definite. Now consider the function

$$2N(x, y, z, w) = a[w + az + (ab - n_1)a^{-1}y + \beta x]^2 + n_1[z + ay + an_1^{-1}h(x)]^2 + va^{-1}(y + an_1\beta v^{-1}x)^2 + n_2 x^2 + 2k_1\int_0^x h(\tau)\,d\tau - dk_1 x^2 + m_1^2 c^{-1}[d^2x^2 - \{h(x)\}^2] + a^2 n_1^{-1}[d^2x^2 - \{h(x)\}^2],$$

where $\beta$ is a constant such that (as possible in view of the condition $ab > c$)

$$0 < \beta < s(ab - c)^{-1}$$

and

$$n_1 = c - \beta, \qquad n_3 = v - \beta n_1,$$

$$v = abn_1 - n_1^2 - a^2 d, \qquad k_1 = ab - n_1 + dm_1^2 c^{-1}.$$

$$n_2 = dsn_1^{-1} + a\beta n_1 n_3 v^{-1} - d\beta(ab - c)n_1^{-1} + d\beta,$$

It is easy to prove that $n_1, v, n_3, k$ are positive, and hence that

$$n_2 > [s - \beta(ab - c)]\, dn_1^{-1} + a\beta n_1 n_3 v^{-1} + d\beta > 0.$$

Also, since $h(0) = 0$, by the mean value theorem and hypothesis (A),

$$d^2x^2 - [h(x)]^2 \geqq 0$$

and

$$P(x) = n_2 x^2 + 2k_1 \int_0^x h(\tau)\, d\tau - dk_1 x^2$$
$$= 2k_1 \int_0^x \left\{ \frac{h(\tau)}{\tau} - \left( d - \frac{n_2}{k_1} \right) \right\} \tau\, d\tau.$$

In order to show that $P(x)$ is positive definite we note that

$$\lim_{\beta \to s(ab-c)^{-1}} n_2 = ds(ab - c)^{-1},$$
$$\lim_{\substack{\beta \to s(ab-c)^{-1} \\ \alpha \to 0}} k_1 = \frac{abs + 2a^2 d(ab - c)}{c(ab - c)};$$

hence

$$\lim \frac{n_2}{k_1} = d \cdot \frac{c}{ab} \cdot \frac{s}{s^*}.$$

Hence we can choose $\alpha$ and $\beta$ so that either

$$d - \frac{ds}{s^*} \cdot \frac{c}{ab} < d - \frac{n_2}{k_1} < \delta_1 < \frac{h(x)}{x} \leqq d$$

or

$$d - \frac{n_2}{k_1} < d - \frac{ds}{s^*} \cdot \frac{c}{ab} < \delta_1 < \frac{h(x)}{x} \leqq d.$$

This implies that in either case

$$\frac{h(x)}{x} - \left( d - \frac{n_2}{k_1} \right) \geqq \delta_1 - \left( d - \frac{n_2}{k_1} \right) > 0$$

and hence that $P(x)$ is positive definite. Thus $N(x, y, z, w)$ is positive definite and consequently $W = M + N$ is positive definite. Further, it is possible to choose $D_3 \equiv D_3(a, b, c, d, \delta_1, \delta_2, \alpha, \beta) > 0$ such that $W(x, y, z, w) \geqq D_3(x^2 + y^2 + z^2 + w^2)$. Also, expanding $W$ into a quadratic in $x$, $y$, $z$, $w$ and using the fact that $|xy| \leqq \frac{1}{2}(x^2 + y^2)$ etc., it follows that there exists $D_4 \equiv D_4(a, b, c, d, \delta_1, \delta_2, \alpha, \beta) > 0$ such that we may write

$$D_3(x^2 + y^2 + z^2 + w^2) \leqq W(x, y, z, w) \leqq D_4(x^2 + y^2 + z^2 + w^2).$$

Using (3) and the definition of $M(x, y, z, w)$ we have

$$\frac{dM}{dt} = \frac{\partial M}{\partial x}\dot{x} + \frac{\partial M}{\partial y}\dot{y} + \frac{\partial M}{\partial z}\dot{z} + \frac{\partial M}{\partial w}\dot{w}$$
$$= \frac{\partial M}{\partial x} y + \frac{\partial M}{\partial y} z + \frac{\partial M}{\partial z} w + \frac{\partial M}{\partial w}[-aw - by - cz - h(x) + p(t)].$$

An easy calculation shows that

$$\frac{dM}{dt} = -m_1\alpha w^2 - m_1c^{-1}m_3z^2 - m_1^2[d - h'(x)]y^2 - m_1[d - h'(x)]yz$$
$$-m_1^2c^{-1}\frac{h(x)}{x}[d - h'(x)]xy + m_1[w + m_1z + m_1dc^{-1}y]p(t).$$

Similarly, using (3) and the definition of $N(x, y, z, w)$, we have

$$\frac{dN}{dt} = -n_3y^2 - a^2[d - h'(x)]y^2 - a\beta\frac{h(x)}{x}x^2 - a[d - h'(x)]yz$$
$$+m_1^2c^{-1}\frac{h(x)}{x}[d - h'(x)]xy + a[w + az + (ab - n_1)a^{-1}y + \beta x]p(t).$$

Thus,

$$\frac{dW}{dt} = -U(x, y, z, w) + [(m_1 + a)w + (m_1^2 + a^2)z + k_1y + a\beta x]p(t),$$

where

$$U(x, y, z, w) = m_1\alpha w^2 + n_3y^2 + a\beta\frac{h(x)}{x}x^2 + T(x, y, z),$$

where

$$T(x, y, z) = m_1c^{-1}m_3z^2 + (m_1^2 + a^2)[d - h'(x)]y^2 + (m_1 + a)[d - h'(x)]yz$$
$$= [d - h'(x)]\left\{\left(m_1y + \frac{z}{2}\right)^2 + \left(ay + \frac{z}{2}\right)^2\right\} + \left\{m_1c^{-1}m_3 - \frac{[d - h'(x)]}{2}\right\}z^2.$$

Note that for all values of $\alpha$ such that for $0 < \alpha < s(bc - ad)^{-1}$

$$m_1m_3c^{-1} = (a - \alpha)[s - \alpha(bc - ad)]c^{-1} < asc^{-1}$$

and that

$$\lim_{\alpha\to 0^+} m_1m_3c^{-1} = asc^{-1}$$

we can, by choosing $\alpha$ small enough but positive, have

$$d - 2asc^{-1} < d - 2m_1m_3c^{-1} < \delta_2 < h'(x) \leqq d.$$

Hence,

$$T(x, y, z) \geqq \{m_1m_3c^{-1} - \tfrac{1}{2}[d - h'(x)]\}z^2$$
$$= \tfrac{1}{2}[h'(x) - (d - 2m_1m_3c^{-1})]z^2$$
$$\geqq \tfrac{1}{2}[\delta_2 - (d - 2m_1m_3c^{-1})]z^2.$$

Thus, there exists $D_5 \equiv D_5(a, b, c, d, \delta_1, \delta_2, \alpha, \beta) > 0$ such that

$$U(x, y, z, w) \geqq D_5(x^2 + y^2 + z^2 + w^2)$$

and

$$\frac{dW}{dt} \leqq -D_5(x^2 + y^2 + z^2 + w^2) + [(m_1 + a)|w| + (m_1^2 + a^2)|z| + k_1|y| + a\beta|x|]|p(t)|$$

$$\leqq -D_5(x^2 + y^2 + z^2 + w^2) + D_6[|w| + |z| + |y| + |x|]|p(t)|,$$

where $D_6 = \max[(m_1 + a), (m_1^2 + a^2), k_1, a\beta]$.

The remainder of the proof of the theorem is the same as the one given by Ezeilo, except that the third order problem is replaced by a fourth order problem. For example, $x^2 + y^2 + z^2$ becomes $x^2 + y^2 + z^2 + w^2$, etc.

## REFERENCE

[1] J. O. C. Ezeilo, *Further results for the solutions of a third order differential equation*, Proc. Cambridge Philos. Soc., 59 (1963), pp. 111–116.

SIAM J. MATH. ANAL.
Vol. 1, No. 1, February 1970

# A REPRESENTATION OF HANKEL TRANSFORMABLE GENERALIZED FUNCTIONS*

E. L. KOH†

**1. Introduction.** In a recent paper, Koh and Zemanian [1] indicated that generalized functions that are Hankel transformable are of "exponential descent." In the present note we show that this name is motivated by a suitable structure theorem.

We recall briefly their definition of the generalized Hankel transformation. Our notation shall be that of [1]. For a real number $\mu$ and a positive real number $a$, $\mathscr{J}_{\mu,a}$ was defined as the space of testing functions $\phi(x)$ which are smooth and for which

$$\tau_k^{\mu,a}(\phi) = \sup_{0<x<\infty} |e^{-ax}x^{-\mu-1/2}S_\mu^k(\phi)| < \infty, \qquad k = 0, 1, 2, \cdots, \tag{1}$$

where

$$S_\mu^k = \left(x^{-\mu-1/2}\frac{d}{dx}x^{2\mu+1}\frac{d}{dx}x^{-\mu-1/2}\right)^k.$$

$\mathscr{J}_{\mu,a}$ is a Hausdorff, locally convex, first countable linear space. It is complete and therefore Fréchet. For each fixed complex $y$ in the strip $\Omega = \{y : |\text{Im } y| < a, y \neq 0 \text{ or a negative number}\}$, $\mathscr{J}_{\mu,a}$ contains the function $\sqrt{xy}\mathscr{J}_\mu(xy)$. The Hankel transformation $\mathfrak{H}_\mu$ is now defined on the dual space $\mathscr{J}'_{\mu,a}$ as follows: Let $\mu$ be restricted to $-\frac{1}{2} \leqq \mu < \infty$. Then, for $f \in \mathscr{J}'_{\mu,a}$,

$$(\mathfrak{H}_\mu f)(y) \triangleq \langle f(x), \sqrt{xy}\mathscr{J}_\mu(xy)\rangle, \qquad y \in \Omega. \tag{2}$$

It is the case that the dual $\mathscr{J}'_{\mu,a}$ contains all distributions of compact support on $(0, \infty)$. Also, any locally integrable function $f(x)$ on $0 < x < \infty$ and such that

$$\int_0^\infty |f(x)e^{ax}x^{\mu+1/2}|\, dx < \infty \tag{3}$$

is a member of $\mathscr{J}'_{\mu,a}$. We now prove that every generalized function belonging to $\mathscr{J}'_{\mu,a}$ can be represented by a finite sum of derivatives of continuous functions decaying exponentially at infinity. Our proof is analogous to the method employed in structure theorems for Schwartz distributions (e.g., [2, p. 317 ff.], [3, p. 272 ff.]).

**2. Theorem.** *Let $f \in \mathscr{J}'_{\mu,a}$. Then $f$ is equal to a finite sum*

$$\sum_{i=0}^k C_i\left(\frac{d}{dx}\right)^i [e^{-ax}x^{-\mu-1/2-k+1}P_i(x)F_i(x)], \tag{4}$$

*where the $F_i(x)$ are continuous on $(0, \infty)$ and the $P_i(x)$ are polynomials of degree $k$.*

*Proof.* By note (viii) of [1], for every $f \in \mathscr{J}'_{\mu,a}$, there exist a nonnegative integer $r$ and a positive constant $C$ such that for all $\phi \in D(I) \subset \mathscr{J}_{\mu,a}$,

$$|\langle f, \phi\rangle| \leqq C \max_{0\leqq k\leqq r} \sup_{0<x<\infty} |e^{-ax}x^{-\mu-1/2}S_\mu^k\phi|, \tag{5}$$

* Received by the editors October 15, 1968.
† Department of Mathematics, University of Saskatchewan, Regina, Saskatchewan.

$D(I)$ being the space of smooth functions with compact supports on $(0, \infty)$. Expanding $S_\mu^k \phi$ by (1) above, we have

(6)
$$\begin{aligned} |\langle f, \phi \rangle| &\leqq C \max_{0 \leqq k \leqq r} \sup_{0 < x < \infty} \left| \sum_{i=0}^{2k} a_{2k,i} e^{-ax} x^{-\mu - 1/2 + i - 2k} \left( \frac{d}{dx} \right)^i \phi \right| \\ &\leqq C' \max_{0 \leqq i \leqq 2r} \sup_{0 < x < \infty} \left| e^{-ax} x^{-\mu - 1/2 - 2r + i} \left( \frac{d}{dx} \right)^i \phi \right|, \end{aligned}$$

where

$$C' = 2rC \max_{0 \leqq k \leqq r} \max_{0 \leqq i \leqq 2k} |a_{2k,i}|.$$

Now set

(7)
$$\phi_r = e^{-ax} x^{-\mu - 1/2 - 2r + i} \phi(x), \qquad i \leqq 2r.$$

Clearly, $\phi_r(x) \in D(I)$ since $\phi(x)$ is. Then $\phi(x) = e^{ax} x^{\mu + 1/2 + 2r - i} \phi_r(x)$. And

$$\frac{d\phi}{dx} = e^{ax} x^{\mu + 1/2 + 2r - i} \left( a + c_{i,r} x^{-1} + \frac{d}{dx} \right) \phi_r,$$

where $c_{i,r}$ is a constant. Let supp $\phi$ = supp $\phi_r = [A, B]$. Then

$$\left| \frac{d\phi}{dx} \right| \leqq A_{i1} e^{ax} x^{\mu + 1/2 + 2r - i} \left( |\phi_r| + \left| \frac{d\phi_r}{dx} \right| \right),$$

where $A_{i1} = \max(a + c_{i,r} A^{-1}, 1)$. Continuing in this fashion, we have

(8)
$$\left| \left( \frac{d}{dx} \right)^1 \phi \right| \leqq A_i e^{ax} x^{\mu + 1/2 + 2r - i} \sum_{q=0}^{i} \left| \left( \frac{d}{dx} \right)^q \phi_r(x) \right|.$$

Substituting (8) into (6) and noting that $e^{-ax} x^{-\mu - 1/2 - 2r + i} > 0$ for $0 < x < \infty$, we have

(9)
$$\begin{aligned} |\langle f, \phi \rangle| &\leqq C'' \max_{0 \leqq i \leqq 2r} \sup_{0 < x < \infty} \left| \sum_{q=0}^{i} \left| \left( \frac{d}{dx} \right)^q \phi_r(x) \right| \right| \\ &\leqq C''' \max_{0 \leqq i \leqq 2r} \sup_{0 < x < \infty} \left| \left( \frac{d}{dx} \right)^i \phi_r(x) \right|, \end{aligned}$$

where $C''$ and $C'''$ are obvious constants.

We can write for every $\psi \in D(I)$,

$$\psi(x) = \int_0^x \frac{d\psi}{dt} \, dt.$$

Hence,

(10)
$$\sup_{0 < x < \infty} |\psi(x)| \leqq \sup_{0 < x < \infty} \int_0^x \left| \frac{d\psi}{dt} \right| dt = \left\| \frac{d\psi}{dt} \right\|_{L_1(0, \infty)},$$

where $L_1(0, \infty)$ is the space of equivalence classes of Lebesgue integrable functions on $(0, \infty)$ whose topology is defined by the norm

$$\| f \|_{L_1(0, \infty)} = \int_0^\infty |f| \, dt < \infty, \quad f \in L_1(0, \infty).$$

The bound (10) enables us to write (9) as

$$|\langle f, \phi\rangle| \leqq C^{(iv)} \max_{1 \leqq i \leqq 2r+1} \left\| \left(\frac{d}{dx}\right)^i \phi_r(x) \right\|_{L_1(0,\infty)} \tag{11}$$

Consider now the linear one-to-one mapping

$$\phi \xrightarrow{M} \left(\frac{d}{dx}\right)^i \phi \Bigg|_{1 \leqq i \leqq 2r+1}$$

of $D(I)$ into $L_1(0, \infty)$. Since $D(I)$ is a linear manifold of $L_1(0, \infty)$, (11) states that the linear functional $f$ is continuous on $MD$ (i.e., $D(I)$) for the topology induced on it by $L_1(0, \infty)$. Hence, by the Hahn-Banach theorem, $f$ can be extended as a continuous linear functional in the whole of $L_1(0, \infty)$. But the dual of $L_1(0, \infty)$ is isomorphic with $L_\infty(0, \infty)$, the space of all equivalence classes (mod = a.e.) of complex-valued integrable functions on $(0, \infty)$ such that, for every $f \in L_\infty(0, \infty)$, there exists an $M$ such that $|f| \leqq M$ a.e. Therefore, there are functions $g_i \in L_\infty(0, \infty)$, $1 \leqq i \leqq 2r + 1$, such that

$$\langle f, \phi\rangle = \sum_{i=1}^{2r+1} \left\langle g_i, \left(\frac{d}{dx}\right)^i \phi_r(x)\right\rangle.$$

Recalling (7), we have

$$\langle f, \phi\rangle = \sum_{i=1}^{2r+1} \left\langle (-1)^i \left(\frac{d}{dx}\right)^i g_i, e^{-ax} x^{-\mu-1/2-2r+i} \phi(x)\right\rangle.$$

Therefore

$$f = \sum_{i=1}^{2r+1} e^{-ax} x^{-\mu-1/2-2r+i} (-1)^i \left(\frac{d}{dx}\right)^i g_i. \tag{12}$$

For each $i$, we set

$$h_i(x) = (-1)^i \int_0^x g_i(t)\, dt.$$

Since $g_i \in L_\infty(0, \infty)$, the functions $h_i$ are continuous on $(0, \infty)$ and

$$|h_i| \leqq \int_0^x |g_i|\, dt \leqq |x| \max_{0<x<\infty} |g_i| = |x|\, \|g_i\|_{L_\infty}.$$

Furthermore, $g_i = (-1)^i (d/dx) h_i$. Hence

$$f = \sum_{i=2}^{2r+2} e^{-ax} x^{-\mu-1/2-2r+i-1} \left(\frac{d}{dx}\right)^i h_i. \tag{13}$$

By letting $2r + 2 = k$ and using the differentiation formulas,

$$u(x)\left(\frac{d}{dx}\right)^i h_i = \sum_{j=0}^{i} (-1)^j \binom{i}{j} [u^{(j)} h_i]^{(i-j)}$$

and

$$(ab)^{(j)} = \sum_{q=0}^{j} \binom{i}{q} a^{(j-q)} b^{(q)},$$

we can write (13) as in (4) where the $F_i$ are continuous functions of $h_i$ and are therefore continuous functions on $(0, \infty)$. This proves the theorem.

## REFERENCES

[1] E. L. KOH AND A. H. ZEMANIAN, *The complex Hankel and I-transformations of generalized functions*, SIAM J. Appl. Math., 16 (1968), pp. 945–957.

[2] R. E. EDWARDS, *Functional Analysis*, Holt, Rinehart, and Winston, New York, 1965.

[3] F. TREVES, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York, 1967.

SIAM J. MATH. ANAL.
Vol. 1, No. 1, February 1970

# ON THE REDUCTION OF DIFFERENTIAL EQUATIONS TO ALGEBRAIC EQUATIONS*

M. J. MORAN† AND R. A. GAGGIOLI‡

**1. Introduction.** The objective of this paper is to extend certain techniques for the reduction of the number of independent variables in systems of partial differential equations developed by Michal [1], Morgan [2] and the present authors [3]–[5]. In review, the techniques currently in use utilize elementary group theory for the purpose of reducing a given system of partial differential equations to a system of differential equations in fewer independent variables. The extension to be presented here is embodied in the theorem of § 2, which in application is aimed at reducing a given system to a system of algebraic equations.

**2. Principal results.** The abovementioned techniques as well as the theorem to follow utilize continuous parameter groups of transformations [6, pp. 13–18]. The groups $G$ to be considered in the present discussion are of the form

$$G:\begin{cases} S:\{\bar{x}^i = F^i(x^1, \cdots, y_n; a_1, \cdots, a_m), & i = 1, \cdots, m, \\ \bar{y}_j = F_j(x^1, \cdots, y_n; a_1, \cdots, a_m), & j = 1, \cdots, n, \end{cases} \tag{2.1}$$

wherein the $a_i$ are the group parameters. Specifically, the class of groups to be considered are those of the form (2.1) which possess $n$, and only $n$, functionally independent absolute invariants $g_j(y_1, \cdots, y_n, x^1, \cdots, x^m) = g_j(\bar{y}_1, \cdots, \bar{y}_n, \bar{x}^1, \cdots, \bar{x}^m)$ [6, pp. 61–62]; moreover, these are to be differentiable in each argument and satisfy the Jacobian condition

$$\frac{\partial[g_1, \cdots, g_n]}{\partial[y_1, \cdots, y_n]} \neq 0. \tag{2.2}$$

THEOREM. *If and only if, for some set of differentiable functions* $\{I_j\}$, $y_j = I_j(x^1, \cdots, x^m)$ *yields* $\bar{y}_j = I_j(\bar{x}^1, \cdots, \bar{x}^m)$ *when transformed under* $G$, *then*

$$g_j(y_1, \cdots, y_n, x^1, \cdots, x^m) = K_j, \qquad j = 1, \cdots, n, \tag{2.3}$$

*where the* $K_j$ *are constants.*

*Proof.* By definition of absolutely invariant,

$$g_j(y_1, \cdots, y_n, x^1, \cdots, x^m) = g_j(\bar{y}_1, \cdots, \bar{y}_n, \bar{x}^1, \cdots, \bar{x}^m). \tag{2.4}$$

If, under $G$, $\bar{y}_j = I_j(\bar{x}^1, \cdots, \bar{x}^m)$ when $y_j = I_j(x^1, \cdots, x^m)$, then

$$g_j(I_1(x^1, \cdots, x^m), \cdots, x^m) = g_j(I_1(\bar{x}^1, \cdots, \bar{x}^m), \cdots, \bar{x}^m). \tag{2.5}$$

---

* Received by the editors August 22, 1968, and in revised form March 28, 1969. This research was sponsored by the Mathematics Research Center, United States Army, University of Wisconsin, under Contract DA-31-124-ARO-D-462.

† Department of Mechanical Engineering, Ohio State University, Columbus, Ohio 43210 and Visiting Member, Mathematics Research Center, University of Wisconsin, Madison, Wisconsin 53706.

‡ Department of Mechanical Engineering, Marquette University, Milwaukee, Wisconsin 53233 and Visiting Fellow, Battelle Memorial Institute, Columbus, Ohio 43201.

Equation (2.5) indicates

$$\gamma_j(x^1, \cdots, x^m) = \gamma_j(\bar{x}^1, \cdots, \bar{x}^m), \tag{2.6}$$

where

$$\begin{aligned} \gamma_j(x^1, \cdots, x^m) &\equiv g_j(I_1(x^1, \cdots, x^m), \cdots, x^m), \\ \gamma_j(\bar{x}^1, \cdots, \bar{x}^m) &\equiv g_j(I_1(\bar{x}^1, \cdots, \bar{x}^m), \cdots, \bar{x}^m). \end{aligned} \tag{2.7}$$

That is, (2.5) indicates that $\gamma_j$ is an absolute invariant of $G$; moreover, since the $g_j$ and $I_j$ are differentiable, $\gamma_j$ is differentiable.

Then since $G$ possesses exactly $n$ functionally independent differentiable invariants $\{g_j\}$, there exists a function $f_j$ of $n$ variables such that

$$\begin{aligned} &\gamma_j(x^1, \cdots, x^m) \\ &\quad = f_j(g_1(y_1, \cdots, y_n, x^1, \cdots, x^m), \cdots, g_n(y_1, \cdots, y_n, x^1, \cdots, x^m)). \end{aligned} \tag{2.8}$$

And the condition (2.2) restricts $f_j$ to be identically a constant: Consider the system of $n$ homogeneous differential equations for $f_j$ obtained from (2.8) upon differentiation with respect to each of the $y$'s in turn; namely, the system

$$\sum_{k=1}^{n} \frac{\partial f_j}{\partial g_k} \frac{\partial g_k}{\partial y_l} = 0, \qquad l = 1, \cdots, n. \tag{2.9}$$

With (2.2), (2.9) is satisfied only if $\partial f_j/\partial g_k \equiv 0$ for all $k$, i.e., only if $f_j(g_1, \cdots, g_n) \equiv K_j$, where $K_j$ is a constant.

Thus, with $\gamma_j(x^1, \cdots, x^m) = K_j$, and $y_j = I_j(x^1, \cdots, x^m)$, (2.7) yields the desired result, (2.3). Moreover, (2.2) assures that the $g$'s may be inverted for the $y$'s as differentiable functions of the $x$'s: Indeed, since $g$ is an absolute invariant, (2.3) yields $y_j = I_j(x^1, \cdots, x^m)$ and $\bar{y}_j = I_j(\bar{x}^1, \cdots, \bar{x}^m)$, which proves the converse.

The utility of the theorem is illustrated in the next section.

**3. Illustrative application.** As in Birkhoff [7, p. 117], an essential feature of the group techniques referred to in § 1 is the notion that whenever a system of equations is transformed invariantly under a group, solutions are to be sought which are also invariant under the group. These concepts as well as the utility of the foregoing theorem will now be illustrated by application to the Helmholtz equation

$$\frac{\partial^2 y}{\partial (x^1)^2} + \frac{\partial^2 y}{\partial (x^2)^2} - \lambda^2 y = 0, \tag{3.1}$$

wherein $\lambda^2$ is a constant. (Equation (3.1) is treated in [8, p. 394]; but as for many such equations the discussion is initiated by assuming a form for the solution: (3.7).)

Equation (3.1) is said to transform invariantly under a group $G$ if, when (3.1) is satisfied, (3.2) is satisfied:

$$\frac{\partial^2 \bar{y}}{\partial (\bar{x}^1)^2} + \frac{\partial^2 \bar{y}}{\partial (\bar{x}^2)^2} - \lambda^2 \bar{y} = 0. \tag{3.2}$$

Means for deducing groups under which an equation transforms invariantly have been developed and are discussed elsewhere [3]–[5], [9]. For the problem at hand it can be found that (3.1) transforms invariantly under the two-parameter group $G'$,

$$G' : \begin{cases} \bar{x}^1 = x^1 + \ln a_1, \\ \bar{x}^2 = x^2 + \ln a_2, \\ \bar{y} = a_1^r a_2^s y. \end{cases} \tag{3.3}$$

For, with the chain rule $[\partial^2 \bar{y}/\partial(\bar{x}^i)^2] = a_1^r a_2^s [\partial^2 y/\partial(x^i)^2]$, $i = 1, 2$, it follows that

$$\left[\frac{\partial^2 \bar{y}}{\partial(\bar{x}^1)^2} + \frac{\partial^2 \bar{y}}{\partial(\bar{x}^2)^2} - \lambda^2 \bar{y}\right] = a_1^r a_2^s \left[\frac{\partial^2 y}{\partial(x^1)^2} + \frac{\partial^2 y}{\partial(x^2)^2} - \lambda^2 y\right]. \tag{3.4}$$

Thus, (3.2) is satisfied when, and only when, (3.1) is satisfied. Clearly, if $y = F(x^1, x^2)$ is any solution to (3.1), then $\bar{y} = F(\bar{x}^1, \bar{x}^2)$ is a solution to (3.2). Furthermore, were $I$ a solution to (3.1) such that $y = I(x^1, x^2)$ transforms under $G'$ to $\bar{y} = I(\bar{x}^1, \bar{x}^2)$, then the conditions of the foregoing theorem would be satisfied, and hence $I$ would be given implicitly via

$$g(y, x^1, x^2) = K, \tag{3.5}$$

wherein $K$ is a constant. In other words, the invariance of (3.1) under $G'$ suggests that solutions be sought which also transform invariantly under $G'$; such solutions are frequently termed invariant solutions. And if invariant solutions to (3.1) exist, they are given implicitly via (3.5) as a result of the foregoing theorem.

In order to apply the theorem—in order to utilize (3.5)—for the purpose of establishing invariant solutions, it is first necessary to determine the function $g$. The authors have utilized general methods for deducing sets of absolute invariants [3]–[5]. For the case at hand, (3.3),

$$g(y, x^1, x^2) = y \exp(-rx^1 - sx^2). \tag{3.6}$$

Combining (3.5), (3.6), and solving the resultant expression yields

$$y = K \exp(rx^1 + sx^2). \tag{3.7}$$

Substitution of (3.7) into the partial differential equation (3.1) reduces it, for $K \neq 0$, to the algebraic equation

$$r^2 + s^2 - \lambda^2 = 0. \tag{3.8}$$

When (3.8) is satisfied, then (3.1) is satisfied by (3.7).

**4. Closure.** This paper has provided a theorem by application of which a system of partial differential equations, say, may in certain instances be reduced to a system of algebraic equations. As illustrated with the Helmholtz equation, solutions yielded by the method evolve in a straightforward manner as an extension of well-known group techniques for partial differential equations [1]–[5]. (And in further analogy to the earlier techniques, failure of the present method to yield a solution generally signifies there is no solution invariant under the particular group being considered.) The method clearly has application to other more complicated situations than (3.1). Indeed, as noted, it is appropriate for systems of partial differential equations (possibly including auxiliary conditions); and for some cases, the particular solutions yielded may even be instrumental, in the manner of [8, pp. 393–394], in obtaining general solutions.

## REFERENCES

[1] A. D. MICHAL, *Differential invariants and invariant partial differential equations under continuous transformation groups in normed linear spaces*, Proc. Nat. Acad. Sci. U.S.A., 37 (1952), pp. 623–627.

[2] A. J. A. MORGAN, *The reduction by one of the number of independent variables in some systems of partial differential equations*, Quart. J. Math. Oxford Ser. (2), 3 (1952), pp. 250–259.

[3] M. J. MORAN AND R. A. GAGGIOLI, *Reduction of the number of variables in systems of partial differential equations, with auxiliary conditions*, SIAM J. Appl. Math., 16 (1968), pp. 202–215.

[4] ———, *Similarity analyses via group theory*, AIAA J., 6 (1968), pp. 2014–2016.

[5] ———, *A new systematic formalism for similarity analysis*, J. Engrg. Math., 3 (1969), pp. 151–162.

[6] L. P. EISENHART, *Continuous Groups of Transformations*, Dover, New York, 1961.

[7] G. BIRKHOFF, *Hydrodynamics*, Princeton University Press, Princeton, 1960.

[8] F. B. HILDEBRAND, *Advanced Calculus for Applications*, Prentice-Hall, Englewood Cliffs, New Jersey, 1962.

[9] G. W. BLUMAN AND J. D. COLE, *The general similarity solution of the heat equation*, J. Math. Mech., 18 (1969), pp. 1025–1042.

SIAM J. MATH. ANAL.
Vol. 1, No. 1, February 1970

# INTERPOLATION IN THE SOLUTION SETS OF ORDINARY DIFFERENTIAL EQUATIONS*

RONALD M. MATHSEN†

**Abstract.** In this paper we give sufficient conditions (Theorem 1) for the existence and uniqueness of solutions of certain boundary value problems for an $n$th order linear ordinary differential on a compact interval $I$ of the reals. Sufficient conditions (Theorem 2) are also given for the existence of certain zero multipoint boundary value problems for an $n$th order nonlinear differential equation.

**1. Introduction.** The notation we now introduce is due mainly to Schoenberg [1]. Let $n$ and $k$ be positive integers with $n \geqq k$. Let $E_n^k = (\varepsilon_{i,j})$, $i = 1, 2, \cdots, k$, $j = 0, 1, \cdots, n-1$, be a $k \times n$ matrix of zeros and ones with $\sum \varepsilon_{i,j} = n$ and no row consisting only of zeros. Let $x_1 < x_2 < \cdots < x_k$ be given points in $I$, and for each ordered pair $(i, j)$ such that $\varepsilon_{i,j} = 1$ let $\alpha_{i,j}$ be a given real number. The nodes $x_1, \cdots, x_k$ and the "incidence" matrix $E_n^k$ describe the interpolation problem of finding a unique solution of

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \tag{1}$$

on $I$ ($y^{(j)}$ is the $j$th derivative of $y$ and each $a_j$ is continuous on $I$) satisfying

$$y^{(j)}(x_i) = \alpha_{i,j} \tag{2}$$

for those $(i, j)$ with $\varepsilon_{i,j} = 1$. The matrix $E_n^k$ will be called *poised* on $I$ (with respect to the solution set of (1) on $I$) in case for any $k$ nodes and any $n$ real numbers $\alpha_{i,j}$ there is a unique solution of (1) on $I$ satisfying (2). Because of the linearity of (1), to show that $E^{kn}$ is poised it will suffice to show that the identically zero solution is the only solution of (1) on $I$ which satisfies (2) when all the numbers $\alpha_{i,j} = 0$.

Some special cases of this "interpolation problem" have been investigated. If (1) is the equation

$$(D^2 + n^2)(D^2 + (n-1)^2) \cdots (D^2 + 1)y' = 0,$$

where $D$ is the derivative operator, then the solution set is the set of trigonometric polynomials of order $\leqq n$. References [2] and [3] give, for specific choices of the nodes $x_i$, some results and further references for this problem. If $a_i = 0$ for all $i = 0, 1, \cdots, n-1$, the solution set is the set of polynomials of degree $\leqq n-1$ in one indeterminate. A good bibliography for this problem is contained in [4]. K. Atkinson and A. Sharma have recently shown in [5] that a matrix $E_n^k$ is poised for these polynomials in case $E_n^k$ is a conservative matrix, i.e., $E_n^k$ is the "direct sum" of irreducible conservative matrices (see [4] or [5] for definitions).

The problem of Lagrange interpolation, i.e., $k = n$ and $\varepsilon_{i,0} = 1$ for $i = 1, 2, \cdots, n$, has been investigated by several authors. Reference [6] contains an extensive bibliography for this problem. O. Aramă in [7] has announced some results for interpolation in the solution set of (1) in the case of two nodes.

* Received by the editors January 23, 1969.

† Department of Mathematics, University of Alberta, Edmonton, Alberta. This research was partially supported by the University of Alberta and the Canadian Mathematical Congress Summer Research Institute.

We say that the matrix $E_n^k$ is of class $D$ if for any $y = y(x)$ satisfying (2) with all $\alpha_{i,j} = 0$ and any integer $j, 0 \leqq j \leqq n - 1$, there is a point $\xi_j$ in $I$ such that $y^{(j)}(\xi_j) = 0$. Theorem II in [8] can be stated as follows:

$E_n^k$ will be poised with respect to solutions of (1) on $I$ in case $E_n^k$ is of class $D$ on $I$ and

$$\sum_{j=1}^{n-1} h^{n-j} \int_I |a_{j-1}(x)|\, dx \leqq 1, \tag{3}$$

where $h$ is the length of $I$.

The class of matrices which in this paper are shown to be poised is a subclass of the class $D$, but inequality (4) is replaced by conditions that are in certain cases less restrictive.

**2. Some poised matrices.** A matrix

$$E_n^k = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \\ & & & & 0 & 0 \\ & & A & & \vdots & \vdots \\ & & & & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad E_n^k = \begin{pmatrix} & 1 & 0 \\ & 0 & 0 \\ A & \vdots & \vdots \\ & 0 & 0 \\ & 1 & 0 \end{pmatrix},$$

where $A$ is a $(k - 2) \times (n - 2)$ (or $k \times (n - 2)$) matrix of class $D$, is said to be of class $C$.

THEOREM 1. *Let the coefficients $a_i(x)$ be continuous on the compact interval $I$ of the reals, let $a_{n-2}(x)$ be nonpositive on $I$ and let $a_{n-1}(x) \not\equiv 0$ on $I$. Then a matrix $E_n^k$ of class $C$ will be poised with respect to* (1) *on any proper subinterval of $I$ provided*

$$H_n(h) N_n E(\tfrac{1}{2}\|a_{n-1}\| h) \leqq 1, \tag{4}$$

*where*

$$E(x) = e^{2x} - e^x - x,$$

$$N_n = (\|a_{n-1}\|)^{-2}(\|a_{n-3}\| + \|a_{n-4}\| + \cdots + \|a_0\|),$$

*$h$ is the length of $I$, $\|\cdot\|$ denotes the supremum norm on $I$ and $H_n(h)$ is a positive number given by* (13) *or* (15).

Since the ordering $x_2 \leqq x_3 \leqq \cdots \leqq x_{k-1}$ will not be used in the proof of Theorem 1, it will suffice to prove the theorem for $E_n^k = \hat{E}_n^k = (\varepsilon_{i,j})$, $\varepsilon_{1,n-1} = \varepsilon_{k,n-1} = 1, \varepsilon_{2,j} = 1$ for $j = 0, 1, \cdots, l$ and $\varepsilon_{i,j} = 1$ for $j = i + l - 2, i = 3, 4, \cdots, k - 1$, i.e.,

$$\hat{E}_n^k = \begin{pmatrix} 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 \\ 1 & 1 & \cdot & \cdot & \cdot & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & & & & & & & \ddots & 0 & & \vdots & \vdots \\ \vdots & \vdots & & 0 & & & & & & \ddots & & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 1 & 0 \end{pmatrix}.$$

For convenience of notation we consider $\hat{E}_n^n$ and reduce it to $\hat{E}_n^k$ by setting $x_2 = x_3 = \cdots = x_{l+2}$. Note that $n = k + l$ and that $l + 1$ is the multiplicity of $x_2$ as a zero of the desired solution to (1).

The following existence theorem is used in the proof of Theorem 1.

THEOREM 2. *Let $f$ be continuous on $I \times R^n$, where $I$ is a compact interval of the set $R$ of real numbers. Let $f$ satisfy the following conditions:*

(i) *$f(x, y_1, \cdots, y_n)$ is nondecreasing in $y_{n-1}$ for fixed $x, y_1, \cdots, y_{n-2}, y_n$;*

(ii) *for each compact set $T$ of $I \times R^n$ there is a constant $K'(T)$ such that*

$$|f(x, y_1, \cdots, y_{n-1}, y_n) - f(x, y_1, \cdots, y_{n-1}, \bar{y}_n)| \leqq K'(T)|y_n - \bar{y}_n|$$

*for all points $(x, y_1, \cdots, y_{n-1}, y_n)$ and $(x, y_1, \cdots, y_{n-1}, \bar{y}_n)$ in $T$;*

(iii) *for each $M > 0$ there is a constant $K = K(M)$ such that*

$$|f(x, y_1, \cdots, y_{n-2}, 0, y_n) - f(x, y_1, \cdots, y_{n-2}, 0, 0)| \leqq K|y_n|$$

*for all $x$ in $I$ and all $|y_n| < \infty$ provided $|y_1| + |y_2| + \cdots + |y_{n-2}| \leqq M$. Then given $M > 0$, the boundary value problem*

$$(5)\qquad \begin{gathered} y^{(n)} = f(x, y, y', \cdots, y^{(n-1)}), \\ 0 = y^{(n-2)}(x_1) = y^{(i-2)}(x_i) \quad \text{for } i = 2, 3, \cdots, n \end{gathered}$$

*has a solution in $C^n[x_1, x_n]$ provided*

$$(6)\qquad H_n(\delta)Q_M E(\tfrac{1}{2}K\delta) \leqq M \cdot K^2,$$

*where $\delta = x_n - x_1$ and*

$$Q_M = \sup\{|f(x, y_1, \cdots, y_{n-2}, 0, 0)| : x \in I, |y_1| + |y_2| + \cdots + |y_{n-2}| \leqq M\}.$$

*Proof.* Let $M > 0$ be given and define

$$B = \{z \in C^{n-2}[x_1, x_n] : \|z\|^* \leqq M\},$$

where $\|z\|^* = \sum_{j=0}^{n-3} \|z^{(j)}\|$ on $[x_1, x_n]$. $B$ is a closed, convex subset of $C^{n-2}[x_1, x_n]$. From the assumptions (i), (ii) and (iii) on $f$ if follows that for each $z \in B$, the boundary value problem

$$(7)\qquad \begin{gathered} u'' = f(x, z(x), z'(x), \cdots, z^{(n-3)}(x), u, u'), \\ u(x_1) = u(x_n) = 0 \end{gathered}$$

has a unique solution in $C^2[x_1, x_n]$ (see [9, Theorem 6.3, p. 1065]). Call $u_z$ this solution to (7). Define the mapping $T$ on $B$ by $T(z) = W$, where

$$(8)\qquad W(x) = \int_{x_2}^{x} \int_{x_3}^{s_3} \int_{x_4}^{s_4} \cdots \int_{x_{n-2}}^{s_{n-2}} \int_{x_{n-1}}^{s_{n-1}} u_z(s)\, ds\, ds_{n-1} \cdots ds^3.$$

Then $W^{(i-2)}(x_i) = 0$ for $i = 2, 3, \cdots, n$ and $W^{(n-2)}(x_1) = 0$. Therefore a fixed point of $T$ will be a solution to (5). By a corollary to the Schauder fixed-point theorem [11, p. 405], $T$ will have a fixed point in $B$ provided $T$ is continuous on $B$, $T$ is compact and $T$ maps $B$ into $B$. That $T$ is compact follows from Ascoli's lemma and the fact that the sets $\{u_z : z \in B\}$ and $\{u_z' : z \in B\}$ are both uniformly bounded (see [10, pp. 628–629]). Continuity of $T$ follows, as in [10], from the fact that there is a positive constant $\gamma$ such that $\|T(u) - T(v)\|^* \leqq \gamma\|u - v\|$. In § 3 we show that

$$\|T(z)\|^* \leqq \|u_z\| H_n(\delta),$$

where $\delta = x_n - x_1$. Now we know (see [10, p. 629]) that $\|u_z\| \leqq Q_M E(\frac{1}{2}K\delta)/K^2$, and hence, by (6), $T(z) \in B$. This proves Theorem 2.

*Proof of Theorem* 1. Let $y = y_0(x)$ be a nontrivial solution to (1) satisfying the boundary conditions in (5) with $x_2 = x_3 = \cdots = x_{l+2}$. Suppose that $\delta = x_n - x_1 < h$ and let $L_n[y]$ denote the left side of (1). For any function $g$ which is continuous on $I$, the function $f$ defined by

$$f(x, y_1, y_2, \cdots, y_n) = -a_{n-1}y_n - a_{n-2}y_{n-1} - \cdots - a_0 y_1 + g \tag{9}$$

satisfies conditions (i), (ii) and (iii) of Theorem 2 with $K = \|a_{n-1}\|$ and $Q_M \leqq (\|a_{n-3}\| + \cdots + \|a_0\|)M + \|g\|$. Now $\delta = x_n - x_1 < h$, so by (4) we have that $H_n(\delta)N_n E(\frac{1}{2}\|a_{n-1}|\delta) < 1$. (We shall show presently that $H_n(h)$ is a decreasing function of $h$.) We prove the theorem by induction on $k = n - l$. Suppose $k = 2$. Choose $g(x)$ so that $V(x) = \varepsilon(x - x_2)^{n-1}$ is a solution to $L_n[y] = g$. Then $\|g\| = O(\varepsilon)$, so for $\varepsilon > 0$ sufficiently small

$$H_n(\delta)N_n E(\tfrac{1}{2}\|a_{n-1}\|\delta) \leqq 1 - \frac{\|g\| H_n(\delta)E(\frac{1}{2}\|a_{n-1}\|\delta)}{M}.$$

Hence (6) holds for $f$ as defined in (9). Then by Theorem 2 the boundary value problem

$$L_n[y] = g,$$

$$0 = y(x_2) = y'(x_2) = \cdots = y^{(n-2)}(x_2) = y^{(n-2)}(x_1)$$

has a solution $y = Y(x)$ on $[x_1, x_2]$. But $Y(x)$ must be of the form $Y(x) = c_0 y_0(x) + V(x)$ where $c_0$ is a constant. Then

$$0 = Y^{(n-2)}(x_1) = c_0 y_0^{(n-2)}(x_1) + V^{(n-2)}(x_1) = V^{(n-2)}(x_1) \neq 0,$$

and this contradiction shows that no such nontrivial $y_0(x)$ can exist if $k = 2$. Suppose now $\hat{E}^k_{n-1}$ is poised. Then there exist $k - 2$ linear independent solutions ($k - 1$ if $y_0$ is counted) $U_3, U_4, \cdots, U_k$ of (1) satisfying $U_i^{(j)}(x_{l+2}) = 0$ for $j = 0, 1, \cdots, l + 1$, $i = 3, 4, \cdots, k$; $U_i^{(n-2)}(x_1) = \delta^i_3$, (the Kronecker $\delta$), and $U_i^{(l+j-2)}(x_{l+j}) = \delta^i_j$ for $i, j = 4, 5, \cdots, k$. These boundary conditions are similar to those in (5) except that $U_i^{(l+1)}(x_{l+3})$ is not specified, and in each case exactly one 1 is specified with the remaining boundary values being 0. Choose $g(x)$ so that

$$V(x) = \varepsilon(x - x_1)^{n-1}(x - x_{l+2})^{l+2}(x - x_{l+3})^{l+1}(x - x_{l+4})^{l+3} \cdot (x - x_{l+5})^{l+4} \cdots (x - x_n)^{n-1}$$

is a solution to $L_n[V] = g$. Note that $V(x)$ satisfies all the boundary conditions of (5) with $x_2 = x_3 = \cdots = x_{l+2}$ except that $V^{(l+1)}(x_{l+3}) \neq 0$. Now again $\|g\| = O(\varepsilon)$ as $\varepsilon \to 0+$, so for $\varepsilon$ sufficiently small (6) will hold for $f$ as given in (9). Hence (5) has a solution $y = Y(x)$ for $f$ as given in (9). Then there exist constants $c_2, c_3, \cdots, c_k$ so that

$$Y(x) = c_2 y_0(x) + c_3 U_3(x) + \cdots + c_k U_k(x) + V(x)$$

for all $x$ in $I$. Now $0 = Y^{(n-2)}(x_1) = c_3$ and $0 = Y^{(l+i+1)}(x_{l+i}) = c_i$ for $i = 4, 5, \cdots, k$.

Hence

$$0 = Y^{(l+1)}(x_{l+3}) = c_2 y_0^{(l+1)}(x_{l+3}) + V^{(l+1)}(x_{l+3}) = V^{(l+1)}(x_{l+3}) \neq 0.$$

This contradiction shows that no such nontrivial $Y_0(x)$ can exist, and therefore $\hat{E}_n^k$ is poised. This proves Theorem 1.

**3.** $H_n(h)$. The basic inequality needed in § 2 is

$$\|W\|^* \leqq \|u_z\| H_n(h), \tag{10}$$

where $W$, $\|W\|^*$ and $u_z$ are defined in the proof of Theorem 2. For $W(x)$ as given in (8) we have $W^{(n-2)}(x) = u_z(x)$ and

$$W^{(n-2-j)}(s_{n-j}) = \int_{x_{n-j}}^{s_{n-j}} W^{(n-1-j)}(t)\,dt \quad \text{for } j = 1, 2, \cdots, n-2. \tag{11}$$

Now

$$|W^{(n-3)}(s_{n-1})| \leqq \int_{x_{n-1}}^{s_{n-1}} u_Z(t)\,dt \leqq \|u_Z\| \cdot |s_{n-1} - x_{n-1}|.$$

Let $(i, j) = |s_{n-i} - x_{n-j}|$ and $[i, j] = |x_{n-i} - x_{n-j}|$. Let $j$ be a positive integer and let $P(j)$ be the set of "ordered partitions of $j$" defined by $p \in P(j)$ if and only if $p = (p_1, p_2, \cdots, p_m)$, $m \geqq 1$, $p_1 + p_2 + \cdots + p_m = j$ and each $p_i$ is a positive integer with $p_i > 1$ for $i > 1$. Let

$$\Delta(j) = \sum_{p \in P(j)} \frac{(j, j - p_1 + 1)^{p_1}}{p_1!} \frac{[j, p_1, j - p_1 - p_2 + 1]^{p_2}}{p_2!} \cdots \frac{[p_m, 1]^{p_m}}{p_m!}$$
$$+ \sum_{p \in P(j)} \frac{[j, j - p_1 + 1]^{p_1}}{p_1!} \frac{[j - p_1, j - p_1 - p_2 + 1]^{p_2}}{p_2!} \cdots \frac{[p_m, 1]^{p_m}}{p_m!},$$

where $p \in P(j)$ is written in the form $p = (p_1, p_2, \cdots, p_m)$. We now claim that

$$|W^{n-2-j}(s_{n-j})| \leqq \Delta(j)\|u_z\| \quad \text{for } j = 1, 2, \cdots, n-2. \tag{12}$$

Equation (12) was established for $j = 1$. Suppose then that (12) holds for some fixed $j$. Then

$$|W^{(n-2-(j+1))}(s_{n-j-1})|$$
$$\leqq \left| \int_{x_{n-j-1}}^{s_{n-j-1}} \left| W^{(n-2-j)}(s_{n-j}) \right| ds_{n-j} \right|$$
$$\leqq \left\{ \sum_{p \in P(j)} \frac{(j+1, j - p_1 + 1)^{p_1+1}}{(p_1+1)!} \frac{[j - p_1, j - p_1 - p_2 + 1]^{p_2}}{p_2!} \cdots \frac{[p_m, 1]^{p_m}}{p_m!} \right.$$
$$\left. + \sum_{p \in P(j)} \frac{(j+1, j)}{1!} \frac{[j, j - p_1 + 1]^{p_1}}{p_1!} \cdots \frac{[p_m, 1]^{p_m}}{p_m!} \right\} \|u_z\|$$
$$+ \sum_{p \in P(j)} \frac{[j+1, j - p_1 + 1]^{p_1}}{p_1!} \frac{[j - p_1, j - p_1 - p_2 + 1]^{p_2}}{p_2!} \cdots \frac{[p_m, 1]^{p_m}}{p_m!}.$$

Observe that in the second sum $p_1 \neq 1$ since $[j, j - 1 + 1] = 0$, so that the first two sums combine to form

$$\sum_{p \in P(j+1)} \frac{(j+1, j+1-p_1+1)^{p_1}}{p_1!} \frac{[j+1-p_1, j+1-p_1-p_2+1]^{p_2}}{p_2!} \cdots \frac{[p_m, 1]^{p_m}}{p_m!}.$$

Also, in the third sum, $p_1 + 1 \geqq 2$, but $[j, j - 1 + 1] = 0$, so the third sum becomes

$$\sum_{p \in P(j+1)} \frac{[j+1, j+1-p_1+1]^{p_1}}{p_1!} \frac{[j+1-p_1, j+1-p_1-p_2+1]^{p_2}}{p_2!} \cdots \frac{[p_m, 1]^{p_m}}{p_m!}.$$

Hence $|W^{(n-2-j-1)}(s_{n-j-1})| \leqq \Delta(j+1)\|u_z\|$, so by induction (12) holds for all $j = 1, 2, \cdots, n-2$. Clearly then one bound for $H_n(h)$ would be

$$H_n(h) = \sum_{j=1}^{n-2} \alpha(j) h^j, \tag{13}$$

where

$$\alpha(j) = \sum_{p \in P(j)} \frac{2}{p_1! p_2! \cdots} + \sum_{q \in P(j-1)} \frac{1}{q_1! q_2! \cdots}.$$

$[r, s] = |x_{n-r} - x_{n-s}| = 0$ if $x_{n-r} = x_{n-s}$. Suppose now that $x_2 = x_3 = \cdots = x_{l+2} \cdot r \geqq s$ so $n - r \leqq n - s$, and $[r, s] = 0$ if $n - s \leqq l + 2$, i.e., $s \geqq n - l - 2 = k - 2$. It is clear then that $\Delta(j)$ does not increase for fixed $j$ as $l$ increases, and in fact, $\Delta(j)$ decreases as $l$ increases for $j \geqq k - 2$. This observation is needed for the proof of Theorem 1 to make the induction step if estimates other than that in (3) are used for $H_n(k)$. Let

$$\beta(j) = \sum_{\substack{p \in P(j) \\ p_1 \neq 1}} \frac{1}{p_1! p_2! \cdots} = \sum_{\substack{p_1 + p_2 + \cdots = j \\ \text{all } p_1 \geqq 2}} \frac{1}{p_1! p_2! \cdots}. \tag{14}$$

Then $\alpha(j) = 2\beta(j) + \beta(j-1)$ for $j = 2, 3, 4, \cdots, n-2$. Let

$$F(z) = \sum_{j=2}^{\infty} \frac{z^j}{j!} = e^z - z - 1.$$

Then

$$\frac{F(z)}{1 - F(z)} = \sum_{j=2}^{\infty} \beta(j) z^j.$$

This power series has radius of convergence $R > 1.1461 = \rho$, so that $\sum_{j=2}^{\infty} \beta(j)\rho^j$ is convergent. This observation is important since it insures that the numbers $\beta(j)$ will decrease exponentially as $j$ gets large.

Suppose that $x_2 = x_3 = \cdots = x_{l+2} < x_{l+3}$ where $1 \leqq l \leqq n - 3$. Then one can estimate $W^{(n-2-j)}(x)$ by $|W^{(n-2-j)}(x)| \leqq h^j$ for $j = 1, 2, \cdots, n - l - 3$ and $|W^{(n-2-j)}(x)| \leqq h^j/(j - n + l + 3)!$ for $j = n - l - 2, \cdots, n - 2$.

Thus

$$\|W\|^* = \sum_{j=0}^{n-3} \|W^{(j)}\| \leqq \|u_z\| \left( \sum_{j=0}^{l} \frac{h^{n-2-j}}{(l+1-j)!} + \sum_{j=1}^{n-l-3} h^j \right),$$

so we may take

(15)
$$H_n(h) = \sum_{j=0}^{l} \frac{h^{n-2-j}}{(l+1-j)!} + \sum_{j=1}^{n-l-3} h^j.$$

For $l = n - 3$ (3 nodes) we have

$$H_n(h) = H_n^{(3)}(h) = \sum_{j=1}^{n-2} \frac{h^j}{j!},$$

and for $l = n - 4$ (4 nodes) we have

$$H_n(h) = H_n^{(4)}(h) = \sum_{j=0}^{n-3} \frac{h^{j+1}}{j!}.$$

Equation (15) is much easier to compute than (13). One may say in a nonprecise fashion that for $l$ "close" to $n - 3$, estimate (15) is preferred to (13), but for $l$ "close" to 0, (13) will give a better result since $\alpha_j < 1$ for $j \geqq 3$.

It is also quite clear that the numbers $\beta(n)$ satisfy the recurrence relation

$$\beta(n) = \frac{1}{n!} + \sum_{k=2}^{n-2} \frac{1}{(n-k)!}\beta_k$$

for $n \geqq 4$.

**4. Some special cases.** Let $l = n - 3$. Then $k = 3$ so that $[r, s] = 0$ if $s \geqq 1$, and thus $[j, j - p_1 + 1] = 0$ if $j - p_1 \geqq 0$, i.e., if $j \geqq p_1$. Hence $\Delta(j)$ reduces to $(j, j)^j/j!$ so that $|\Delta(j)| \leqq h^j/j!$ and $H_n(h)$ reduces to

$$H_n(h) = H_n^{(3)}(h) = \sum_{j=1}^{n-2} \frac{h^j}{j!}.$$

Let $l = n - 4$. Then $k = 4$, so $[r, s] = 0$ if $s \geqq 2$. Thus $[j, j - p + 1] = 0$ if $j - p_1 + 1 \geqq 2$, i.e., if $j \geqq p_1 + 1$. Also $[j - p_1, j - p_1 - p_2 + 1] = 0$ if $j - 1 \geqq p_1 + p_2$, so $\Delta(j)$ reduces to

$$\sum_{\substack{r=0 \\ r \neq j-1}}^{j} \frac{(j, j-r+1)^r}{r!} \frac{[j-r, 1]^{j-r}}{(j-r)!} + \frac{[j, 1]^j}{j!}$$

$$= \frac{|x_{n-j} - x_{n-1}|^j}{j!} + \sum_{\substack{r=0 \\ r \neq j-1}}^{j} \frac{|x - x_{n-j+r-1}|^r}{r!} \frac{|x_{n-j+r} - x_{n-1}|^{j-r}}{(j-r)!}.$$

The maximum value of this sum occurs when $x = x_{n-1}$ and $x_{n-2} = x_{n-3} = \cdots = x_{n-j-1}$, so that

$$\Delta(j) \leqq \frac{h^j}{j!}\left(1 + \sum_{r=0}^{j-2} \binom{j}{r}\right),$$

where $\binom{j}{r}$ is the binomial coefficient. The resulting bound for $H_n(h)$ is not as good as the bound $\sum_{j=1}^{n-2} h^j/(j-1)!$, which is given by (15) for this special case. This is in accordance with the comment at the end of § 3.

Consider next the 4-point problem obtained by putting $x_2 = x_3 = \cdots = x_{l+1} = x_{l+2} < x_{l+3} = x_{l+4} = \cdots = x_{n-1}$. Then $[r, s] = 0$ if $n - s \leqq l + 2$ or $n - r \geqq l + 3$, i.e., $[r, s] = 0$ if $s \geqq n - l - 2$ or $r \leqq n - l - 3$. For $1 \leqq j \leqq n - l - 3$ we have $\Delta(j) = [j, 1]_j^j!$. For $j \geqq n - l - 2$ we have

$$\Delta(j) = \frac{[j, 1]^j}{j!} + \sum_{\substack{r=n-l-2 \\ r \neq j-1}}^{j} \frac{(j, j-r+1)^r}{r!} \frac{[r, 1]^{j-r}}{(j-r)!}.$$

So

$$\Delta(j) \leqq \begin{cases} \dfrac{h^j}{j!}, & 1 \leqq j \leqq n - l - 3, \\ \dfrac{h^j}{j!}\left(1 + \displaystyle\sum_{\substack{r=n-l-1 \\ r \neq j-1}}^{j} \binom{j}{r}\right), & j \geqq n - l - 2. \end{cases}$$

Also of interest is the special case where $x_2 = x_4 = x_6 = \cdots$ and $x_3 = x_5 = \cdots$. In this case $[r, s) = 0$ if $r - s$ is an even integer. Hence

$$\Delta(j) = \sum_{\substack{p_1+p_2+\cdots=j \\ p_2, p_3, \cdots \text{ all even}}} \frac{(j, j-p_1+1)^{p_1}}{p_1!} \frac{[j-p_1, j-p_1-p_2+1]^{p_2}}{p_2!} \cdots \frac{[p_m, 1]^{p_m}}{p_m!}$$
$$+ \sum_{\substack{p_1+p_2\cdots=j \\ p_1, p_2, \cdots \text{ all even}}} \frac{[j, j-p_1+1]^{p_1}}{p_1!} \frac{[j-p_1, j-p_1-p_2+1]^{p_2}}{p_2!} \cdots \frac{[p_m, 1]^{p_m}}{p_m!}.$$

This is a special case of the problem obtained by putting $x_i = x_{i+p_0}$, $i = 2, 3, \cdots, n - 1 - p_0$ where $p_0$ is a positive integer. In this case

$$\Delta(j) = \sum_{p_1+p_2+\cdots=j} \frac{(j, j-p_1+1)^{p_1}}{p_1!} \frac{[j-p_1, j-p_1-p_2+1]^{p_2}}{p_2!} \cdots \frac{[p_m, 1]^{p_m}}{p_m!}$$
$$+ \sum_{p_1+p_2+\cdots=j} \frac{[j, j-p_1+1]^{p_1}}{p_1!} \frac{[j-p_1, j-p_1-p_2+1]^{p_2}}{p_2!} \cdots \frac{[p_m, 1]^{p_m}}{p_m!},$$

the first sum being taken over all partitions with $p_i \not\equiv 1 \pmod{p_0}$ for $i \geqq 2$ and the second sum with $p_1 \not\equiv 1 \pmod{p_0}$ as well. These will give bounds $H_n(h)$ for the special problems.

**5. Some poised matrices.** We consider matrices $E_n^4$ of the two types $B_1$ and $B_2$.

*Type* $B_1$. $\varepsilon_{2,j} = 1$ for $j = 0, 1, \cdots, n-4$ and $j = n - 2$; $\varepsilon_{3,n-3} = \varepsilon_{4,n-2} = \varepsilon_{1,n-2} = 1$ (or the given rows 2 and 3 are interchanged), i.e.,

$$E_n^4 = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 & 0 \\ 1 & \cdots & 1 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{or} \quad E_n^4 = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & \cdots & 0 & 1 & 0 & 0 \\ 1 & \cdots & 1 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix}.$$

*Type* $B_2$. $\varepsilon_{2,j} = 1$ for $j = 0, 1, \cdots, n - 4$; $\varepsilon_{1,n-2} = 1$; $\varepsilon_{3,j} = 1$ for $j = j_0$, $0 \leqq j_0 < n - 2$, and $\varepsilon_{4,j} = 1$ for $j = j_1$, $j_0 \leqq j_1 \leqq n - 2$, with not both $\varepsilon_{3,n-3} = \varepsilon_{4,n-2} = 1$ (or the given rows 1 and 4 and the rows 2 and 3 interchanged).

THEOREM 3. *Let* (1) *be as in Theorem* 1. *Let* $E_n^4$ *satisfy condition* $B_1$ *and let* (4) *hold with*

$$H_n(h) = H_n^{(4)}(h) = \sum_{j=1}^{n-2} \frac{h^j}{(j-1)!}.$$

*Then* $E_n^4$ *is poised with respect to* (1) *on any proper subinterval of* $I$.

This theorem is simply a restatement of Theorem 1 for the case $k = 4$. The corollary of this theorem involving matrices satisfying condition $B_2$ is of interest since matrices of type $B_2$ are not included in class $C$.

COROLLARY 1. *Let the condition of Theorem* 3 *be satisfied with* $B_1$ *replaced by* $B_2$. *Then* $E_n^4$ (*of Type* $B_2$) *is poised on* $I$.

*Proof.* Suppose $y = Y(x)$ is a solution of (1) on $I$ satisfying (2) with $\alpha_{i,j} = 0$. Apply Rolle's theorem to $Y^{(j_0)}(x)$ on $[x_2, x_3]$ to get a point $\gamma_1 \in (x_2, x_3)$ such that $Y^{(j_0+1)}(\gamma_1) = 0$. Then apply Rolle's theorem to $Y^{(j_0+1)}(x)$ and continue until there exist points $\hat{x}_3 < \hat{x}_4$, both in $(x_2, x_4)$, so that $y^{(n-3)}(\hat{x}_3) = 0$ and $y^{(n-2)}(\hat{x}_4) = 0$. Then the matrix determined with $x_1$, $x_2$, $\hat{x}_3$ and $\hat{x}_4$ as nodes, $0 = y^{(n-2)}(x_1) = y^{(n-2)}(\hat{x}_4) = y^{(n-3)}(\hat{x}_3)$ and $y^{(j)}(x_2) = 0$ for $j = 0, 1, 2, \cdots, n-4$, is of type $B_1$ on $[x_1, \hat{x}_4]$, and hence by Theorem 3 ($\hat{x}_4 < x_4$) is poised on $[x_1, \hat{x}_4]$. Hence $Y(x) \equiv 0$ on $I$, and therefore $E_n^4$ is poised on $I$.

Similar corollaries can be stated and proved in analogous fashion for the cases $k \neq 4$ also.

In the general case it suffices that the matrix $A$ which forms part of $E_n^k$ (see the beginning of § 3) should be conservative since in [5] it is shown that if $A$ is conservative, then $A$ is of class $D$.

**6. Zeros of solutions.** Suppose that $y = y_0(x)$ is a nontrivial solution to (1) on the compact interval $I$ and assume that $a_{n-2}(x) \leqq 0$ on $I$. Then the following situations cannot occur when inequality (4) holds:

(a) $y_0(x)$ has a zero $z$ of multiplicity $k$ with $n - k - 1$ zeros (counting multiplicity) to the left of $z$ and $n - k - 1$ zeros (counting multiplicity) to the right of $z$ in $I$;

(b) $y_0(x)$ has $2n - 3$ zeros (counting multiplicity) in $I$;

(c) $y_0(x)$ has a zero of multiplicity $n - 2$ which separates two other zeros of $y_0(x)$ in $I$;

(d) $y_0(x)$ has a zero of multiplicity $n - 1$ and one other zero in $I$.

*Note.* For cases (a) and (b) let $H_n(h)$ be defined by (13); for cases (c) and (d) let $H_n(h)$ be given by

$$\sum_{j=1}^{n-2} \frac{h^j}{j!}.$$

Cases (c) and (d) are impossible by Theorem 1. Case (a) follows from Theorem 1 by observing that if $I = [a, b]$, then there are points $z = a_0 \geqq a_1 \geqq a_2 \geqq \cdots \geqq a_{n-2}$ and $z = b_0 \leqq b_1 \leqq b_2 \leqq \cdots \leqq b_{n-2}$ with $a_{n-2} < b_{n-2}$ so that $y_0^{(j)}(a_j) = y_0^{(j)}(b_j) = 0$ for $j = 0, 1, \cdots, n-2$. This of course violates Theorem 1 when $y_0(x)$ is not identically zero on $I$. Case (b) is a special case of (a). Note that as $k$ increases in case (a), more terms are zero in $\Delta(j)$ and hence a smaller estimate can be given for $H_n(h)$.

**7. Miscellany.** Suppose that $a_{n-1}(x) \equiv 0$ on $I$. In Theorem 2 this corresponds to $f$ not depending on $y_n$. Then condition (iii) of Theorem 2 holds for any $K > 0$. Also

$$\lim_{K \to 0+} \frac{E(\frac{1}{2}hK)}{K^2} = \frac{3h^2}{8};$$

so if (7) is replaced by

$$H_n(\delta) Q_M 3h^2 \leqq 8M, \tag{16}$$

then Theorem 2 is still valid. Similarly Theorem 1 remains valid if (5) is replaced by

$$H_n(h)(\|a_{n-3}\| + \|a_{n-4}\| + \cdots + \|a_0\|)3h^2 \leqq 8$$

in case $a_{n-1}(x) \equiv 0$ on $I$. Also notice that if $Q_M E(\frac{1}{2}\delta K(M))/(K(M))^2 = O(M^2)$ as $M \to \infty$, then (6) (or (16) in the case $f$ does not depend on $y_n$) is valid for $M$ large enough, so the given boundary value problem (5) has a solution.

Suppose that $I$, $a_{n-1}$ and $a_{n-2} \leqq 0$ on $I$ are given. Then $a_{n-3}, a_{n-4}, \cdots, a_0$ can be chosen with small enough norms so that $N_n$ is sufficiently small for (4) to hold. Hence it is clear that the set of equations (1) for which (4) holds is not vacuous. In fact if $A$ is conservative and irreducible, then

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ & & & & 0 & 0 & 0 \\ & & A & & \vdots & \vdots & \vdots \\ & & & & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} & 1 & 0 \\ & 0 & 0 \\ A & \vdots & \vdots \\ & 0 & 0 \\ & & \end{pmatrix}$$

are both poised with respect to solutions of (1). Also,

$$\sum_{j=2}^{\infty} \alpha(j) \leqq 3 \sum_{j=2}^{\infty} \beta(j) = 3\left(\frac{e-2}{3-e}\right),$$

so

$$H_n(1) \leqq \sum_{j=1}^{\infty} \alpha(j) \leqq 1 + 3\left(\frac{e-2}{3-e}\right) = \frac{2e-3}{3-e}.$$

Thus we have the following corollaries of Theorem 1.

COROLLARY 2. *The matrix $E_n^k$ of class C is poised on any proper subinterval of* [0, 1] *with respect to* (1) *provided*

$$\frac{2e-3}{3-e} \frac{(\|a_{n-3}\| + \cdots + \|a_0\|)}{\|a_{n-1}\|^2} E(\tfrac{1}{2}\|a_{n-1}\|) \leqq 1.$$

Of course we assume that $a_{n-2} \leqq 0$ on [0, 1] and that all coefficients are continuous.

COROLLARY 3. *Suppose that $a(x)$ and $b(x)$ are continuous on* [0, 1] *with $a(x) \leqq 0$ on* [0, 1]. *Then the matrix $E_n^k$ of class C is poised on any proper subinterval of* [0, 1] *with respect to*

$$y^{(n)} + ay^{(n-2)} + by = 0,$$

*provided*

$$\frac{3}{8}\left(\frac{2e-3}{3-e}\right)\|b\| \leqq 1.$$

**Acknowledgment.** Most of the results of this paper were obtained while the author was a postdoctoral fellow at the University of Alberta. Thanks are due to the University of Alberta and the Canadian Mathematical Congress Summer Research Institute for support of this research.

REFERENCES

[1] I. J. SCHOENBERG, *On Hermite-Birkhoff interpolation*, J. Math. Anal. Appl., 16 (1966), pp. 538–543.
[2] A. SHARMA AND A. K. VARMA, *Trigonometric interpolation*, Duke Math. J., 32 (1965), pp. 341–358.
[3] ———, *Trigonometric interpolation, (0, 2, 3) case*, Ann. Polon. Math., to appear.
[4] J. PRASAD AND A. SHARMA, *On Abel-Hermite-Birkhoff interpolation*, SIAM J. Numer. Anal., 5 (1968), pp. 864–881.
[5] K. ATKINSON AND A. SHARMA, *A partial characterization of poised Hermite-Birkhoff interpolation problems*, Ibid., 6 (1969), pp. 230–235.
[6] A. LASOTA AND Z. OPIAL, *L'existence et l'unicité des solutions du problème d'interpolation pour l'équation différentielle ordinaire d'ordre n*, Ann. Polon. Math., 15 (1964), pp. 253–271.
[7] O. ARAMĂ, *Un problème d'interpolation lacunaire avec des solution différentielles*, Mathematica (Cluj), 6 (1964), pp. 353–354.
[8] Z. NEHARI, *On an inequality of Lyapunof*, Studies in Mathematical Analysis and Related Topics, Stanford University Press, Stanford, California, 1962, pp. 256–261.
[9] J. W. BERBENES, *A subfunction approach to a boundary value problem for ordinary differential equations*, Pacific J. Math., 13 (1963), pp. 1053–1066.
[10] R. M. MATHSEN, *A disconjugacy condition for* $y''' + a_2y'' + a_1y' + a_0y = 0$, Proc. Amer. Math. Soc., 17 (1966), pp. 627–632.
[11] P. HARTMAN, *Ordinary Differential Equations*, John Wiley, New York, 1964.

SIAM J. MATH. ANAL.
Vol. 1, No. 1, February 1970

# LEAST SQUARES METHODS FOR ILL-POSED PROBLEMS WITH A PRESCRIBED BOUND*

KEITH MILLER†

**1. Introduction.** Ill-posed problems in partial differential equations are those in which the solution depends uniquely but not continuously on the data. The Cauchy problem for elliptic equations, backward solution of parabolic equations, elliptic continuation, and complex analytic continuation are a few of the classical problems which are not well-posed in the sense of Hadamard. It has been found for a large class of these problems, however, that continuous dependence on data may be restored on compact subsets by restricting attention to those solutions satisfying a prescribed global bound; see the initial papers by Pucci [14] and John [9], for example. This bound is often of physical origin, and such problems are thereby restored to physical usefulness.

The study of ill-posed problems seems to divide naturally into two tasks: firstly, establishment of a priori stability estimates which assure continuous dependence on data with the prescribed bound, and secondly, development of adequate computational methods. This paper is directed primarily toward the latter task.

In two previous papers, [11] and [12], the author presented a method useful for solving ill-posed problems in cases, such as those occurring when separation of variables applies, where an eigenfunction expansion of the solutions and the data is known. The present paper introduces methods which are much more general and more numerical in nature. These methods are analogous to the versatile linear programming method introduced by J. Douglas and applied to many ill-posed problems by Douglas [5], J. Cannon and the author [3] and others. However, the $l_2$ norm instead of the $l_\infty$ norm is used, which leads to great computational advantages. This switch in norms is also justified by the fact that in all our examples the stability estimates for the quadratic norm are only slightly higher than those for the uniform norm.

The secondary purpose of this paper is to present the results and methods it contains for the problem of analytic continuation. Analytic continuation is used throughout as the example for illustration because it is typical, because the notation and a priori estimates are simpler than for other examples, but also because it is of great interest in its own right. Analytic continuation is being constantly used (and often abused, with little understanding of its instability) in the physical sciences these days, in particular in nuclear scattering theory. See, for example, the work of T. Regge [15], the copious publications on $S$-matrix theory of the G. F. Chew school at Berkeley [4], and work by other physicists too numerous to mention here.

---

* Received by the editors December 24, 1968.

† Department of Mathematics, University of California, Berkeley, California 94720. This research was supported by the Air Force Office of Scientific Research and the National Academy of Sciences through a postdoctoral research fellowship for 1964–1965 at the University of Genova, and also in the write-up stage by the Air Force Office of Scientific Research under Grant 553-64.

Sections 2–4 deal with examples and methods in a Hilbert space setting. Section 2 introduces the example of analytic continuation from data given everywhere on an interior arc and rephrases it in terms of operators on Hilbert space. Section 3 considers the Hilbert space problem and methods in general. The methods developed here, involving as they do solution of normal equations in infinite dimensions, are not exactly numerical in character; however, the exposition is simplest and most general in this setting. We see that a satisfactory approximation policy for ill-posed problems requires knowledge of a bound for the data accuracy (as in Method 3), or knowledge of a bound for the global constraint of the solution (as in Method 4), or, preferably, both these bounds (as in Methods 1 and 2). The fortunate feature of these methods is that they do not depend upon stability estimates being known and that they are "almost best possible" methods, independent of the choice of norm for measuring the error. A brief observation on compactness explains qualitatively the success of prescribed bounds in restoring stability in so many instances. Section 4 compares the present methods with the previous eigenfunction expansion methods, again in the general Hilbert space setting.

Sections 5–9 deal with numerical methods, with the problem of approximately determining an analytic function on the unit disc from approximate data for its values at a finite number of interior data points as example. The results and methods for this problem and the computer program whose sample output is shown in Tables 1–4 all go under the title of "Stabilized Numerical Analytic Continuation", or "SNAC" for short. SNAC begins in § 5 with discrete data; § 6 further discretizes the problem, approximating the unknown analytic function by an unknown polynomial and discretizing the boundary constraint. The original problem is thereby replaced by a fully discretized problem for which all the operators and normal equations of § 3 are finite-dimensional and subject to direct numerical computation. Section 7 discusses discretized problems in general.

Method 1 involves considering the global constraint as an additional piece of data, then applying the natural least squares procedure to the resulting problem with overspecified data. The general method itself, that of minimizing a quadratic combination of the fit to the data plus a small weighting parameter times a regularizing function, seems to have been devised and used independently in many different unstable situations; see for example [1, p. 137] and [16]. What is of present interest is the fact that the method is essentially "best possible" within the context of a prescribed bound, the fact that making the prescribed bound explicit allows precise choice of the weighting parameter, the relation of Method 1 to Methods 2–4, and the ease and generality of these as numerical methods. One feature of Method 1 which should be emphasized is that it allows "best possible" stability estimates for the discretized problem to be easily generated *by the computer*. This is of especially practical importance in the case of ill-posed problems, for as a general rule the restored continuity is still extremely poor. Whereas the best possible estimate may give some useful information, a theoretically derived estimate may be so large as to be totally useless.

The a priori stability estimates for SNAC are derived in § 8. The data points may be arbitrarily distributed; however, the a priori estimates derived are in terms of their "radial density" on a fixed compact subset of positive capacity. One fortunate feature of SNAC is that the interpolation error must go down exponentially with the square root of the number of data points. We reiterate that as a practical matter one need not strain too hard for quantitative precision in derivation of a priori estimates, for as a rule the computer-generated "best possible" estimates will be much more precise. The only a priori analysis required is the relatively simple discretization analysis needed to insure that the discretization error is negligible.

Section 9 gives a comparison between the $l_2$ and $l_\infty$ methods. Finally, we discuss some sample output from the computer program SNAC.

**2. An example with nondiscrete data.** Consider the problem of approximately determining an unknown analytic function $f_0$ on the open unit disc $D$ from approximate values $h(z)$ for $f_0(z)$ given everywhere on a smooth arc $\Gamma$ compactly contained in $D$. We assume that $h$ is an $L_2$ data function given on $\Gamma$, and that $f_0$ satisfies the following data error bound on $\Gamma$ and prescribed bound on $\partial D$:

$$\|f_0 - h\|_\Gamma \leqq \varepsilon, \tag{2.1}$$

$$\|f_0\|_{\partial D} \leqq E, \tag{2.2}$$

where we assume for now that both $\varepsilon$ and $E$ are known numbers. The norms indicated are the $L_2$ norms on $\Gamma$ and $\partial D$, with integration with respect to normalized arclength.

An analytic function is uniquely determined by its exact values on an interior arc, as is well known. However, this problem, without the prescribed bound (2.2), would be completely unstable. As the simplest example, suppose that $\Gamma$ is the circle $\{|z| = b\}$, $0 < b < 1$. Let $c$ be any number between $b$ and 1, and consider the sequence of functions $(z/c)^n$ which tend to zero together with all their derivatives on the data circle yet tend to infinity at any point with $|z| > c$. Adding such functions onto $f_0$ as error functions, we see that an arbitrarily small error in the data can induce an arbitrarily large error in the solution at any point exterior to the data circle.

The prescribed bound (2.2) restores stable dependence on the data on compact subsets of $D$, as is shown in the following two lemmas, which give estimates for the difference between two functions satisfying (2.1) and (2.2). The first lemma is basic to our analysis; it gives a simple and quite precise a priori stability estimate provided uniform bounds are assumed in (2.1) and (2.2). This variation [7] of the classical Carleman inequality includes the Hadamard 3-circle theorem, for example, as a special case. The second lemma extends the estimate to the desired case of $L_2$ bounds in (2.1) and (2.2). We delay its proof to § 8, however.

LEMMA 1. *Suppose that $f$ is a function analytic on $D - \Gamma$, continuous on $\bar{D}$, and satisfying*

$$|f| \leqq \varepsilon \quad \textit{on } \Gamma, \tag{2.3}$$

$$|f| \leqq E \quad \textit{on } \partial D. \tag{2.4}$$

*Then on $D$,*

$$|f(z)| \leqq \varepsilon^{w(z)} E^{1 - w(z)}, \tag{2.5}$$

*where w is the harmonic measure of* $\Gamma$ *with respect to* $D - \Gamma$, *that is, the solution of the following Dirichlet problem*:

$$\begin{aligned} &w \text{ is harmonic on } D - \Gamma, \text{ continuous on } \bar{D}, \\ &w \equiv 1 \quad \text{on } \Gamma, \\ &w \equiv 0 \quad \text{on } \partial D. \end{aligned} \tag{2.6}$$

*Proof.* The function $\log|f(z)|$, being the real part of any branch of $\log f(z)$, is harmonic except at zeros of $f$, at which points it tends to $-\infty$. Now, this subharmonic function $\log|f(z)|$ is less than or equal the harmonic function $(\log \varepsilon)w(z) + (\log E)(1 - w(z))$ on the boundary of $D - \Gamma$; hence the same inequality extends to all $D - \Gamma$ by the maximum principle, which completes the proof.

LEMMA 2. *Suppose that* $f$ *is analytic on* $D$ *and satisfies the* $L_2$ *bounds*

$$\|f\|_\Gamma \leqq \varepsilon, \tag{2.7}$$

$$\|f\|_{\partial D} \leqq E. \tag{2.8}$$

*Then on* $D$,

$$|f(z)| \leqq \left(c \log\left(\frac{E}{\varepsilon}\right) + \frac{1}{\sqrt{1 - |z|^2}}\right)\varepsilon^{w(z)}E^{1-w(z)}, \tag{2.9}$$

*where* $w$ *is the harmonic function of* (2.8) *and* $c$ *is a constant depending only on the length of* $\Gamma$, *on* $b = \max\{|z| : z \in \Gamma\}$, *and on the "radial span" of* $\Gamma$ (see § 8).

We now rephrase the problem in terms of operators on Hilbert space. The Hilbert space $H^2$ of functions $f$ analytic on the open unit disc with finite $L^2$ norm on $\partial D$ is topologically isomorphic to the Hilbert space $l_2$ of infinite complex sequences $x = (x_0, x_1, \cdots)$ with the usual $l_2$ norm under the Taylor expansion $f(z) = \sum_0^\infty x_j z^j$. The fact that this is an isometric isomorphism, i.e.,

$$\|f\|_{\partial D} = \|x\| \equiv \left(\sum_0^\infty |x_j|^2\right)^{1/2}, \tag{2.10}$$

is only incidental. Thus, instead of dealing with analytic functions $f$, we shall be dealing with their coefficient sequences or "parameter vectors" $x$. Corresponding to each $l_2$ sequence $x$, let $Ax$ and $Bx$ denote the trace functions of its Taylor series on $\Gamma$ and $\partial D$; that is, if $\Gamma$ is parametrized by $z = \gamma(s)$, and of course $\partial D$ by $z = e^{i\theta}$, then

$$Ax(s) = \sum_0^\infty x_j(\gamma(s))^j, \tag{2.11}$$

$$Bx(\theta) = \sum_0^\infty x_j e^{ij\theta}. \tag{2.12}$$

We see that $A$ is a bounded linear transformation on the "parameter space" $l_2$ into (not onto) the "data space" $L_2(\Gamma)$ (it is incidentally compact and 1–1, with an unbounded inverse on its range). $B$ is a bounded linear transformation on $l_2$ into (not onto) the "constraint space" $L_2(\partial D)$ (it is incidentally an isometry). Notice that the data function $h(s)$ is an element of the data space.

We were not sufficiently precise in the original statement of the problem; we asked only that we determine an "approximation" (let us say it is also an analytic function $f_1$, with parameter vector $x^1$) to $f_0$ (with parameter vector $x^0$) without specifying a norm or seminorm $\langle \cdot \rangle$ with which to measure the error $x^1 - x^0$. We certainly cannot use the $l_2$ norm, for our problem is unstable with respect to this norm, as we can see by considering $f_0(z) = z^n$ and $f_1(z) = -z^n$ with $n$ large. By Lemma 2 the problem is stable with respect to the uniform norm on any compact subset. In particular it is often convenient to consider the seminorm

$$\langle x \rangle_{z_0} \equiv |f(z_0)| = \left| \sum_0^\infty (x_j) z_0^j \right|, \tag{2.13}$$

where $z_0$ is any fixed point in $D$, since this seminorm can be written simply as an inner product in the parameter space:

$$\langle x \rangle_{z_0} = |(x, v)|, \qquad v = (1, \bar{z}_0, \bar{z}_0^2, \cdots). \tag{2.14}$$

**3. The infinite-dimensional problem and method in general.** In general we have the following situation: $x^0$ is an unknown element in a Hilbert space $X$, the *parameter space*. $\langle \cdot \rangle$ denotes a seminorm, or sometimes a family of seminorms on $X$. $A$, the *data operator*, is a bounded linear operator on $X$ into a Hilbert space $Y$, the *data space*, and $h$, the *data vector*, is a given element in $Y$ which approximately equals $Ax^0$. $B$, the *constraint operator*, is a bounded linear operator (with bounded inverse on its range) on $X$ into a Hilbert space $Z$, the *constraint space*.

*Problem.* Suppose that $x^0$ satisfies

$$\|Ax^0 - h\| \leqq \varepsilon, \tag{3.1}$$

$$\|Bx^0\| \leqq E, \tag{3.2}$$

where $E$ is a "fixed" number and $\varepsilon$ is a "small" number. We assume for the moment that both $\varepsilon$ and $E$ are known. We want to find an element $x^1 \in X$ which "approximates" $x^0$, in the sense that $\langle x^1 - x^0 \rangle$ is small when $\varepsilon$ is small. Our methods will luckily turn out to be completely independent of the seminorm under consideration.

We assume that Problem (3.1), (3.2) is stable with respect to the seminorm $\langle \cdot \rangle$; that is,

$$\mathcal{M}(\varepsilon, E) \equiv \sup \{\langle x \rangle : x \in X \text{ and } \|Ax\| \leqq \varepsilon,\ \|B(x)\| \leqq E\} \tag{3.3}$$

tends to zero as $\varepsilon$ tends to zero, for fixed $E$. Finding an approximation to $x^0$ then reduces to the problem of finding any other element $x^1$ satisfying the constraints (3.1) and (3.2); for then $\langle x^1 - x^0 \rangle \leqq 2\mathcal{M}(\varepsilon, E)$. We call any upper bound for $\mathcal{M}(\varepsilon, E)$ a *stability estimate* for Problem (3.1), (3.2), and $\mathcal{M}(\varepsilon, E)$ itself we call the *best possible stability estimate.*

Instead of dealing with the two constraints (3.1) and (3.2) separately, we combine them quadratically into a single constraint, as is often done in such cases of overspecified data (data $h$ for $Ax^0$ and data 0 for $Bx^0$). We lose at most a factor of $\sqrt{2}$ in the constraints in the process.

LEMMA 3. *If $x^0$ satisfies* (3.1) *and* (3.2), *then it also satisfies*

$$\|Ax^0 - h\|^2 + \left(\frac{\varepsilon}{E}\right)^2 \|Bx^0\|^2 \leqq 2\varepsilon^2. \tag{3.4}$$

*Conversely, any* $x^0$ *satisfying* (3.4) *also satisfies* (3.1) *and* (3.2) *except for a factor of at most* $\sqrt{2}$. *If we let* $\mathscr{M}_1(\varepsilon, E)$ *denote the supremum of* $\langle x \rangle$ *with respect to* (3.4) *with* $h = 0$, *then*

$$\mathscr{M}(\varepsilon, E) \leqq \mathscr{M}_1(\varepsilon, E) \leqq \sqrt{2}\mathscr{M}(\varepsilon, E). \tag{3.5}$$

*Method* 1 (A least squares method). Let our approximation $x^1$ be that element of $X$ which minimizes

$$\|Ax - h\|^2 + \left(\frac{\varepsilon}{E}\right)^2 \|Bx\|^2. \tag{3.6}$$

It is the solution of the normal equations

$$\left(A^*A + \left(\frac{\varepsilon}{E}\right)^2 B^*B\right)x^1 = A^*h. \tag{3.7}$$

The derivation of the normal equations for the minimization of (3.6) is easily accomplished by introducing the direct sum Hilbert space $Y \oplus Z$ whose elements are the pairs $[y, z]$, $y \in Y$, $z \in Z$, and whose inner product is $([y_1, z_1], [y_2, z_2]) = (y_1, y_2) + (z_1, z_2)$. The composite operator $[A, (\varepsilon/E)B]$ is defined by

$$\left[A, \left(\frac{\varepsilon}{E}\right)B\right]x = \left[Ax, \left(\frac{\varepsilon}{E}\right)Bx\right],$$

and the problem is to minimize

$$\left\|\left[A, \left(\frac{\varepsilon}{E}\right)B\right]x - [h, 0]\right\|^2.$$

Our solution $x^1$ then is the solution of the normal equation

$$\left[A, \left(\frac{\varepsilon}{E}\right)B\right]^*\left[A, \left(\frac{\varepsilon}{E}\right)B\right]x = \left[A, \frac{\varepsilon}{E}B\right]^*[h, 0],$$

as is well known, which can be written as (3.7). The existence of a minimum and the fact that the operator

$$C \equiv A^*A + \left(\frac{\varepsilon}{E}\right)^2 B^*B \tag{3.8}$$

of (3.7) is invertible follow from the fact that $B$ has a bounded inverse on its range and hence $B^*B$ is positive definite on $X$.

*Note.* We have required $B$ to be bounded and with a bounded inverse on its range mainly for ease of exposition. All that is really required is that it be densely defined, so that $B^*$ exists, plus some other combination of hypotheses to insure that the normal operator $C$ has a bounded inverse. See for instance the examples of [11, p. 131] where we let $B$ be various derivative and integral operators on $\partial D$.

*Note.* In our example of § 2, $B^*B = I$ since $B$ is an isometry. Moreover, using the Schwarz inequality on the defining formula (2.12) we see that the uniform norm (and hence the $L_2$ norm) of $Ax$ is bounded by $(1 - b^2)^{-1/2}\|x\|$, where as before $b = \max\{|z| : z \in \Gamma\}$. Hence $\|A\| \leqq (1 - b^2)^{-1/2}$ and the spectrum of $A^*A$ lies in the interval $[0, (1 - b^2)^{-1}]$. The operator $C$ of the normal equation therefore has its spectrum in the interval $[(\varepsilon/E)^2, (1 - b^2)^{-1} + (\varepsilon/E)^2]$. Thus we have quite a good estimate for the "condition number" for $C$ (the supremum of the spectrum divided by the infimum of the spectrum).

LEMMA 4 (A posteriori compatibility check and a priori error bound). *If there exists an* $x^0$ *satisfying* (3.4), *then* $x^1$ *must satisfy*

$$\|Ax^1 - h\|^2 + \left(\frac{\varepsilon}{E}\right)^2 \|Bx^1\|^2 \leqq 2\varepsilon^2. \tag{3.9}$$

*Moreover,*

$$\|A(x^1 - x^0)\|^2 + \left(\frac{\varepsilon}{E}\right)^2 \|B(x^1 - x^0)\|^2 \leqq 2\varepsilon^2; \tag{3.10}$$

*hence,*

$$\langle x^1 - x^0 \rangle \leqq \mathscr{M}_1(\varepsilon, E). \tag{3.11}$$

*Proof.* Inequality (3.9) is automatic of course. Inequality (3.10) follows from the fact that $[A, (\varepsilon/E)B]x^1$ is the perpendicular projection in $Y \oplus Z$ of $[h, 0]$ on the range of $[A, (\varepsilon/E)B]$. Hence

$$\begin{aligned} &\left\|\left[A, \frac{\varepsilon}{E}B\right](x^1 - x^0)\right\|^2 + \left\|\left[A, \frac{\varepsilon}{E}B\right]x^1 - [h, 0]\right\|^2 \\ &\qquad = \left\|\left[A, \frac{\varepsilon}{E}B\right]x^0 - [h, 0]\right\|^2 \leqq 2\varepsilon^2, \end{aligned} \tag{3.12}$$

which completes the proof.

Inequalities (3.10) and (3.11) show that *the error* $\langle x^1 - x^0 \rangle$ (*independently of what seminorm is used*) *is as small as can be expected except for the factor of* $\sqrt{2}$.

Notice that (3.10) is twice as good as could be obtained by merely using the triangle inequality in $Y \oplus Z$ on (3.9) and (3.4). Inequality (3.9) gives an a posteriori check (to be computed after computing $x^1$) on the claimed accuracy $\varepsilon$ and bound $E$ for the given data function $h$.

In the following lemma we see that in case the seminorm is in the form of a known inner product then a computable formula can be given for $\mathscr{M}_1(\varepsilon, E)$, the "best possible stability estimate with respect to (3.4)."

LEMMA 5 (A formula for the best possible stability estimate). *If* $\langle \cdot \rangle$ *is of the form* $\langle x \rangle = |(x, v)|$, *then* $\mathscr{M}_1(\varepsilon, E)$ *is given by*

$$\mathscr{M}_1(\varepsilon, E) = \sqrt{2}\varepsilon(C^{-1}v, v)^{1/2}. \tag{3.13}$$

*Proof.* $\mathscr{M}_1$ is the maximum of $(x, v)$ with respect to the quadratic constraint

$$\left\|\left[A, \left(\frac{\varepsilon}{E}\right)B\right]x\right\|^2 = (Cx, x) \leqq 2\varepsilon^2.$$

We then just apply the Schwarz inequality, but with respect to the new inner product $[x, y] \equiv (Cx, y)$. We have

$$\begin{aligned}(x, v) &= [x, C^{-1}v] \\ &\leqq [x, x]^{1/2}[C^{-1}v, C^{-1}v]^{1/2} = (Cx, x)^{1/2}(C^{-1}v, v)^{1/2} \\ &\leqq \sqrt{2}\varepsilon(C^{-1}v, v)^{1/2},\end{aligned}$$

which inequality is precise.

The preceding formulation has the disadvantage that the error bound $\varepsilon$ and the constraint $E$ must both be known. We show now that a satisfactory approximation policy requires a knowledge of only one of these numbers. The following considerations are extremely similar to those discussed on pages 141 and 142 of [11] for the author's previous eigenfunction expansion method.

Given numbers $\varepsilon$ and $E$ may clearly be too small for the given data $h$, and $x$ satisfying (3.1) and (3.2) may fail to exist. Let us call a pair $(\varepsilon, E)$ *permissible* if there exists an $x$ in $X$ satisfying (3.1) and (3.2). Now it turns out that the factor $\lambda^2 = (\varepsilon/E)^2$ in the normal equation is really a Lagrangian multiplier, and the solutions of the normal equation, as $\lambda$ increases from 0 to $\infty$, give complete information concerning which pairs are permissible.

Let $x_\lambda$ denote the solution of the minimization problem (3.6), i.e., of the normal equation (3.7), with $(\varepsilon/E)^2$ replaced by $\lambda^2$. Let

$$\text{(3.14)} \qquad \varepsilon_\lambda \equiv \|Ax_\lambda - h\|, \qquad E_\lambda \equiv \|Bx_\lambda\|.$$

Clearly $x_\lambda$ minimizes $\|Ax - h\|$ with respect to the constraint $\|Bx\| \leqq E_\lambda$. Likewise $x_\lambda$ minimizes $\|Bx\|$ with respect to the constraint $\|Ax - h\| \leqq \varepsilon_\lambda$. It is also easily seen that $\varepsilon_\lambda$ and $E_\lambda$ are continuously increasing and continuously decreasing functions of $\lambda$. Thus, the set $\mathscr{E}$ of permissible pairs is exactly the set of points which are above and to the right of the curve $(\varepsilon_\lambda, E_\lambda)$, $0 \leqq \lambda \leqq \infty$. Here the case $\lambda = 0$ corresponds to minimization of $\|Ax - h\|$ alone, in which case $E_\lambda$ may be $\infty$, and the case $\lambda = \infty$ corresponds to minimization of $\|Bx\|$ alone, in which case $\varepsilon_\lambda = \|h\|$ and $E_\lambda = 0$. Moreover, one easily shows that $\mathscr{E}$ is a convex set, hence computation of only a finite number of points on its boundary curve $(\varepsilon_\lambda, E_\lambda)$, coupled with linear interpolation inbetween, should give a good idea of its shape.

Suppose now we are looking for a particular parameter vector $x^0$ and are given approximate data $h$ for $Ax^0$. Let $\bar{\varepsilon}$ and $\bar{E}$ denote $\|Ax^0 - h\|$ and $\|Bx^0\|$ respectively.

*Method* 2. Consider first the case where upper bounds $\varepsilon$ and $E$ are known for both $\bar{\varepsilon}$ and $\bar{E}$. This was our assumption in (3.1) of Problem (3.1), (3.2). By our previous discussion the $(\bar{\varepsilon}, \bar{E})$ for $x^0$ must lie in the shaded area of Fig. 1. Further, *any* $x_\lambda$ whose corresponding $(\varepsilon_\lambda, E_\lambda)$ touches the shaded area will be a satisfactory approximation to $x^0$, for then the error is bounded by $\langle x_\lambda - x^0 \rangle \leqq \mathscr{M}(\varepsilon + \bar{\varepsilon}, E + \bar{E}) \leqq 2\mathscr{M}(\varepsilon, E)$.

Our least squares Method 1 of course calls for us to take $x_\lambda$ with $\lambda = (\varepsilon/E)$ as our approximation; in that case we saw that $\varepsilon_\lambda \leqq \sqrt{2}\varepsilon$, $E_\lambda \leqq \sqrt{2}E$, and $\langle x_\lambda - x^0 \rangle \leqq \mathscr{M}_1(\varepsilon, E) \leqq \sqrt{2}\mathscr{M}(\varepsilon, E)$.

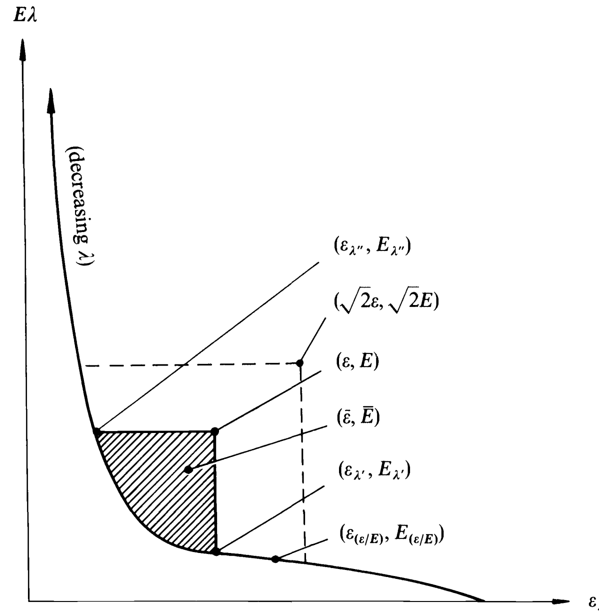


FIG. 1

*Method* 3. Consider next the case where only an upper bound $\varepsilon$ for $\bar{\varepsilon}$ is known. Let our approximation $x^1$ then be the element in $X$ which minimizes $\|Bx\|$ with respect to $\|A_x - h\| \leqq \varepsilon$. That is, let $x^1 = x_{\lambda'}$, where $\lambda'$ is the value such that $\varepsilon_{\lambda'} = \varepsilon$, as shown in Fig. 1. This method then involves merely solving a sequence of least squares problems; because of the continuity and monotonicity of $\varepsilon_\lambda$, we can solve for the desired Lagrangian multiplier $\lambda'$ by a variety of iterative root solving methods, interval halving for example. Now $\bar{E}$ must lie above $E_{\lambda'}$. Thus the error is bounded by $\langle x^1 - x^0 \rangle \leqq \mathscr{M}(\varepsilon + \bar{\varepsilon}, E_{\lambda'} + \bar{E}) \leqq 2\mathscr{M}(\varepsilon, \bar{E})$, which is essentially optimal with respect to the given information, even though $\bar{E}$ is unknown.

*Method* 4. Consider finally the case where only an upper bound $E$ for $\bar{E}$ is known. Let our approximation $x^1$ be the element in $X$ which minimizes $\|Ax - h\|$ with respect to $\|Bx\| \leqq E$. That is, let $x^1 = x_{\lambda''}$, where $\lambda''$ is the value such that $E_{\lambda''} = E$, as is shown in Fig. 1. Here the error is bounded by $\langle x^1 - x^0 \rangle \leqq 2\mathscr{M}(\bar{\varepsilon}, E)$, which is essentially optimal with respect to the given constraint, even though $\bar{\varepsilon}$ is unknown.

*Remark.* The success of prescribed bounds in restoring stability to so many of the classical ill-posed problems is *qualitatively* explained by the following standard theorem on compactness [10, p. 141]: Let $\sigma$ be a continuous mapping on the topological space $X$ into the Hausdorff topological space $Y$; if $\sigma$ is 1–1 and $X$ is compact, then $\sigma^{-1}$ is continuous. In our example of analytic continuation we see that analytic functions satisfying the prescribed bound (2.2) on $\partial D$ form a bounded equicontinuous family, which is hence precompact with respect to the uniform norm $|\cdot|_{D_1}$ on any subdomain $D_1$ with $\bar{D}_1 \subset D, \Gamma \subset D_1$. Taking the closure with respect to $|\cdot|_{D_1}$ we still have a compact family $X$ of analytic functions.

This, plus the uniqueness of analytic continuation, therefore restores continuity to the problem of continuation from $\Gamma$ to $D_1$.[1]

**4. Comparison with the method of partial eigenfunction expansion.** It is instructive to compare our present methods with the previous methods of [11] and [12], again in the general Hilbert space setting. We shall see that the previous methods have two serious restrictions. In the first place $A^*A$ and $B^*B$ must commute. This is not too great a disadvantage since we can often contrive to make $B^*B = I$ by a different choice of the parameter space, as noted in the remarks at the end of the section. A more serious disadvantage, however, is that the method requires knowledge of the spectral decompositions of $A^*A$ and $B^*B$. In certain instances these are given to us naturally, such as in many problems of partial differential equations where separation of variables is possible. As a general rule, however, even in discretized problems where the operators involved are finite-dimensional and subject to numerical computations, the finding of spectral decompositions is an exceedingly more difficult operation than is the mere solution of normal equations.

We assume now that $A^*A$ and $B^*B$ commute, leaving all other hypotheses and notation from Problem (3.1), (3.2) unchanged. $A$ and $B$ have the polar decompositions

$$A = U\sqrt{A^*A}, \qquad B = V\sqrt{B^*B}, \tag{4.1}$$

where $\sqrt{A^*A}$ and $\sqrt{B^*B}$ are positive semidefinite and positive definite operators on $X$, $U$ is an isometry of $X$ into $Y$ and $V$ is an isometry of $X$ into $Z$. Because $\sqrt{A^*A}$ and $\sqrt{B^*B}$ commute, they have spectral decompositions with respect to the same real spectral measure $E$; that is,

$$\sqrt{A^*A} = \int \psi(\lambda)\, dE(\lambda), \qquad \sqrt{B^*B} = \int \varphi(\lambda)\, dE(\lambda),$$

where $\psi$ and $\varphi$ are nonnegative functions on the spectrum of $E$. See [13, p. 67] and [6] as references for the spectral theory used here. Now, let $P$ be the projection in $X$ corresponding to that portion of the spectrum where

$$\frac{1}{\varepsilon}\psi(\lambda) \geqq \frac{1}{E}\varphi(\lambda),$$

and let $Q = I - P$; that is,

$$P = \int_\Omega dE(\lambda), \qquad \Omega = \left\{ \lambda : \lambda \in \text{spectrum } E \text{ and } \frac{1}{\varepsilon}\psi(\lambda) \geqq \frac{1}{E}\varphi(\lambda) \right\}. \tag{4.2}$$

We have chosen this particular orthogonal decomposition of $X$ because of the following property.

LEMMA 6. *For any $x$ in $X$, $\|APx\| \leqq \varepsilon$ implies $\|BPx\| \leqq E$, and $\|BQx\| \leqq E$ implies $\|AQx\| \leqq \varepsilon$.*

[1] I am indebted for this observation to a mathematician in the audience at Pisa in 1965 when I was presenting these numerical methods there.

Let $R$ denote the projection in $Y$ onto the range of $AP$, and let $S$ denote the projection in $Z$ onto the range of $BQ$.

$$APx = RAx, \qquad BQx = SBx, \tag{4.3}$$

$$AQx = (I - R)Ax, \qquad BPx = (I - S)Bx, \tag{4.4}$$

$$\|Ax\|^2 = \|APx\|^2 + \|AQx\|^2, \tag{4.5}$$

$$\|Bx\|^2 = \|BPx\|^2 + \|BQx\|^2. \tag{4.6}$$

Instead of keeping the constraints in the form

$$\|Ax^0 - h\| \leqq \varepsilon, \tag{4.7}$$

$$\|Bx^0\| \leqq E, \tag{4.8}$$

we use for our method only the information

$$\|R(Ax^0 - h)\| = \|APx^0 - Rh\| \leqq \varepsilon, \tag{4.9}$$

$$\|SBx^0\| = \|BQx^0\| \leqq E. \tag{4.10}$$

*Method* 5. Since $A$, as an operator on the range of $P$ onto the range of $R$, is invertible, let our approximation be that element $x^1$ such that

$$APx^1 = Rh, \qquad x^1 = Px^1. \tag{4.11}$$

The following lemma corresponds to the "three norm lemma" of [11] and Lemmas 8 and 9 here then correspond to Lemmas 7 and 8 there.

LEMMA 7. *Let* $\mathscr{M}(\varepsilon, E)$ *be defined as in* (3.3) *and let*

$$L = L(\varepsilon, E) = \sup\{\langle Px\rangle : x \in X, \|APx\| \leqq \varepsilon\}, \tag{4.12}$$

$$H = H(\varepsilon, E) = \sup\{\langle Qx\rangle : x \in X, \|BQx\| \leqq E\}. \tag{4.13}$$

*Then*

$$\tfrac{1}{2}(L + H) \leqq \max(L, H) \leqq \mathscr{M}(\varepsilon, E) \leqq L + H. \tag{4.14}$$

*Proof*. The lower bounds on $\mathscr{M}$ by $L$ and $H$ follow from Lemma 6, and the upper bound on $\mathscr{M}$ by $L + H$ follows by writing $\langle x\rangle \leqq \langle Px\rangle + \langle Qx\rangle$ for any $x$ satisfying both constraints.

Notice that if $\langle \cdot \rangle$ is of the form of a known inner product $\langle x\rangle = |(x, v)|$, as in Lemma 5, then we can give formulas for $L$ and $H$. Using the Schwarz inequality as in the proof of Lemma 5 we have

$$L = \varepsilon((A^*A)^{-1}Pv, Pv)^{1/2}, \tag{4.15}$$

$$H = E((B^*B)^{-1}Qv, Qv)^{1/2}, \tag{4.16}$$

where the inverses indicated for $A^*A$ and $B^*B$ are their inverses as operators restricted to the range of $P$ and the range of $Q$, respectively, on which spaces they are invertible.

LEMMA 8 (A priori bound). *The error* $x^1 - x^0$ *satisfies*

$$\|AP(x^1 - x^0)\| \leqq \varepsilon, \qquad \|BQ(x^1 - x^0)\| \leqq E; \tag{4.17}$$

*hence,*

$$\langle x^1 - x^0 \rangle \leqq L(\varepsilon, E) + H(\varepsilon, E). \tag{4.18}$$

*Alternatively,*

$$\|A(x^1 - x^0)\| \leqq \sqrt{2}\varepsilon, \qquad \|B(x^1 - x^0)\| \leqq \sqrt{2}E; \tag{4.19}$$

*hence,*

$$\langle x^1 - x^0 \rangle \leqq \sqrt{2}\mathscr{M}(\varepsilon, E). \tag{4.20}$$

*Proof.* The first inequality of (4.17) follows from (4.11) substituted in (4.9). The second follows from (4.10) and the fact that $BQx^1 = 0$. Now Lemma 6 with (4.17) gives the companion inequalities

$$\|BP(x^1 - x^0)\| \leqq E, \qquad \|AQ(x^1 - x^0)\| \leqq \varepsilon. \tag{4.21}$$

These with (4.17) and with (4.5) and (4.6) then yield (4.19) as desired.

LEMMA 9 (A posteriori compatibility check). *If there exists an $x^0$ satisfying* (4.7), (4.8) *for the given data h, then $x^1$ must satisfy*

$$\|Ax^1 - h\| \leqq 2\varepsilon, \qquad \|Bx^1\| \leqq 2E. \tag{4.22}$$

*Proof.* We have

$$\begin{aligned}\|Ax^1 - h\| &= \|(I - R)(-h)\| = \|(I - R)(Ax^0 - h) - (I - R)Ax^0\| \\ &\leqq \|Ax^0 - h\| + \|AQx^0\| \leqq 2\varepsilon,\end{aligned}$$

where the term $\|AQx^0\|$ is $\leqq\varepsilon$ by Lemma 6 since $\|BQx^0\|$ is $\leqq E$. Then

$$\|Bx^1\| = \|BPx^1\| \leqq \|BP(x^1 - x^0)\| + \|BPx^0\| \leqq 2E,$$

where the first term is $\leqq E$ by Lemma 6 since

$$\|AP(x^1 - x^0)\| = \|Rh - RAx^0\| = \|R(Ax^0 - h)\| \leqq \varepsilon.$$

This completes the proof.

*Note.* When $B$ is an isometry, as is often the case, then $A^*A$ commutes with $B^*B = I$ automatically. If commutativity should fail, however, then theoretically we can just take $y = Bx$ as our parameter vector instead, since we have assumed $B$ is invertible on its range, itself a Hilbert space. The constraints then read $\|AB^{-1}y - h\| \leqq \varepsilon$, $\|y\| \leqq E$, and one is required to consider the spectral decomposition of $(AB^{-1})^*(AB^{-1})$.

We finish this section by comparing Method 1 with Method 5 in a concrete example for which both can be exactly executed. This is an example which we have used previously in [12, p. 402] for Method 5. Consider the problem of analytic continuation (2.1), (2.2) when $\Gamma$ is the circle $|z| = b$, $0 < b < 1$. In that case $X = l_2$ and $Y = Z = L_2[0, \pi]$. Taking the natural basis for $l_2$ and the Fourier basis $\{\cdots, e^{-i2\theta}, e^{-i\theta}, 1, e^{i\theta}, e^{i2\theta}, \cdots\}$ for $L_2$ we see that $A$ and $B$ have the infinite matrices

$$\begin{aligned} a_{ij} &= b^j\delta_{ij}, \\ b_{ij} &= \delta_{ij}, \qquad -\infty < i < \infty, \quad 0 < j < \infty, \end{aligned} \tag{4.23}$$

where $\delta_{ij}$ is the Kronecker delta. Thus $B^*B$ is the identity and $A^*A$ has the diagonal matrix

$$(a^*a)_{ij} = b^{2j}\delta_{ij}, \qquad 0 \leqq i < \infty, \quad 0 \leqq j < \infty. \tag{4.24}$$

Suppose that the $L_2$ data function $h$ has the Fourier expansion

$$h(\theta) = \sum_{j=-\infty}^{\infty} h_j e^{ij\theta}. \tag{4.25}$$

Method 1 then gives us the approximation

$$f_1(z) = \sum_{j=0}^{\infty} h_j\left(\frac{b^j}{b^{2j} + (\varepsilon/E)^2}\right) z^j \tag{4.26}$$

and the a priori error bound from (3.11), (3.13) and (2.16):

$$|(f_1 - f_0)(z)| \leqq \mathscr{M}_1(\varepsilon, E) = \sqrt{2}\varepsilon\left(\sum_{j=0}^{\infty} \frac{|z|^{2j}}{b^{2j} + (\varepsilon/E)^2}\right)^{1/2}. \tag{4.27}$$

On the other hand Method 5 gives us the approximation

$$f_1(z) = \sum_{j=0}^{[\alpha]} h_j(b^{-j})z^j, \tag{4.28}$$

where $\alpha$ is defined by $(\varepsilon/E) = a^\alpha$ and $[\alpha]$ denotes the greatest integer $\leqq \alpha$. The a priori error bound for this method from (4.18), (4.15), (4.16) and (2.16) is

$$\begin{aligned} |(f_1 - f_0)(z)| &\leqq L(\varepsilon, E) + H(\varepsilon, E) \\ &= \varepsilon\left(\sum_{j=0}^{[\alpha]}\left(\frac{|z|}{b}\right)^{2j}\right)^{1/2} + E|z|^{[\alpha+1]}(1 - |z|^2)^{-1/2} \\ &\leqq \varepsilon^{w(z)}E^{(1-w(z))}\left\{\left[\frac{\log(\varepsilon/E)}{\log b}\right]^{1/2} + \left[\frac{1}{1 - |z|^2}\right]^{1/2}\right\}, \end{aligned} \tag{4.29}$$

where $w(z) = (\log|z|)/(\log b)$ in this case.

A similar simplification of (4.27) gives a similar bound there.

**5. Stabilized numerical analytic continuation: a numerical example.** We return to the problem of analytic continuation, but this time we only require data values given at a finite number of points. We suppose that $\Gamma$ is a closed set of positive capacity contained in $D$, that the *data set* $\dot{\Gamma} = \{d_1, \cdots, d_k\}$ is a discrete subset of $\Gamma$, that $f_0$ is an unknown analytic function on $D$, that $h$ is a discrete data function given on $\Gamma$, and that

$$\|f_0 - h\|_{\dot{\Gamma}} \leqq \varepsilon, \tag{5.1}$$

$$\|f_0\| \leqq E. \tag{5.2}$$

The problem again is to approximately determine $f_0$ on compact subsets of $D$. The norm indicated on $\dot{\Gamma}$ is the normalized $l_2$ norm, that is,

$$\|f_0 - h\|_{\dot{\Gamma}} \equiv \left(\frac{1}{k}\sum_{j=1}^{k} |(f_0 - h)(d_j)|^2\right)^{1/2}. \tag{5.3}$$

Later, in § 8, we shall develop a priori stability estimates which show that the difference between two analytic functions satisfying (5.1) and (5.2) must tend uniformly to zero on compact subsets of $D$ as both $\varepsilon$ and the "radial density" of the data points on $\Gamma$ tend to zero. We consider first, however, the much simpler discretization analysis.

**6. Discretization analysis for** SNAC. We want to reduce problem (5.1), (5.2) to a discrete or finite-dimensional one in which all the computations, and even the stability estimates, can be carried out by computer.

In the first place we approximate $f_0$ by a polynomial $F_0$ of degree $n$. Because we are on the disc we can let $F_0$ be the $n$th partial sum of $f_0$'s Taylor series. Using the prescribed bound (5.2) and the Schwarz inequality we get the precise truncation error bound

$$|(f_0 - F_0)(z)| \leqq \frac{E}{\sqrt{1 - |z|^2}} |z|^{n+1}. \tag{6.1}$$

Taking $n$ sufficiently large that this is $\leqq .1\varepsilon$ on $\{|z| \leqq b\}$ we have that

$$\| f_0 - F_0 \|_{\dot{\Gamma}} \leqq .1\varepsilon, \qquad \| f_0 - F_0 \|_{\partial D} \leqq E, \tag{6.2}$$

and hence the truncation error $(f_0 - F_0)(z)$ is less at each point than any stability estimate for the problem (5.1), (5.2). More explicitly, we obtain from (6.1) the bound

$$\begin{aligned} |(f_0 - F_0)(z)| &\leqq \frac{1}{\sqrt{1 - |z|^2}} (.1\varepsilon)^{(\log|z|)/(\log b)} E^{1 - (\log|z|)/(\log b)} \\ &\leqq \frac{1}{\sqrt{1 - |z|^2}} (.1\varepsilon)^{w(z)} E^{1 - w(z)}, \end{aligned} \tag{6.3}$$

where $w$ is the harmonic function of Lemma 1. We have written $.1\varepsilon$ instead of just $\varepsilon$ to emphasize the fact that the effect of the truncation error, since it goes down exponentially with $n$, can be essentially wiped out by taking $n$ a little bit larger than is absolutely necessary.

Finally, we replace the $L_2$ norm on all $\partial D$ by the normalized $l_2$ norm on a discrete subset $\partial \dot{D} = \{c_1, \cdots, c_l\}$, $l > n$. This is not really necessary in the present example since the $L_2$ norm can be written as the sum of the squares of the polynomial coefficients, but we do it anyway to emphasize the discretization of everything in sight and to point out the generalization of the method to regions other than discs. In the present case we take the *constraint points* $c_j$ equally spaced, for then the $l_2$ norm equals the $L_2$ norm for polynomials of order $< l$. This is easily seen since the basis functions $z^j$, $0 \leqq j < l$, are orthonormal with respect to both the $L_2$ and the $l_2$ inner product.

We have therefore reduced the original problem to the following one.

*Discretized problem.* Approximately determine the polynomial $F_0$ of order $n$, where $F_0$ satisfies

$$\|F_0 - h\|_{\dot{\Gamma}} \leqq 1.1\varepsilon, \tag{6.4}$$

$$\|F_0\|_{\partial \dot{D}} \equiv \|F_0\|_{\partial D} \leqq E. \tag{6.5}$$

**7. The discretized problem and methods in general.** In terms of the general terminology of § 3 we have reduced the original problem to one in which the parameter space, the data space, and the constraint space are all finite-dimensional. In fact, $X$ is $C^{n+1}$, $(n + 1)$-dimensional complex space with the usual $l_2$ inner product. $Y$ is $C^k$, and $Z$ is $C^l$ with $l > n$. The *data matrix* $A$ is given by

$$(a_{ij}) = \left(\frac{1}{\sqrt{k}}(d_i)^j\right), \qquad 1 \leqq i \leqq k, \quad 0 \leqq j \leqq n.$$

The *constraint matrix* $B$ is given by

$$(b_{ij}) = \left(\frac{1}{\sqrt{l}}(c_i)^j\right), \qquad 1 \leqq i \leqq l, \quad 0 \leqq j \leqq n.$$

The *parameter vector* $x$ has the polynomial coefficients $x_j$, $0 \leqq j \leqq n$, as components. The *data vector* $h$, has the components $h_i = h(c_i)/\sqrt{k}$. Notice that the normalization constants $1/\sqrt{k}$ and $1/\sqrt{l}$ have been included in $A$, $B$ and $h$ since we are using the unnormalized $l_2$ inner product on $Y$ and $Z$. The seminorm mentioned in (2.14) is given by the same formula, $\langle x \rangle_{z_0} = |(x, v)|$ with $v = (1, \bar{z}_0, \cdots, \bar{z}_0^n)$.

All of the methods of § 3 now merely involve matrix computations which are easily carried out by computer. The matrix $C$ of the normal equation must still have its spectrum in the interval $[(\varepsilon/E)^2, (1 - b)^{-1} + (\varepsilon/E)^2]$; hence we have an estimate on its condition number, which indicates the roundoff difficulties involved in its inversion.

The computation of $C^{-1}$ involves only approximately double the computational work required for just solving the single normal equation (3.7); hence with little extra work we can use the formula (3.13) to compute the precise stability estimate $\mathscr{M}_1(\varepsilon, E)$ at a whole grid of points $z_0$ across the region $D$, thereby giving a very good picture of the dependence of the stability estimate upon the geometry.

Generalization of these discretized least squares methods, illustrated here in the example SNAC, to other ill-posed problems should be fairly evident now. We take a moment though to speak loosely about certain requirements on the discretization method used. In the general situation we are confronted with the problem of approximately determining a "solution" (of a certain partial differential equation say) which closely fits given data and which satisfies a prescribed global constraint. Discretization of the problem involves finding a *linear approximate representation*, of sufficient accuracy for all solutions of interest, in terms of a finite number of parameters. (For example, in the problem of harmonic continuation, one can imagine approximately representing solutions of Laplace's equation by the discrete solutions of the Laplace difference equation; such discrete solutions depending linearly on their discrete boundary values as parameters.) We must insure that the approximation for the desired solution also closely fits the (discretized) data and also satisfies a prescribed (discretized) constraint. The problem then shifts to solving the discretized problem.

It should be emphasized that *it is the discretized problem which must be stable*, the stability estimates being independent of the dimensionality of the discretization used. This is the case in our example SNAC because the discretized problem is really just a subproblem of the original problem (5.1), (5.2). A polynomial $F$ satisfying (6.4) and (6.5) is itself an analytic function satisfying (5.1) and (5.2); hence any stability estimate for the original problem must also hold for the discretized problem. The situation would be different, however, for the above-mentioned example of a finite difference equation discretization of the harmonic continuation problem. Stability with a prescribed bound for discrete harmonic continuation, independent of the grid size, does not follow from the stability of harmonic continuation, but would have to be proved independently.

**8. A priori stability estimates for** SNAC. The task now is to obtain estimates similar to (2.5) of Lemma 1 when the infinite data set $\Gamma$ is replaced by the finite subset $\dot{\Gamma}$, and when the uniform norm is replaced by the quadratic norm. The basic interpolation error analysis of Lemma 10 is quite similar to that in a previous paper by Cannon and the author [3]; however, we now bound the interpolation error only on $\Gamma$ and go from there using the more natural bound of Lemma 1.

We wish to point out that the sets $D$ and $\Gamma$ for Lemma 1 can be more general than mentioned there. The proof of (2.5) still holds if we let $D$ be any bounded open set, the data set $\Gamma$ be any closed subset of $\bar{D}$ (it may for example be a portion of the boundary rather than an interior subset), and if we assume (2.4) holds on $\partial D - \Gamma$. The Dirichlet problem (2.5) may be nonsolvable if $D - \Gamma$ has exceptional boundary points; in such a case we have to reinterpret $w(z)$ to be the "lower" generalized Dirichlet solution, that is, the supremum of all subharmonic functions on $D - \Gamma$ which are continuous on $\bar{D}$, $\leqq 1$ on $\Gamma$, and $\leqq 0$ on $\partial D - \Gamma$.

Notice that the bound (2.5) includes Hadamard's three circle theorem as a special case. In the three circle case, the bound is precise, at least for a sequence of $\varepsilon$'s tending to zero, as is seen by considering the power functions $z^n$. In many other cases when $\Gamma$ is an arc, it can be shown that the bound is almost best possible. When $\Gamma$ is a set of zero capacity however, for example a finite set of points, the function $w(z)$ is identically zero on $D - \Gamma$; (2.5) then gives us no information at all, and the bound is clearly not best possible. The problem is that the method of proof does not use the fact that $f$ is single-valued. It uses merely the maximum principle and the fact that $|f|$ is single-valued on $D - \Gamma$. For example, the best possible bound when $D$ is the unit disc, $\Gamma$ is a discrete set of interior points $d_1, \cdots, d_k$, $\varepsilon = 0$ and $E = 1$ is $|B(z)|$, where $B$ is the Blaschke product

$$B(z) = \prod_{j=1}^{k} \frac{z - d_j}{1 - z\bar{d}_j}.$$

Now $|B^\alpha(z)|$ is still single-valued on $D - \Gamma$ and satisfies (2.3) and (2.4), yet $B^\alpha(z) \to 1$ as $\alpha \to 0$. Notice also that no bound such as (2.5) can hold unless $\Gamma$ is at least a set of unique analytic continuation, for the bound is zero when $\varepsilon$ is zero.

We assume for the rest of this section that $D$, $\Gamma$ and $\dot{\Gamma}$ are as mentioned in § 5. The requirement that $\Gamma$ be of positive capacity is of course exactly the condition that the harmonic measure $w(z)$ not be identically zero. We suppose now that $\dot{\Gamma}$ has a "*radial density*" $\delta$ on $\Gamma$; that is, for each point $t$ on $\Gamma$ there exists a point $d_i$ in $\dot{\Gamma}$ such that $|T_t(d_i)| \leqq \delta$, where $T_t$ denotes any linear fractional transformation mapping the disc onto itself and $t$ into the origin. We also suppose that $\Gamma$ has "*radial span*" $a$ about each of its points, $0 < a < 1$; that is, for every $t$ in $\Gamma$ the set $\{|T_t(z)| : z \in \Gamma\}$ contains the whole real segment $[0, a]$.

LEMMA 10. *Suppose $f$ is analytic on the unit disc $D$, continuous on $\bar{D}$, and*

$$|f(z)| \leqq E \quad \text{on } \partial D, \tag{8.1}$$

$$|f(z)| \leqq \varepsilon \quad \text{on } \dot{\Gamma}, \tag{8.2}$$

*where $\dot{\Gamma}$ has radial density $\delta$ on $\Gamma$ and $\Gamma$ has radial span $a$ about each of its points, $a > 2\delta$. Then*

$$|f(z)| \leqq \left(2\varepsilon + \frac{2E}{\sqrt{1-a^2}} a^{\sqrt{a/4\delta}}\right) \quad \text{on } \Gamma; \tag{8.3}$$

*hence, by Lemma* 1,

$$|f(z)| \leqq \left(2\varepsilon + \frac{2E}{\sqrt{1-a^2}} a^{\sqrt{a/4\delta}}\right)^{w(z)} E^{1-w(z)} \quad \text{on } D. \tag{8.4}$$

*Proof*. Let $\|f\|_\Gamma$ denote the uniform norm of $f$ on $\Gamma$. Let $t$ be any point on $\Gamma$ where $|f|$ assumes this maximum; since we could map $t$ into the origin by a linear fractional transformation, we may assume that $t = 0$. The geometrical hypotheses now assure us that there exists a point $d_i$ in $\dot{\Gamma}$ with $|d_i| \leqq \delta$ and that the set $|\Gamma_a| = \{|z| : z \in \Gamma, |z| \leqq a\}$ is all $[0, a]$.

We intuitively expect that the constraints (8.1) and (8.2) would give a worse bound for $|f(0)|$ if all the data points of $\dot{\Gamma}$ were rotated around to the positive real axis, maintaining their same moduli. This in fact can be proved; but we find it simpler to do a similar analysis for a polynomial approximation to $f$.

Let $f_n$ denote the $n$th partial sum of $f$'s Taylor series, $n$ to be chosen later. Using the fact that the uniform norm, and hence the $L_2$ norm, of $f$ is bounded by $E$ on $\partial D$, we obtain the truncation error bound as in (6.1),

$$|(f - f_n)(z)| \leqq \frac{E}{\sqrt{1-|z|^2}} |z|^{n+1} \leqq \frac{E}{\sqrt{1-a^2}} a^{n+1}, \qquad |z| < a. \tag{8.5}$$

Now $f_n$ has the factored form

$$f_n(z) = k(z - \xi_1) \cdots (z - \xi_n). \tag{8.6}$$

We rotate the zeros of $f_n$ around to the real axis; that is, we let

$$f_n^*(z) = k(z - |\xi_1|) \cdots (z - |\xi_n|). \tag{8.7}$$

Clearly

$$f_n^*(|z|) \leqq |f_n(z)| \leqq f_n^*(-|z|) \tag{8.8}$$

for all $z$. Thus

$$|f_n^*(0)| = |f(0)| = \|f\|_\Gamma, \tag{8.9}$$

$$|f_n^*(|d_i|)| \leqq |f_n(d_i)| \leqq \varepsilon + \frac{E\delta^{n+1}}{\sqrt{1-\delta^2}}, \tag{8.10}$$

$$\|f_n^*\|_{[0,a]} \leqq \|f_n\|_{\Gamma_a} \leqq \|f\|_\Gamma + \frac{Ea^{n+1}}{\sqrt{1-a^2}}, \tag{8.11}$$

where $\Gamma_a$ is the set $\{z : z \in \Gamma, |z| \in [0, a]\}$. By interpolation from the point $(|d_i|)$ we have

$$(8.12) \qquad |f_n^*(0)| \leqq \varepsilon + \frac{\delta^{n+1}}{\sqrt{1+\delta^2}} + \delta \|f_n^{*\prime}\|_{[0,a]}.$$

From Chebyshev theory [2, p. 7] we know that an $n$th order polynomial bounded by 1 on $[-1, +1]$ has its derivative bounded by $n^2$ there. After normalization, and using (8.9) and (8.11) we have

$$(8.13) \qquad \|f\|_\Gamma \leqq \varepsilon + \frac{E\delta^{n+1}}{\sqrt{1-\delta^2}} + n^2\left(\frac{2\delta}{a}\right)\left(\|f\|_\Gamma + \frac{Ea^{n+1}}{\sqrt{1-a^2}}\right).$$

Choose $n$ such that

$$(8.14) \qquad 0 \leqq n^2\left(\frac{2\delta}{a}\right) \leqq \frac{1}{2}, \qquad (n+1)^2\left(\frac{2\delta}{a}\right) > \frac{1}{2}.$$

Therefore we obtain

$$(8.15) \qquad \|f\|_\Gamma \leqq 2\varepsilon + 2\frac{E\delta^{n+1}}{\sqrt{1-\delta^2}} + \frac{Ea^{n+1}}{\sqrt{1-a^2}}.$$

As a rule we have $\delta \ll a$, $n$ large, and the second term in (8.15) is exceedingly small. However, for simplicity's sake we have merely assumed enough ($\delta < \frac{1}{2}a$) to insure that the second term is no greater than the third. Use of the inequality (8.14) for $n + 1$ then completes the proof.

LEMMA 11. *If $F$ is a polynomial of order $n$ satisfying*

$$(8.16) \qquad \|F\|_{\dot{\Gamma}} \leqq \varepsilon,$$

$$(8.17) \qquad \|F\|_{\partial \dot{D}} \equiv \|F\|_{\partial D} \leqq E,$$

*where these are the $l_2$ norms considered in the discretized problem, $\dot{\Gamma}$ containing $k$ points and $\partial\dot{D}$ containing $l > n$ equally spaced points, then $|f(z)|$ is bounded by* (8.3) *and* (8.4) *with $\varepsilon$ replaced by $\sqrt{k}\varepsilon$ and $E$ replaced by $\sqrt{n}E$.*

*Proof.* We just use the facts that the uniform norm on $k$ points is no more than $\sqrt{k}$ times greater than the normalized $l_2$ norm and that the uniform norm of an $n$th order polynomial on a circle is no more than $\sqrt{n}$ times greater than its $L_2$ norm.

The above transition from uniform to $l_2$ norm is quite crude, but it is completely adequate for our purposes. We may have introduced unnecessary factors of up to $\sqrt{n}$ and $\sqrt{k}$. However, we shall not have to take $n$ and $k$ very large since the truncation error (see (6.1)) goes down exponentially with $n$ and since the interpolation error (the second term in (8.3)) can be made to go down exponentially with $\sqrt{k}$.

We first show that the $\sqrt{n}$ term does not hurt us, thereby obtaining a stability estimate for problem (5.1), (5.2), and hence also for the discretization problem independently of $n$.

LEMMA 12. *If $f$ is analytic on $D$ and*

$$(8.18) \qquad \|f\|_{\dot{\Gamma}} \leqq \varepsilon,$$

$$(8.19) \qquad \|f\|_{\partial D} \leqq E,$$

*then*

$$|f(z)| \leqq \left[\frac{(.1\varepsilon)^{w(z)}}{\sqrt{1-|z|^2}} + (2\sqrt{k}\varepsilon)^{w(z)}\sqrt{c\log\left(\frac{E}{\varepsilon}\right)}^{1-w(z)} + \sqrt{c\log\left(\frac{E}{\varepsilon}\right)}\left(\frac{Ea^{\sqrt{a/4\delta}}}{\sqrt{1-a^2}}\right)^{w(z)}\right]E^{1-w(z)}, \tag{8.20}$$

*where c depends only on* $b = \max\{|z| : z \in \Gamma\}$.

*Proof*. We approximate $f$ by its $n$th order Taylor sum $f_n$, taking $n$ just large enough to make the truncation error $|f - f_n|$ less than $.1\varepsilon$ on $\Gamma$, as called for in (6.1). This can be accomplished by taking $n = c\log(E/\varepsilon)$, where $c$ depends only on $b$. Then using (6.3) to bound $|(f - f_n)(z)|$, and Lemma 11 to bound $|f_n(z)|$, we obtain (8.20) as desired.

We next show that the factor $\sqrt{k}$ does not hurt us. We make the additional assumptions that $\Gamma$ is an arc and that the data points are taken fairly evenly spaced on $\Gamma$ (say for example that their maximum spacing is less than twice their minimum spacing.) Their radial density $\delta$ on $\Gamma$ is then bounded by $c/k$, where $c$ depends only on $b$ and on the length of $\Gamma$. We take $k$ just large enough that the interpolation error on $\Gamma$ is less than $.1\varepsilon$,

$$\frac{Ea^{\sqrt{a/4\delta}}}{\sqrt{1-a^2}} \leqq \frac{E(a^{\sqrt{a/4c}})^{\sqrt{k}}}{\sqrt{1-a^2}} \leqq .1\varepsilon, \tag{8.21}$$

which can be accomplished by taking $k = c[\log(E/\varepsilon)]^2$. This choice of $k$ in (8.20) yields

$$|f(z)| \leqq \left[\frac{1}{\sqrt{1-|z|^2}} + c\log\left(\frac{E}{\varepsilon}\right)\right]\varepsilon^{w(z)}E^{1-w(z)}, \tag{8.22}$$

where $c$ depends only on $b$, on $a$, and on the length of $\Gamma$.

The above analysis requires that we not take $k$ "too large." As a practical consideration for the numerical use of SNAC though, the upper bound we have placed on $k$ is just not very critical. We could overshoot greatly on $k$, making extremely sure that the interpolation error bound (8.21) is satisfied, without the factor of $\sqrt{k}$ doing us much harm.

As a matter of fact, closer analysis will show that no upper bound on $k$ is necessary. We again assume that the data points are fairly evenly spaced on the arc $\Gamma$ (alternatively we could introduce a weighted $l_2$ norm in which the data points are weighted according to their spacing on $\Gamma$, but that would involve more complicated notation). We can suppose that $k > 2k_0$, where $k_0$ is the least integer such that (8.21) is satisfied. Let $N$ now be the integer such that $2k_0N < k < 2k_0(N + 1)$. Notice that $N \geqq 1$. We group the points of $\dot{\Gamma}$ into $2k_0$ groups of either $N$ or $N + 1$ successive points, as needed to make the grouping come out even. In each group let $d_j^*$ be the data point at which $f$ is minimum, thus picking out a subset $\Gamma^*$. Now by design the radial density of $\Gamma^*$ on $\Gamma$ is less than the $\delta_0$ necessary to insure that

(8.21) holds. Moreover, the $l_2$ norm of $f$ on $\Gamma^*$ is no greater than $\sqrt{2}$ times its $l_2$ norm on $\dot{\Gamma}$:

$$\frac{1}{2k_0}\sum_{d_j^*\in\Gamma^*}|f(d_j^*)|^2 \leqq \frac{1}{2k^0}\left(\frac{1}{N}\sum_{j=1}^{k}|g(d_j)|^2\right) \leqq \frac{N+1}{N}\left(\frac{1}{k}\sum_{j=1}^{k}|f(d_j)|^2\right).$$

Therefore inequality (8.22) holds as before.

LEMMA 13. *Let $f$ be as in Lemma* 12. *Let $\Gamma$ be an arc, let the points of $\dot{\Gamma}$ be "fairly evenly spaced" on $\Gamma$, and let $k$ be sufficiently large that* (8.21) *holds. Then $|f(z)|$ satisfies the bound* (8.22), *where $c$ depends only on $b$, on $a$, and on the length of* $\Gamma$.

*Proof of Lemma* 2. We note that the $l_2$ norm on $\dot{\Gamma}$ with the points being equally spaced tends to the $L_2$ norm on $\Gamma$ as we let $k \to \infty$. Hence (8.22) holds in the limit and the proof is completed.

**9. Comparison between $l_2$ and $l_\infty$ methods.** The linear programming method introduced by J. Douglas uses uniform norms rather than quadratic norms. That method may be paraphrased as follows: One must first reduce the original problem, much as we did in § 6, to a discrete problem in the form

$$\|Ax - h\|_\infty \leqq \varepsilon,$$

$$\|Bx\|_\infty \leqq E,$$

where $x$, $A$, $B$, $h$ are just as in § 7 and where $\|\cdot\|_\infty$ denotes the $l_\infty$ norm $\|y\|_\infty = \max_j |y_j|$ if $y$ has real components, or

$$\|y\|_\infty = \max_j \{|\mathrm{Re}\, y_j|, |\mathrm{Im}\, y_j|\}$$

if $y$ has complex components. One then takes as our approximation the vector $x^1$ which minimizes $\|Ax - h\|_\infty$ subject to the constraint $\|Bx\|_\infty \leqq E$. This minimization can be stated as a linear programming problem. Notice that our discrete Method 4 is just the exact $l_2$ analogy of the $l_\infty$ method.

The switch in norms is justifiable on two grounds. In the first place the stability estimates for the quadratic norm are usually not much worse than for the uniform norm. We saw in the a priori estimates for SNAC that the $l_2$ norm gave an added factor of only $\log(E/\varepsilon)$ at the data set $\Gamma$. This is typical of many of these problems where the data set is embedded deep inside the solution domain; see for example all the examples in [11]. The explanation, loosely, is that the solutions are very smooth there, and for smooth functions the uniform norm is not much greater than the quadratic norm. On the other hand the $l_2$ norm gave an added factor of $1/\sqrt{1-|z|^2}$ at the boundary. This also is characteristic of all the examples in [11]. As a practical matter, though, this boundary behavior does us little harm, for one usually will not be trying to do continuation close to the boundary, the Hölder continuity there being so poor. For example, accuracy $\varepsilon = 10^{-6}$ at the data set gives accuracy of only $\varepsilon^{w(z)} = 10^{-1}$ at a point $z$ sufficiently close to the boundary that $w(z) = \frac{1}{6}$.

In the second place there are great computational advantages to the $l_2$ methods. The least squares approach involves only solution of equations with an $(n + 1) \times (n + 1)$ complex matrix (or a $2(n + 1) \times 2(n + 1)$ real matrix). The $l_\infty$ approach, however, requires solution at each step of the linear programming algorithm of equations with a $4(l+k) \times 4(l+k)$ real matrix. Since in general $l + k \gg n$, *the least squares approach involves much less computation.* The number of equations does *not* go up as the number of data and constraint points goes up; thus the possibility of solving really large scale problems is at hand. Cannon and the author, in unpublished numerical trials in conjunction with [3], found that even trivially small linear programming problems, say 15 data points, 15 constraint points, and $n = 9$, soon filled the 32,000 word storage of a large computer. On the other hand, the sample output for SNAC presented in Tables 1–4

TABLE 1
$-\log_{10}$ *of observed error*, $n = 14$, $\varepsilon = 10^{-4}$, $E = 2.30$

| $y$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 4 | −0.4 | −0.5 | −0.2 | 0.3 | 1.0 | 1.7 | 1.2 | 1.0 | 0.0 |
| 3 | −0.3 | −0.2 | 0.1 | 0.7 | 1.7 | 3.0 | 2.9 | 2.1 | 0.7 |
| 2 | −0.1 | −0.1 | 0.3 | 1.0 | 2.3 | 4.3 | 4.6 | 3.4 | 1.2 |
| 1 | −0.1 | 0.0 | 0.4 | 1.2 | 2.5 | 4.8 | 5.6 | 4.5 | 1.7 |
| 0 | −0.1 | 0.1 | 0.5 | 1.3 | 2.5 | 4.5 | 5.8 | 4.8 | 1.9 |
| $x$ | −8. | −6. | −4. | −2. | 0. | 2. | 4. | 6. | 8. |

TABLE 2
$-\log_{10}$ *of theoretical error bound*, $n = 14$, $\varepsilon = 10^{-4}$, $E = 2.30$

| $y$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 4 | −1.5 | −1.3 | −1.0 | −0.6 | 0.0 | 0.6 | 0.8 | 0.2 | −1.0 |
| 3 | −1.3 | −1.0 | −0.7 | −0.2 | 0.6 | 1.5 | 1.9 | 1.1 | −0.5 |
| 2 | −1.3 | −0.8 | −0.5 | 0.1 | 1.0 | 2.4 | 3.2 | 2.0 | 0.0 |
| 1 | −1.3 | −0.7 | −0.4 | 0.3 | 1.4 | 3.0 | 3.6 | 2.9 | 0.4 |
| 0 | −1.3 | −0.7 | −0.2 | 0.4 | 1.5 | 3.2 | 3.7 | 3.2 | 0.5 |
| $x$ | −8. | −6. | −4. | −2. | 0. | 2. | 4. | 6. | 8. |

TABLE 3
$-\log_{10}$ *of the observed error*, $n = 24$, $\varepsilon = 10^{-4}$, $E = 2.30$

| $y$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 5 | −0.4 | −0.3 | −0.3 | −0.1 | 0.5 | 1.2 | 1.4 | 1.1 | −0.4 |
| 3 | −0.3 | −0.2 | −0.0 | 0.4 | 1.2 | 2.4 | 3.4 | 2.4 | 0.4 |
| 2 | −0.3 | −0.1 | 0.2 | 0.7 | 1.8 | 3.5 | 4.6 | 3.6 | 1.2 |
| 1 | −0.2 | −0.0 | 0.3 | 0.9 | 2.2 | 4.8 | 5.6 | 4.4 | 1.7 |
| 0 | −0.2 | −0.0 | 0.3 | 1.0 | 2.3 | 4.6 | 5.8 | 4.6 | 1.8 |
| $x$ | −8. | −6. | −4. | −2. | 0. | 2. | 4. | 6. | 8. |

TABLE 4
$-\log_{10}$ *of theoretical error bound*, $n = 24$, $\varepsilon = 10^{-4}$, $E = 2.30$

| $y$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 4 | −1.7 | −1.5 | −1.4 | −1.2 | −0.7 | −0.1 | 0.1 | −0.5 | −1.5 |
| 3 | −1.6 | −1.1 | −0.9 | −0.6 | 0.1 | 1.0 | 1.6 | 0.6 | −1.1 |
| 2 | −1.5 | −0.9 | −0.6 | −0.2 | 0.7 | 2.1 | 3.2 | 1.8 | −0.6 |
| 1 | −1.5 | −0.8 | −0.5 | −0.0 | 1.1 | 3.0 | 3.6 | 2.8 | −0.3 |
| 0 | −1.4 | −0.8 | −0.4 | 0.1 | 1.3 | 3.2 | 3.7 | 3.1 | −0.1 |
| $x$ | −8. | −6. | −4. | −2. | 0. | 2. | 4. | 6. | 8. |

involves problems with 20 data points, 40 constraint points, and $n$ up to 24. The $25 \times 25$ complex Hermitian matrix for the normal equations, solved by Cholesky elimination, then requires less than 1000 words, with plenty of storage left in core for much larger problems.

We turn finally to a description of the sample output.[2] All four tables involve continuation of the trial function

$$f_0(z) = [(z - 4)^2 + (9/2)^2]^{-1} + \sqrt{z - 4.5i} \tag{9.1}$$

on the rectangle $D = \{z : -8 < x < 8, \ -4 < y < 4\}$ from data on the circle $\Gamma = \{z : |z - 4| = 2\}$. $\dot{\partial D}$ consists of 40 constraint points, 11 evenly spaced across each of the four sides. $\dot{\Gamma}$ consists of 20 constraint points spaced fairly evenly on $\Gamma$ (but not exactly evenly, in fact not exactly symmetrically across the real axis). Approximate data $h$ was generated by adding to $f$ a "random" error vector with mean $l_2$ norm of $10^{-4}$. Method 1 was then applied with parameters $\varepsilon$, $E$ chosen as

$$\begin{aligned} \varepsilon &= \|f_0 - h\|_{\dot{\Gamma}} = 10^{-4}, \\ E &= \|f_0\|_{\dot{\partial D}} = 2.305, \end{aligned} \tag{9.2}$$

and with $n = 14$ and 24. Notice that no concession is made to truncation error in the choice of $\varepsilon$ and $E$, the tacit assumption being made that $n$ is sufficiently large that $f_0$ could be replaced in (9.2) by a polynomial approximate $F_0$ of order $n$ without $\varepsilon$ and $E$ being noticeably increased.

Tables 1 and 2 correspond to $n = 14$ and Tables 3 and 4 correspond to $n = 24$. In both cases $\varepsilon$, $E$ and, in fact, the "random" error vector are the same.

Tables 1 and 3 give values of $-\log_{10}$ of the observed error $|f_0(z) - G(z)|$, where $G$ is the polynomial computed by Method 1. Tables 2 and 4 give values of $-\log_{10}$ of the theoretical error bound $\mathcal{M}_1(\varepsilon, E, z)$; notice that here also no concession has been made to truncation error, $\mathcal{M}_1$ being a bound for $|F_0(z) - G(z)|$ rather than for $|f_0(z) - G(z)|$. In all four tables $z$ varies over a rectangular grid of 45 points on the upper half of $D$, with $x = \operatorname{Re} z$ varying through $-8, -6, \cdots, 6, 8$ and $y = \operatorname{Im} z$ varying through $0, 1, \cdots, 4$. The normalized variable $(z - 4)/13$ is used throughout the computational core of the program to avoid roundoff and overflow difficulties. The FORTRAN program was run on a CDC 6400 with approximately 14 decimal digit word length. When run previously on an 8 decimal digit machine, definite roundoff errors dominated in Table 4, but not in the other three tables.

The observed fit to the data and constraint for $G$ were

$$\|G - h\|_{\dot{\Gamma}} = 1.51 \times 10^{-5} \quad \text{and} \quad \|G\|_{\dot{\partial D}} = 2.55 \quad \text{for } n = 14,$$

$$\|G - h\|_{\dot{\Gamma}} = .998 \times 10^{-5} \quad \text{and} \quad \|G\|_{\dot{\partial D}} = 2.34 \quad \text{for } n = 24.$$

Notice that the error bound tends to remain approximately 10 to 50 times greater than the observed error and that $-\log$ of both does seem to drop off as one would expect the harmonic function $w$ to do. Notice, most importantly, that except where the boundary behavior dominates there is little change between $n = 14$ and $n = 24$.

---

[2] I wish to thank Dr. Len Schlessinger for programming and running all of these examples.

## REFERENCES

[1] R. Bellman, R. Kalaba and K. Lockett, *Numerical Inversion of the Laplace Transform*, American Elsevier, New York, 1966.

[2] S. Bernstein, *Leçons sur les proprietés extremales*, Gauthier-Villars, Paris, 1926.

[3] J. Cannon and K. Miller, *Some problems in numerical analytic continuation*, J. Soc. Indust. Appl. Math. Ser. B Numer. Anal., 2 (1965), pp. 87–96.

[4] G. F. Chew, *S-Matrix Theory of Strong Interactions*, Benjamin, New York, 1963.

[5] J. Douglas, *A numerical method for analytic continuation*, Boundary Problems in Differential Equations, Univ. of Wisconsin Press, Madison, 1960, pp. 179–189.

[6] N. Dunford and J. Schwartz, *Linear Operators*, vol. II, Interscience, New York, 1963, p. 1249.

[7] A. Gorny, *Contribution á l'étude des fonctions dérivables d'une variable réele*, Acta Math., 71 (1939), pp. 317–358.

[8] P. Henrici, *An algorithm for analytic continuation*, Mathematical Note 320, Boeing Scientific Research Laboratories, Seattle, 1963.

[9] F. John, *Continuous dependence on data for solutions of partial differential equations with a prescribed bound*, Comm. Pure Appl. Math., 13 (1960), pp. 551–585.

[10] J. Kelley, *General Topology*, Van Nostrand, Princeton, 1955.

[11] K. Miller, *Three circle theorems in partial differential equations and applications to improperly posed problems*, Arch. Rational Mech. Anal., 16 (1964), pp. 126–154.

[12] ———, *Eigenfunction expansion methods for problems with overspecified data*, Ann. Scuola Norm. Sup. Pisa, 19 (1965), pp. 397–405.

[13] Bela Sz.-Nagy, *Spectraldarstellung linearer Transformationen des Hilbertschen Raumes*, Springer-Verlag, Berlin, 1942.

[14] C. Pucci, *Sui problemi di Cauchy non "ben posti,"* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), 18 (1955), pp. 473–477.

[15] T. Regge and G. A. Viano, *The interpolation problem in the theory of complex angular momentum*, Nuovo Cimento, 25 (1962), pp. 709–722.

[16] A. N. Tikhonov, *On the solution of incorrectly put problems and the regularization method*, Dokl. Akad. Nauk SSSR, 151 (1963), pp. 501–504.

SIAM J. MATH. ANAL.
Vol. 1, No. 1, February 1970

# CERTAIN RESULTS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTIONS*

H. M. SRIVASTAVA†

**1. Introduction.** Making use of the familiar notation

$$(\lambda)_n = \lambda(\lambda + 1)(\lambda + 2) \cdots (\lambda + n - 1), \qquad n \geqq 1, \quad (\lambda)_0 = 1,$$

we write the power series definition of the generalized hypergeometric ${}_pF_q$ function in the form [6, p. 41]

$$ {}_pF_q\left[\begin{matrix} a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix} \; z\right] = \sum_{n=0}^{\infty} \frac{((a_j))_n}{((b_j))_n} \frac{z^n}{n!}, \tag{1.1}$$

where, for the sake of brevity,

$$((a_j))_n \equiv (a_1)_n(a_2)_n \cdots (a_p)_n, \text{ etc.},$$

it being assumed that there are always $p$ of the $a$ products and $q$ of the $b$ products. For the usual restrictions on the $b$ parameters and the conditions of convergence of the general series (1.1), see Slater [6, p. 45].

In a recent paper, Bhattacharya [2, p. 179] proved that if $\operatorname{Re}(s) > 0$ and $\alpha \neq -1, -2, -3, \cdots$, then

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(\frac{s}{s+1}\right)^{n+1} {}_2F_1\left[\begin{matrix} 1, n + \alpha + 1; \\ \alpha + 1; \end{matrix} \; \frac{1}{s+1}\right] \frac{\beta^n}{n!} \\ &\quad = \exp(\beta)\, {}_1F_1\left[\begin{matrix} \alpha; \\ \alpha + 1; \end{matrix} \; -\frac{\beta}{s+1}\right] \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} &\sum_{n=0}^{\infty} \left(\frac{1}{s+1}\right)^{n} {}_1F_1\left[\begin{matrix} n + \alpha + 1; \\ \alpha + 1; \end{matrix} \; \frac{\beta s}{s+1}\right] \\ &\quad = \left(\frac{s+1}{s}\right) \exp(\beta)\, {}_1F_1\left[\begin{matrix} \alpha; \\ \alpha + 1; \end{matrix} \; -\frac{\beta}{s+1}\right], \end{aligned} \tag{1.3}$$

where, in the notation of (1.1), ${}_2F_1[z]$ and ${}_1F_1[z]$ are Gauss's and Kummer's hypergeometric functions respectively.

Elsewhere [7] we have given rapid proofs of the formulas (1.2) and (1.3). The object of the present note is to derive their generalizations in the following elegant forms:

$$\begin{aligned} &\sum_{n=0}^{\infty} {}_{p+1}F_q\left[\begin{matrix} \lambda + n, a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix} \; x\right] \frac{z^n}{n!} \\ &\quad = \exp(z) \sum_{n=0}^{\infty} \frac{(\lambda)_n((a_j))_n}{((b_j))_n} \frac{x^n}{n!} \cdot {}_{p+1}F_{q+1}\left[\begin{matrix} \lambda + n, a_1 + n, \cdots, a_p + n; \\ \lambda, b_1 + n, \cdots, b_q + n; \end{matrix} \; xz\right] \end{aligned} \tag{1.4}$$

---

* Received by the editors January 16, 1969, and in revised form March 27, 1969.

† Department of Mathematics, West Virginia University, Morgantown, West Virginia 26506. Now at Department of Mathematics, University of Victoria, Victoria, British Columbia, Canada.

and

$$\sum_{n=0}^{\infty} {}_{p+1}F_q\left[\begin{matrix}\lambda+n, a_1, \cdots, a_p; \\ b_1, \cdots, b_q;\end{matrix} \; x\right] z^n$$

(1.5)
$$= (1-z)^{-1} \sum_{n=0}^{\infty} \frac{(\lambda)_n((a_j))_n}{n!((b_j))_n}\left(\frac{x}{1-z}\right)^n$$

$$\cdot {}_{p+2}F_{q+1}\left[\begin{matrix}\lambda-1, \lambda+n, a_1+n, \cdots, a_p+n; \\ \lambda, b_1+n, \cdots, b_q+n;\end{matrix} \; -\frac{xz}{1-z}\right],$$

where, for convergence, $|z| < 1$ and the nonnegative integers $p$ and $q$ satisfy $p \leqq q$, equality holding when $|x| < 1$.

In the next two sections, we exhibit the fact that the formulas (1.4) and (1.5) are consequences of the rather obvious result

(1.6)
$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_{p+1}F_q\left[\begin{matrix}\lambda+n, a_1, \cdots, a_p; \\ b_1, \cdots, b_q;\end{matrix} \; x\right] z^n = (1-z)^{-\lambda} {}_{p+1}F_q\left[\begin{matrix}\lambda, a_1, \cdots, a_p; \\ b_1, \cdots, b_q;\end{matrix} \; \frac{x}{1-z}\right],$$

which holds when $|z| < 1$ and the nonnegative integers $p$ and $q$ are constrained by the inequality $p < q$, or by $p = q$ with $|x| < 1$. The formula (1.6) corresponds to the limiting case $x \to 0$ of our recent bilinear generating relations (3.4) and (3.5) in [8, § 3] which, when $p = q = r = s = 1$, give us the results of Meixner ([5], see also [3, p. 84]) who obtained them two decades ago by transforming the Pochhammer contour integral associated with Gauss's hypergeometric function.

The results presented here find an interesting application in the evaluation of certain infinite integrals whose specialized forms arise frequently in a number of applied problems. With this point in view we cite, in the last section, some examples of the possible applications in statistics and certain areas of physics and engineering.

**2. Proof of (1.4).** In formula (1.6), replace $z$ by $z/t$, multiply both sides by $t^{-\lambda}$ and take their inverse Laplace transforms using the known result

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_{\tau-i\infty}^{\tau+i\infty} e^t t^{-z}\, dt, \qquad \operatorname{Re}(z) > 0.$$

On substituting the power series definition (1.1) on the right-hand side of (1.6), we thus find that, for $\operatorname{Re}(\lambda) > 0$,

(2.1)
$$I \equiv \sum_{n=0}^{\infty} {}_{p+1}F_q\left[\begin{matrix}\lambda+n, a_1, \cdots, a_p; \\ b_1, \cdots, b_q;\end{matrix} \; x\right]\frac{z^n}{n!} = \sum_{m=0}^{\infty} \frac{(\lambda)_m((a_j))_m}{((b_j))_m}\frac{x^m}{m!}\sum_{n=0}^{\infty}\frac{(\lambda+m)_n}{(\lambda)_n}\frac{z^n}{n!}.$$

The inner series in (2.1) is an ${}_1F_1$ which can be transformed by Kummer's first theorem [3, p. 253]

$$ {}_1F_1\left[\begin{matrix} a; \\ c; \end{matrix}\ z\right] = e^z {}_1F_1\left[\begin{matrix} c-a; \\ c; \end{matrix}\ -z\right], \tag{2.2} $$

and we have

$$ \begin{aligned} I &= \exp(z) \sum_{m=0}^{\infty} \frac{(\lambda)_m((a_j))_m}{((b_j))_m} \frac{x^m}{m!} \sum_{n=0}^{m} \frac{(-m)_n}{(\lambda)_n} \frac{(-z)^n}{n!} \\ &= \exp(z) \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(\lambda)_m((a_j))_m}{(\lambda)_n((b_j))_m} \frac{x^m}{(m-n)!} \frac{z^n}{n!}, \end{aligned} \tag{2.3} $$

since

$$ (-m)_n = \frac{(-1)^n m!}{(m-n)!}, \qquad m \geqq n \geqq 0. \tag{2.4} $$

In (2.3) we now write $m + n$ for $m$, make use of the identity

$$ (\lambda)_{m+n} = (\lambda)_m(\lambda + m)_n, \tag{2.5} $$

and we readily get

$$ \begin{aligned} I &= \exp(z) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\lambda)_{m+n}((a_j))_{m+n}}{(\lambda)_n((b_j))_{m+n}} \frac{x^m}{m!} \frac{(xz)^n}{n!} \\ &= \exp(z) \sum_{m=0}^{\infty} \frac{(\lambda)_m((a_j))_m}{((b_j))_m} \frac{x^m}{m!} \sum_{n=0}^{\infty} \frac{(\lambda + m)_n((a_j + m))_n}{(\lambda)_n((b_j + m))_n} \frac{(xz)^n}{n!}, \end{aligned} $$

whence the right-hand side of (1.4) follows immediately. The final result is then obtained by an appeal to the principle of analytic continuation.

**3. Proof of (1.5).** If in (2.1) we replace $z$ by $zt$ and take the Laplace transforms of both sides, using the well-known formula

$$ \Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt, \qquad \operatorname{Re}(z) > 0, $$

we obtain

$$ \begin{aligned} J &\equiv \sum_{n=0}^{\infty} {}_{p+1}F_q\left[\begin{matrix} \lambda + n, a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix}\ x\right] z^n \\ &= \sum_{m=0}^{\infty} \frac{(\lambda)_m((a_j))_m}{((b_j))_m} \frac{x^m}{m!} \sum_{n=0}^{\infty} \frac{(\lambda + m)_n(1)_n}{(\lambda)_n} \frac{z^n}{n!}. \end{aligned} \tag{3.1} $$

By Euler's transformation [6, (1.3.15), p. 10]

$$ {}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix}\ z\right] = (1 - z)^{c-a-b} {}_2F_1\left[\begin{matrix} c-a, c-b; \\ c; \end{matrix}\ z\right], \tag{3.2} $$

the inner series in (3.1) can be reduced to

$$(1-z)^{-m-1}\sum_{n=0}^{m}\frac{(-m)_n(\lambda-1)_n}{(\lambda)_n}\frac{z^n}{n!},$$

giving us, in view of (2.4),

$$J=(1-z)^{-1}\sum_{m=0}^{\infty}\frac{(\lambda)_m((a_j))_m}{((b_j))_m}\left(\frac{x}{1-z}\right)^m\sum_{n=0}^{m}\frac{(\lambda-1)_n}{(m-n)!(\lambda)_n}\frac{(-z)^n}{n!}. \tag{3.3}$$

On setting $m - n = k$ in (3.3), we find that

$$\begin{aligned}J&=(1-z)^{-1}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\lambda-1)_n(\lambda)_{n+k}((a_j))_{n+k}}{(\lambda)_n((b_j))_{n+k}n!k!}\left(\frac{x}{1-z}\right)^k\left(\frac{xz}{z-1}\right)^n\\&=(1-z)^{-1}\sum_{k=0}^{\infty}\frac{(\lambda)_k((a_j))_k}{((b_j))_k k!}\left(\frac{x}{1-z}\right)^k\sum_{n=0}^{\infty}\frac{(\lambda-1)_n(\lambda+k)_n((a_j+k))_n}{(\lambda)_n((b_j+k))_n n!}\left(\frac{xz}{z-1}\right)^n,\end{aligned} \tag{3.4}$$

by means of the identity (2.5), and (3.4) evidently leads us to the formula (1.5).

We remark in passing that it is not difficult to construct direct proofs of the formulas (1.4) and (1.5) without using (1.6).

**4. Particular cases.** When $p = q = 1, a_1 = \nu$ and $b_1 = \lambda$, the second member of (1.4) equals

$$\exp(z)\sum_{n=0}^{\infty}\frac{(\nu)_n x^n}{n!}\,{}_1F_1\left[\begin{matrix}\nu+n;\\ \lambda;\end{matrix}\ xz\right]. \tag{4.1}$$

On writing the power series for ${}_1F_1$, if we interchange the order of the double summation and make use of the binomial expansion

$$(1-z)^{-\lambda}=\sum_{n=0}^{\infty}\frac{(\lambda)_n z^n}{n!},\qquad |z|<1,$$

(4.1) assumes the form

$$\begin{aligned}&(1-x)^{-\nu}\exp(z)\,{}_1F_1\left[\begin{matrix}\nu;\\ \lambda;\end{matrix}\ \frac{xz}{1-x}\right]\\&=(1-x)^{-\nu}\exp\left(\frac{z}{1-x}\right){}_1F_1\left[\begin{matrix}\lambda-\nu;\\ \lambda;\end{matrix}\ -\frac{xz}{1-x}\right],\end{aligned}$$

by Kummer's theorem (2.2), and we finally have

$$\sum_{n=0}^{\infty}{}_2F_1\left[\begin{matrix}\nu,\lambda+n;\\ \lambda;\end{matrix}\ x\right]\frac{z^n}{n!}=(1-x)^{-\nu}\exp\left(\frac{z}{1-x}\right){}_1F_1\left[\begin{matrix}\lambda-\nu;\\ \lambda;\end{matrix}\ -\frac{xz}{1-x}\right], \tag{4.2}$$

provided $|x| < 1$ and $|z| < 1$. The formula (4.2) is indeed a generalization of (1.2), to which it would obviously reduce when the free parameter $\nu = 1$, and $\lambda$, $x$, $z$ are replaced by $\alpha + 1$, $1/(s + 1)$ and $\beta s/(s + 1)$, respectively.

On the other hand, if in (1.5) we let $p = q - 1 = 0$ and $b_1 = \lambda$, then the right-hand side simplifies to

$$(1-z)^{-1}\exp\left(\frac{x}{1-z}\right){}_1F_1\left[\begin{matrix}\lambda-1;\\ \lambda;\end{matrix}\ -\frac{xz}{1-z}\right]$$

and we get

$$(4.3)\quad \sum_{n=0}^{\infty} {}_1F_1\begin{bmatrix} \lambda + n; & \\ & x \\ \lambda; & \end{bmatrix} z^n = (1 - z)^{-1} \exp\left(\frac{x}{1 - z}\right) {}_1F_1\begin{bmatrix} \lambda - 1; & \\ & -\dfrac{xz}{1 - z} \\ \lambda; & \end{bmatrix},$$

$$|z| < 1,$$

which yields the formula (1.3) when $\lambda = \alpha + 1$, $x = \beta s/(s + 1)$ and $z = 1/(s + 1)$.

A number of particular cases of our formulas (1.4) and (1.5) can be deduced in this manner.

**5. Applications.** In this section we first cite an instance from statistics that gives rise to the problem of evaluation of infinite integrals of the type

$$(5.1)\quad I^{m,\alpha}_{\lambda,\mu}[a, b] = \int_{-\infty}^{\infty} \exp\{-(at + b)^2\} {}_1F_1\begin{bmatrix} \lambda; & \\ & \alpha t^2 \\ \mu; & \end{bmatrix} t^{2m+1}\, dt, \quad m = 0, 1, 2, \cdots.$$

We then evaluate this integral by using the results presented above and discuss its possible applications in certain areas of physics and engineering.

Let $X$, $Y$ be two random variables which have a bivariate normal distribution (see, e.g., [1, Chap. 2]). Let $\mu_X$ and $\mu_Y$ denote the respective means, $\sigma_X^2$ and $\sigma_Y^2$ the variances, and $\rho$ the correlation coefficient. Suppose that we wish to determine the probability of an event of the type $X \leqq aY + b$.

We have [1, p. 18]

$$(5.2)\quad \begin{aligned} P[X \leqq aY + b] &= \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{(1 - \rho^2)}} \int_{-\infty}^{\infty} \int_{-\infty}^{ay+b} \exp\left\{ -\frac{1}{2(1 - \rho^2)} \right. \\ &\quad \cdot \left. \left[ \frac{(x - \mu_X)^2}{\sigma_X^2} - \frac{2\rho(x - \mu_X)(y - \mu_Y)}{\sigma_X\sigma_Y} + \frac{(y - \mu_Y)^2}{\sigma_Y^2} \right] \right\} dx\, dy. \end{aligned}$$

On introducing a sequence of elementary substitutions we observe that

$$(5.3)\quad P[X \leqq aY + b] = \frac{\theta}{\pi} \int_{-\infty}^{\infty} \exp\{-(a^*t + b^*)^2\} \int_{-\infty}^{t} e^{-z^2}\, dz\, dt,$$

where $\theta$, $a^*$ and $b^*$ are constants depending on $a, b, \rho, \sigma_X, \sigma_Y, \mu_X$ and $\mu_Y$.

Now, since [4, p. 17]

$$(5.4)\quad \int_0^{\infty} e^{-z^2}\, dz = \frac{\sqrt{\pi}}{2}$$

and

$$(5.5)\quad \int_0^{t} e^{-z^2}\, dz = \operatorname{Erf}(t),$$

where Erf($t$) denotes the error function (see also [3, p. 266]), it follows at once that

$$(5.6)\quad P[X \leqq aY + b] = \frac{\theta}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\{-(a^*t + b^*)^2\} \left[ 1 + \frac{2}{\sqrt{\pi}} \operatorname{Erf}(t) \right] dt.$$

In view of the known relationship [4, p. 272]

$$\text{(5.7)} \qquad \operatorname{Erf}(z) = z\,{}_1F_1\left[\begin{matrix}\tfrac{1}{2}; \\ \tfrac{3}{2};\end{matrix}\ -z^2\right],$$

the second integral on the right-hand side of (5.6) assumes the same form as the general integral (5.1).

To evaluate (5.1), we expand the exponential function and then integrate term by term. We thus derive

$$\text{(5.8)}\ I_{\lambda,\mu}^{m,\alpha}[a,b] = -e^{-b^2}\sum_{n=0}^{\infty}\frac{\Gamma(m+n+\frac{3}{2})}{(2n+1)!}\frac{(2b)^{2n+1}}{a^{2m+2}}\,{}_2F_1\left[\begin{matrix}\lambda, m+n+\tfrac{3}{2}; \\ \mu;\end{matrix}\ \frac{\alpha}{a^2}\right],$$

provided $\operatorname{Re}(a^2) > \operatorname{Re}(\alpha)$, $m$ being a nonnegative integer.

In particular, when $m = 0$ and $\mu = 3/2$, we get

$$\text{(5.9)}\quad \begin{aligned} I_{\lambda,3/2}^{0,\alpha}[a,b] &= -\frac{b\sqrt{\pi}}{a^2}e^{-b^2}\sum_{n=0}^{\infty}{}_2F_1\left[\begin{matrix}\lambda, n+\tfrac{3}{2}; \\ \tfrac{3}{2};\end{matrix}\ \frac{\alpha}{a^2}\right]\frac{(b^2)^n}{n!} \\ &= -\frac{a^{2(\lambda-1)}b\sqrt{\pi}}{(a^2-\alpha)^{\lambda}}\exp\left(\frac{b^2\alpha}{a^2-\alpha}\right){}_1F_1\left[\begin{matrix}\tfrac{3}{2}-\lambda; \\ \tfrac{3}{2};\end{matrix}\ -\frac{b^2\alpha}{a^2-\alpha}\right],\end{aligned}$$

by using our formula (4.2); and on applying Kummer's theorem (2.2) once again, we find that

$$\text{(5.10)}\ \int_{-\infty}^{\infty}\exp\{-(at+b)^2\}\,{}_1F_1\left[\begin{matrix}\lambda; \\ \tfrac{3}{2};\end{matrix}\ \alpha t^2\right]t\,dt = -\frac{a^{2(\lambda-1)}b\sqrt{\pi}}{(a^2-\alpha)^{\lambda}}\,{}_1F_1\left[\begin{matrix}\lambda; \\ \tfrac{3}{2};\end{matrix}\ \frac{b^2\alpha}{a^2-\alpha}\right],$$

where, as before, $\operatorname{Re}(a^2) > \operatorname{Re}(\alpha)$.

The formulas (5.8) and (5.10) are indeed useful in various other statistical problems. Note also that by assigning special values to the free parameter $\lambda$, the hypergeometric ${}_1F_1$ function occurring on either side of (5.10) can be replaced by the Whittaker function $M_{k,m}(z)$, the Laguerre polynomial $L_n^{(\nu)}(z)$, the parabolic cylinder function $D_\nu(z)$, the Hermite function $H_\nu(z)$, the Bessel function $I_\nu(z)$, the incomplete gamma function $\gamma(\nu, z)$ and of course the error function $\operatorname{Erf}(z)$, and so on (see [3, pp. 268–269] and [4, pp. 271–274]). Since these special functions are of frequent occurrence in problems of physics and engineering, our formulas (5.8) and (5.10) might find applications in these areas as well.

For instance, in a number of boundary value problems of potential theory, like the Dirichlet problem for a parabolic cylinder discussed by Lebedev [4, pp. 293–296], we may be required to express a real-valued function $f(x)$, defined on the interval $(-\infty, \infty)$ and piecewise smooth on every finite subinterval $[-\delta, \delta]$, as a Fourier-Hermite series

$$\text{(5.11)} \qquad f(x) = \sum_{n=0}^{\infty} c_n H_n(x), \qquad -\infty < x < \infty,$$

where $H_n(x)$ denotes the Hermite polynomial of degree $n$. In order to determine the unknown coefficients $c_n$, we multiply both sides of (5.11) by $\exp(-x^2)H_m(x)$ and integrate term by term over the infinite interval $(-\infty, \infty)$. Making use of the orthogonality property [4, p. 66]

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = 2^n n! \sqrt{\pi}\, \delta_{mn}, \tag{5.12}$$

where $\delta_{mn}$ is the Kronecker delta, we readily have

$$c_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} f(x) H_n(x)\, dx, \qquad n = 0, 1, 2, \cdots. \tag{5.13}$$

If we let $f(x) = x^r \exp(Ax^2 + Bx + C)$, and recall the known formula [3, p. 267]

$$H_n(x) = 2^n x \Psi(\tfrac{1}{2} - \tfrac{1}{2}n, \tfrac{3}{2}; z^2), \tag{5.14}$$

where $\Psi(a, c; z)$ is the Tricomi function defined by [3, p. 257]

$$\begin{aligned} \Psi(a, c; z) &= \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\, {}_1F_1\left[\begin{matrix} a; \\ c; \end{matrix}\ z\right] \\ &+ \frac{\Gamma(c-1)}{\Gamma(a)} z^{1-c}\, {}_1F_1\left[\begin{matrix} a-c+1; \\ 2-c; \end{matrix}\ z\right], \end{aligned} \tag{5.15}$$

then substitutions in (5.13) will at once lead us to integrals of the type (5.1), provided $\mathrm{Re}(A) < 0$ and $r$ is an appropriate integer.

**Acknowledgment.** I should like to thank my colleague, Professor Vincent Uthoff, for fruitful discussions. Thanks are also due to the referee for kindly suggesting a number of improvements in the original version of the paper.

REFERENCES

[1] T. W. Anderson, *An Introduction to Multivariate Statistical Analysis*, John Wiley, New York, 1958.

[2] S. K. Bhattacharya, *A result on hypergeometric functions*, Proc. Cambridge Philos. Soc., 63 (1967), pp. 179–181.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, vol. I, McGraw-Hill, New York, 1953.

[4] N. N. Lebedev, *Special Functions and Their Applications*, Prentice-Hall, Englewood Cliffs, New Jersey, 1965.

[5] J. Meixner, *Umformung gewisser Reihen, deren Glieder Produkte hypergeometrischer Funktionen sind*, Deutsche Math., 6 (1942), pp. 341–349.

[6] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.

[7] H. M. Srivastava, *A note on certain formulas involving hypergeometric functions*, Glasnik Mat. Ser. III, 4 (1969), no. 24, to appear.

[8] ———, *An extension of the Hille–Hardy formula*, Math. Comp., 23 (1969), pp. 305–311. See also Notices Amer. Math. Soc., 15 (1968), pp. 634–635.

SIAM J. MATH. ANAL.
Vol. 1, No. 1, February 1970

# ON A PRIORI BOUNDS IN THE CAUCHY PROBLEM FOR ELLIPTIC EQUATIONS*

L. E. PAYNE†

**1. Introduction.** In an earlier paper, the author [6] gave a method for computing bounds for the solution of the Cauchy problem for the Laplace equation when the solution was restricted to lie in the class of uniformly bounded functions. Although, in theory, the method could be used to obtain bounds at arbitrary points in the domain $D$, the bounds were, in fact, impractical at points in $D$ which were not sufficiently close to the portion $\Sigma$ of the boundary on which Cauchy data were given. The author's method was generalized by Trytten [9] and applied to Cauchy problems for certain classes of second order quasi-linear elliptic equations, and further generalized by Schaefer [10] and applied to certain nonlinear elliptic systems. A somewhat different system was subsequently considered by Conlan and Trytten [2].

In the work of Trytten, Schaefer, and Conlan and Trytten the same difficulty in application at points away from the Cauchy surface still remains. The reason is that in all of these papers, bounds at distant points were dependent on bounds at intermediate points and the approximation error could accumulate quite rapidly. In this paper, we show how, by introducing an approximate class of auxiliary surfaces, one may include the point at which bounds are sought (at least in theory) in an initial estimate and thus avoid the error accumulation problem.

Let $D$ denote an open region in $R_n$. The boundary $\partial D$ of $D$ consists of a portion $\Sigma$ on which Cauchy data are to be prescribed and a remainder $\partial D - \bar{\Sigma}$ on which no data are given. For the purpose of this paper, we shall assume that $\bar{\Sigma}$ is a $C'$ surface and that $\partial D$ is a Lyapunov boundary.

Let $L$ denote the elliptic operator

$$Lu \equiv (a_{ij}u_{,i})_{,j} \tag{1.1}$$

where we have adopted the summation convention over repeated indices and the comma denotes partial differentiation. The problem which we shall consider is the following:

$$\begin{aligned} &Lu = \mathscr{F} \quad \text{in } D, \\ &u = g, \quad \frac{\partial u}{\partial x_i} = h_i \quad \text{on } \Sigma, \qquad i = 1, 2, \cdots, n, \end{aligned} \tag{1.2}$$

where $\mathscr{F}$, $g$, and $h_i$ are prescribed data. As pointed out in [6], [9] and [10], one must allow for error in measurement of the data; however, since this will be done as indicated in these earlier papers, we do not go into the question here, but rather restrict our consideration to the determination of the a priori inequalities themselves.

* Received by the editors February 10, 1969, and in revised form May 5, 1969.

† Cornell University, Ithaca, New York 14850. This work was carried out while the author was on sabbatical leave at the Eidg. Technische Hochschule, Zürich, and was supported in part by the National Science Foundation under Grant GP-8413.

One could, in fact, handle the nonlinear problems discussed in [2], [9], and [10] by the present method, but by doing so he would become unnecessarily involved in details and perhaps obscure the method. Hence we limit our attention to solutions of (1.2).

We shall assume that $L$ is symmetric and strongly elliptic, i.e., the matrix $a_{ij}$ is symmetric and there exists a positive constant $a_0$ such that for all vectors $\xi_i$, the inequality

$$a_{ij}\xi_i\xi_j \geqq a_0 \sum_{i=1}^{n} \xi_i^2 \tag{1.3}$$

holds at every point in $D$.

**2. Inequalities and error bounds.** Let us now define a set of (not necessarily closed) surfaces $f = \text{const}$. This set is to be so chosen that for each $\alpha$ satisfying

$$0 < \alpha \leqq 1, \tag{2.1}$$

the surface $f(x) = \alpha$ intersects $D$ and forms a closed region $D_\alpha$ whose boundary points consist only of points of $\Sigma$ and points on the surface $f = \text{const}$. In [6], $f$ was chosen for $n > 2$ as

$$f = \frac{1 + (r_0/r)^{n-2}}{1 - (r_0/r)^{n-2}}. \tag{2.2}$$

(We have made a slight alteration in order to make $f$ satisfy (2.1).) In (2.2), $r_0$ and $R$ are appropriately chosen constants.

We shall assume that $f(x)$ has continuous second derivatives in $\bar{D}_1$. We assume further that if $f$ satisfies (2.1), then

$$\beta \leqq \gamma \quad \text{implies} \quad D_\beta \subset D_\gamma, \qquad 0 < \beta \leqq \gamma \leqq 1, \tag{2.3}$$

$$|\text{grad} f| > \delta > 0 \quad \text{in } D_1, \tag{2.4}$$

$$Lf \leqq 0, \quad |Lf| \leqq \alpha_0 \delta^2 d \quad \text{in } D_1. \tag{2.5}$$

We shall also assume that the surfaces have been so chosen that for $\alpha$ satisfying (2.1), $D_\alpha$ has nonzero volume measure, but that $D_0$ has zero measure. Using this set of surfaces, we shall indicate how to obtain bounds for the solution $u$ at points in $D_\alpha$, $\alpha < 1$.

We approximate $u$ by a function $\phi$ which is assumed to have bounded second derivatives in $D$ and bounded first derivatives in $D \cup \Sigma$. We make no further assumptions on $\phi$ at this point. Let us now set

$$w = u - \phi \tag{2.6}$$

and let

$$F(\alpha) = \int_0^\alpha (\alpha - \eta) \left\{ \int_{D_\eta} [a_{ij} w_{,i} w_{,j} + wLw] \, dx \right\} d\eta + Q, \tag{2.7}$$

where $Q$ is given by

$$Q = k_0 \int_\Sigma w^2 \, ds + k_1 \int_\Sigma w_{,i} w_{,i} \, ds + k_2 \int_{D_1} (Lw)^2 \, dx. \tag{2.8}$$

Here $k_0, k_1$ and $k_2$ are explicit constants to be determined later. We shall show that as a function of $\alpha$, $F$ satisfies a differential inequality of the form

$$FF'' - (F')^2 \geqq -K_1FF' - K_2F^2 \tag{2.9}$$

for explicit constants $K_1$ and $K_2$. The solution of this differential inequality will then lead to the desired bounds.

A simple calculation gives

$$\begin{aligned} F'(\alpha) &= \int_0^\alpha \int_{D_\eta} [a_{ij}w,_iw,_j + wLw]\, dx\, d\eta, \\ F''(\alpha) &= \int_{D_\alpha} [a_{ij}w,_iw,_j + wLw]\, dx. \end{aligned} \tag{2.10}$$

We now write $F(\alpha)$ and $F'(\alpha)$ in more useful forms; e.g., using the divergence theorems, we have

$$\begin{aligned} F'(\alpha) &= \int_0^\alpha \left\{ \int_{S_\eta} \frac{a_{ij}w,_iwf,_j\, ds}{|\mathrm{grad}\, f|} + \int_{\Sigma_\eta} w\frac{\partial w}{\partial \nu}\, ds \right\} d\eta \\ &= \int_{D_\alpha} a_{ij}w,_iwf,_i\, dx + \int_0^\alpha \int_{\Sigma_\eta} w\frac{\partial w}{\partial \nu}\, ds\, d\eta. \end{aligned} \tag{2.11}$$

Here $S_\eta$ denotes the portion of the surface $f(x) = \eta$ and $\Sigma_\eta$ is the portion of $\Sigma$ which lies on the boundary of $D_\eta$. We have also made use of the fact that on $S_\eta$ the component $n_j$ of the unit normal is given by $f,_j|\mathrm{grad}\, f|^{-1}$ and have introduced the expression $\partial/\partial\nu$ for the conormal derivative $a_{ij}n_j(\partial/\partial x_i)$ on the boundary of $D_\alpha$. Using (2.11) we see that

$$\begin{aligned} F(\alpha) &= \int_0^\alpha F'(\eta)\, d\eta + Q = \int_0^\alpha \left\{ \int_{D_\eta} a_{ij}w,_if,_jw\, dx + \int_0^\eta \int_{\Sigma_\sigma} w\frac{\partial w}{\partial \nu}\, ds\, d\sigma \right\} d\eta + Q \\ &= \int_0^\alpha \left\{ \frac{1}{2}\int_{S_\eta} \frac{a_{ij}f,_if,_jw^2}{|\mathrm{grad}\, f|}\, ds + \frac{1}{2}\int_{\Sigma_\eta} a_{ij}f,_jn_iw^2\, ds - \int_{D_\eta} w^2L\, dx \right. \\ &\qquad \left. + \int_0^\eta \int_{\Sigma_\sigma} w\frac{\partial w}{\partial \nu}\, ds\, d\sigma \right\} d\eta + Q \\ &\geqq \frac{1}{2}\int_{D_\alpha} a_{ij}f,_if,_jw^2\, dx - \gamma_1 \int_\Sigma w^2\, ds - \gamma_2 \int_\Sigma w,_iw,_i\, ds + Q \end{aligned} \tag{2.12}$$

for computable constants $\gamma_1$ and $\gamma_2$. It is clear then that we can choose the $k_i$ in $Q$ such that

$$\frac{d+1}{2}\left[\int_{D_\alpha} \rho w^2\, dx + Q\right] \geqq F(\alpha) \geqq \frac{1}{2}\left[\int_{D_\alpha} \rho w^2\, dx + Q\right], \tag{2.13}$$

where

$$\rho = a_{ij}f,_if,_j. \tag{2.14}$$

Before proceeding with the derivation of (2.9), let us first prove some auxiliary lemmas.

LEMMA 1. *If $F(\alpha)$ is given by* (2.7), *then*

$$|F'| \leqq F' + K_2 F \tag{2.15}$$

*for a computable constant $K_2$.*

From (2.10) it easily follows that

$$|F'| \leqq F' + 2\left|\int_0^\alpha \int_{D_\eta} wLw\,dx\,d\eta\right|. \tag{2.16}$$

But by the arithmetic-geometric mean inequality, we have, for some positive constant $\beta$,

$$2\left|\int_0^\alpha \int_{D_\eta} wLw\,dx\,d\eta\right| \leqq \beta \int_0^\alpha \int_{D_\eta} w^2\,dx\,d\eta + \frac{1}{\beta}\int_0^\alpha \int_{D_\eta} [Lw]^2\,dx\,d\eta. \tag{2.17}$$

But Bramble and Payne [1] have derived the explicit a priori inequality

$$\int_{D_\eta} w^2\,dx \leqq k_4\left\{\int_{S_2} w^2\,ds + \int_{\Sigma_\eta} w^2\,ds\right\} + k_5 \int_{D_2} [Lw]^2\,dx, \tag{2.18}$$

from which it follows by integration and use of (2.13) that

$$\int_0^\alpha \int_{D_\eta} w^2\,dx\,d\eta \leqq k_4 \int_{D_\alpha} w^2|\operatorname{grad} f|\,dx + k_4\int_0^\alpha \int_{\Sigma_\eta} w^2\,ds\,d\eta + k_5\int_0^\alpha \int_{D_2} (Lw)^2\,dx\,d\eta$$

$$\leqq k_4\{\rho^{-1}|\operatorname{grad} f|\} \max_{x\in D_\alpha}\{2F(\alpha) - Q\} + k_4 \int_0^\alpha \int_{\Sigma_2} w^2\,ds\,d\eta \tag{2.18a}$$

$$+k_5 \int_0^\alpha \int_{D_\eta} (Lw)^2\,dx\,d\eta.$$

Inserting (2.18a) into (2.17) and choosing all arbitrary constants appropriately, one is led directly to (2.15).[1]

LEMMA 2. *If $F(\alpha)$ is given by* (2.7), *then*

$$\int_{D_\alpha} a_{ij}w_{,i}w_{,j}\,dx - 2\int_{D_\alpha} \rho^{-1}[a_{ij}w_{,i}f_{,j}]^2\,dx \geqq K_3F' - K_4F \tag{2.19}$$

*for computable constants $K_3$ and $K_4$.*

To establish the result, we consider the identity

$$\begin{aligned}\int_{D_\alpha} (\alpha - \eta)\rho^{-1}a_{kl}f_{,k}w_{,l}Lw\,dx &= \frac{1}{2}\int_0^\alpha \oint_{S_\eta+\Sigma_\eta} \{2a_{kl}f_{,k}w_{,l}a_{ij}w_{,j}n_i - a_{kl}f_{,k}n_l a_{ij}w_{,i}w_{,j}\}\rho^{-1}\,ds\,d\eta \\ &\quad -\frac{1}{2}\int_{D_\alpha} (\alpha - \eta)\left\{2\left[\frac{a_{kl}f_{,l}}{\rho}\right]_{,j} a_{ij}w_{,i}w_{,k} \right. \\ &\qquad \left. -\left[\frac{a_{kl}f_{,k}a_{ij}}{\rho}\right]_{,l} w_{,i}w_{,j}\right\}dx.\end{aligned} \tag{2.20}$$

---

[1] Note that from (1.3) and (2.4), it follows that $\max_{x\in D_\alpha}\{\rho^{-1}|\operatorname{grad} f|\} \leqq (a_0\delta)^{-1}$.

The expressions involving integrals over $S_\eta$ may be simplified as follows:

$$\int_0^\alpha \int_{S_\eta} \{2a_{kl}f_{,k}w_{,l}a_{ij}w_{,j}n_i - a_{kl}f_{,k}n_l a_{ij}w_{,i}w_{,j}\}\rho^{-1}\,ds\,d\eta = \int_{D_\alpha} [2(a_{ij}f_{,i}w_{,j})^2\rho^{-1} - a_{ij}w_{,i}w_{,j}]\,dx. \tag{2.21}$$

Solving (2.20) for this expression and using obvious inequalities we obtain at once

$$\begin{aligned}\int_{D_\alpha} a_{ij}w_{,i}w_{,j}\,dx - 2\int_{D_\alpha} \rho^{-1}[a_{ij}f_{,i}w_{,j}]^2\,dx &\geqq -k_7\int_\Sigma w_{,i}w_{,i} - k_8\int_{D_1}[Lw]^2\,dx \\ &\quad -k_9\int_{D_\alpha}(\alpha-\eta)a_{ij}w_{,i}w_{,j}\,dx.\end{aligned} \tag{2.22}$$

We have made use of the arithmetic-geometric mean inequality and have employed the ellipticity constant in computing the last term. As in the proof of the previous lemma, we note that since

$$\int_{D_\alpha}(\alpha-\eta)a_{ij}w_{,i}w_{,j}\,dx \leqq F' + \left|\int_{D_\alpha}(\alpha-\eta)wLw\,dx\right|, \tag{2.23}$$

we may again use the results of [1] to complete the proof of (2.19).

We now form (using (2.13))

$$\begin{aligned}FF'' - (F')^2 &\geqq \left\{\frac{1}{2}\int_{D_\alpha}\rho w^2\,dx\int_{D_\alpha}a_{ij}w_{,i}w_{,j}\,dx - \left(\int_{D_\alpha}a_{ij}w_{,i}f_{,j}w\,dx\right)^2\right\} \\ &\quad + F\int_{D_\alpha}wLw\,dx - 2|F'|\left|\int_0^\alpha\int_{\Sigma_\eta}w\frac{\partial w}{\partial \nu}\,ds\,d\eta\right|.\end{aligned} \tag{2.24}$$

In arriving at (2.24), we have dropped a number of nonnegative terms on the right. By use of the arithmetic-geometric mean inequality, the last two terms may be easily handled in the sense that

$$\begin{aligned}\left|\int_{D_\alpha}wLw\,dx\right| &\leqq K_5F, \\ \left|\int_0^\alpha\int_\Sigma w\frac{\partial w}{\partial\nu}\,ds\,d\eta\right| &\leqq K_6F\end{aligned} \tag{2.25}$$

for explicitly determinable constants $K_5$ and $K_6$. It is clear that, with a particular choice of $f$, one might want to combine terms in different ways, thus obtaining inequalities that are sharper than those we have indicated. For the term in braces, we have, by Schwarz's inequality,

$$\begin{aligned}\int_{D_\alpha}\rho w^2\,dx\int_{D_\alpha}a_{ij}w_{,i}w_{,j}\,dx - 2\left(\int_{D_\alpha}a_{ij}w_{,i}wf_{,j}\,dx\right)^2 \\ \geqq \int_{D_\alpha}\rho w^2\,dx\left\{\int_{D_\alpha}a_{ij}w_{,i}w_{,j}\,dx - 2\int_{D_\alpha}\rho^{-1}[a_{ij}w_{,i}f_{,j}]^2\,dx\right\}\end{aligned} \tag{2.26}$$

and Lemma 2 may be used to complete the bound. Thus, by (2.25) and (2.26), together with Lemmas 1 and 2, we are led to (2.9) with computable $K_1$ and $K_2$.

It is well known (see, e.g., Levine [5]) that a solution of (2.9) which vanishes for one value of $\alpha$ in the interval [0, 1] must vanish identically. Thus, without loss, we may assume that $F(\alpha) > 0$ for all $\alpha$ ($0 \leqq \alpha \leqq 1$). Then setting

$$\sigma = e^{-K_1\alpha}, \tag{2.27}$$

we find (regarding $F$ temporarily as a function of $\sigma$)

$$\frac{d^2}{d\sigma^2}\{\log[F\sigma^{-K_2/K_1^2}]\} \geqq 0, \tag{2.28}$$

from which it follows, by Jensen's inequality, that

$$F(\alpha)\sigma^{-K_2/K_1^2} \leqq [F(1)\sigma_1^{-K_2/K_1^2}]^{(1-\sigma)/(1-\sigma_1)}[F(0)]^{(\sigma-\sigma_1)/(1-\sigma_1)}, \tag{2.29}$$

where

$$\sigma_1 = e^{-K_1}$$

and $F$ is now regarded as a function of $\alpha$. We note, by (2.7), that $F(0) \equiv Q$, an expression involving only data terms.

The obvious method for choosing $\phi$ in order to make $F(0)$ ($\equiv Q$) small is the Rayleigh-Ritz method. The term $F(0)$ represents the error made in the approximation of the data. As indicated in [6], one must in this expression also allow for error in the measurement of the data. Since the approximation procedure was discussed in [6], we do not go into it here.

As has been noted in earlier papers (see, e.g., John [3], Pucci [8]), in order to make $F(\alpha)$ small for $0 \leqq \alpha < 1$, it is not sufficient to make $F(0)$ small. One must be sure that at the same time $F(1)$ does not become so large that the product is no longer small. To stabilize the problem we therefore assume that the solution $u$ lies in a class $\mathscr{M}$ defined by the condition that

$$\int_{D_1} u^2\,dx \leqq M^2 \tag{2.30}$$

for some prescribed $M$. Thus we first choose $\phi$ so as to make $Q$ small; then we compute $\int_{D_1} \phi^2\,dx$ and thus (using (2.30)) we can compute an $M_1$ such that

$$F(1)\sigma_1^{-K_2/K_1^2} \leqq M_1^2. \tag{2.31}$$

Insertion into (2.31) now gives

$$F(\alpha) \leqq \sigma^{-K_2/K_1^2}\{M_1^{2(1-\sigma)/(1-\sigma_1)}Q^{(\sigma-\sigma_1)/(1-\sigma_1)}\}. \tag{2.32}$$

If we think of $\phi$ as a solution $u_1$ of (1.2) corresponding to different data $\mathscr{F}_1$, $g_1$, and $\hat{h}_i$, then we have the following stability theorem.

THEOREM. *If* $u \in \mathscr{M}$ *and* $u_1 \in \mathscr{M}$ *are solutions to* (1.2) *corresponding to different data, then the difference satisfies the following continuous dependence inequality* ($\alpha < 1$)

$$\int_{D_\alpha} [u_1 - u]^2\,dx \leqq KM_1^{2\nu(\alpha)}Q^{1-\nu(\alpha)} \tag{2.33}$$

*for computable* $K$, $M_1$, *and* $\nu(\alpha)$ *with* $0 \leqq \nu(\alpha) \leqq 1$.

Here $Q$ is given by

$$(2.34)\quad Q = k_0 \int_\Sigma [g - g_1]^2 \, ds + k_1 \int_\Sigma (h_i - \hat{h}_i)(h_i - \hat{h}_i) \, dS + k_3 \int_{D_1} (\mathscr{F} - \mathscr{F}_1)^2 \, dx.$$

With a bound for $F(\alpha)$ we may use the Dirichlet problem estimates of Bramble and Payne [1] to compute pointwise bounds for $u$ in $D_\alpha$.

Of prime importance in applications is the choice of the surfaces $f = \alpha$. In [2], [6], [9] and [10] the particular choice (2.2) was made (with the exponent $n - 2$ replaced by $p$). With this choice the surfaces were a family of hyperspheres with origin outside of $D$. Thus the initial bounds were obtained only at points sufficiently close to $\Sigma$. These surfaces were particularly simple to work with but clearly such a choice might not always be a good one. One could easily envisage problems in which, for instance, ellipsoidal surfaces would be a more appropriate choice. A possible choice for $f$ in two dimensions would be the level curves of the first eigenfunction $\psi$ of the following Stekloff problem (assuming they are known):

$$(2.35)\quad \begin{aligned} L\psi &= 0 \quad \text{in } D, \\ \psi &= 0 \quad \text{on } \partial D - \Sigma, \\ \frac{\partial \psi}{\partial n} - \lambda\psi &= 0 \quad \text{on } \Sigma. \end{aligned}$$

The first eigenfunction is clearly positive in $D$, the level curves cannot intersect, and all of the level curves begin and terminate on $\Sigma$. If one knew $\psi$ and chose $f = \psi/\psi_m$ (where $\psi_m = \sup_{x \in \Sigma} \psi$), then he could obtain pointwise bounds at any point in the open region $D$ with this single choice for $f$. As is usual in such problems, the bounds become inapplicable as one approaches a point on the boundary.

**3. Concluding remarks.** With our bound (2.32) we could compute bounds for $\int_{D_\beta} a_{ij} w_{,i} w_{,j} \, dx$ for $\beta < \alpha$ as follows: We define a function $\eta(f)$ by

$$(3.1)\quad \eta = \begin{cases} 1, & f \leqq \beta, \\ \left(\dfrac{\alpha - f}{\alpha - \beta}\right), & \beta < f \leqq \alpha, \end{cases}$$

and consider the identity

$$(3.2)\quad \int_{D_\alpha} \eta^2 a_{ij} w_{,i} w_{,j} \, dx = \int_{\Sigma_\alpha} \eta^2 a_{ij} w_{,j} w n_j \, dS - \int_{D_\alpha} \eta^2 w L w \, dx - 2 \int_{D_\alpha} \eta a_{ij} \eta_{,i} w_{,j} w \, dx.$$

Clearly, we can compute a $\gamma$ such that

$$(3.3)\quad \int_{D_\alpha} \eta^2 a_{ij} w_{,i} w_{,j} \, dx \leqq \gamma F(\alpha) + \frac{1}{2} \int_{D_\alpha} \eta^2 a_{ij} w_{,i} w_{,j} \, dx.$$

One merely makes use of the arithmetic-geometric mean inequality and the right-hand side of (2.13). Thus from (3.3) we have (using (3.1)):

$$(3.4)\quad \begin{aligned} \int_{D_\beta} a_{ij} w_{,i} w_{,j} \, dx &\leqq \int_{D_\alpha} a_{ij} w_{,i} w_{,j} \eta^2 \, dx \\ &\leqq 2\gamma F(\alpha) \leqq 2\gamma \sigma^{K_1^2/K^2} M_1^{2[(1-\sigma)/(1-\sigma_1)]} Q^{(\sigma - \sigma)/(1-\sigma_1)}, \end{aligned}$$

the desired inequality.

In this paper, we have made strong use of the fact that $L$ is elliptic. It is clear from results of Payne and Sather [7], Knops and Payne [4] and Levine [5], that for certain special classes of differential equations and geometries, the ellipticity requirement can be relaxed. In all of the above cases, however, the problems were such that the surfaces $f =$ const. could be chosen as hyperplanes. We propose now to see to what extent this requirement may be relaxed or eliminated.

## REFERENCES


[1] J. H. Bramble and L. E. Payne, *Bounds for solutions of second order elliptic partial differential equations*, Contributions to Differential Equations, 1 (1963), pp. 95–127.

[2] J. Conlan and G. N. Trytten, *Pointwise bounds in the Cauchy problem for elliptic systems of partial differential equations*, Arch. Rational Mech. Anal., 22 (1966), pp. 143–152.

[3] F. John, *Continuous dependence on the data for solutions of partial differential equations with a prescribed bound*, Comm. Pure Appl. Math., 13 (1960), pp. 551–585.

[4] R. Knops and L. E. Payne, *Uniqueness in classical elastodynamics*, Arch. Rational Mech. Anal., 27 (1968), pp. 349–355.

[5] H. Levine, *Convexity and differential inequalities in Hilbert space*, Doctoral thesis, Cornell University, Ithaca, 1969.

[6] L. E. Payne, *Bounds in the Cauchy problem for the Laplace equation*, Arch. Rational Mech. Anal., 5 (1960), pp. 35–45.

[7] L. E. Payne and D. Sather, *On some improperly posed problems for quasilinear equations of mixed type*, Trans. Amer. Math. Soc., 128 (1967), pp. 135–141.

[8] C. Pucci, *Discussione del problema di Cauchy per le equazione di tipo ellitico*, Ann. Mat. Pura Appl., 46 (1958), pp. 131–153.

[9] G. N. Trytten, *Pointwise bounds for solutions in the Cauchy problem for elliptic equations*, Arch. Rational Mech. Anal., 13 (1963), pp. 222–244.

[10] P. W. Schaefer, *On the Cauchy problem for elliptic systems*, Ibid., 20 (1965), pp. 391–412.

SIAM J. MATH. ANAL.
Vol. 1, No. 1, February 1970

# APPLICATIONS OF A CLASS OF SINGULAR PARTIAL DIFFERENTIAL EQUATIONS TO GEGENBAUER SERIES WHICH CONVERGE TO ZERO*

DAVID COLTON†

**1. Introduction.** Expansions in series of hypergeometric polynomials arise frequently when the method of separation of variables is applied to a partial differential equation and the resulting solutions are superimposed in an attempt to solve certain boundary value problems. As was pointed out in [8] care must be used in this approach since the solutions obtained by such a procedure will not necessarily be unique due to the existence of nontrivial representations of zero. In particular this occurs in the study of the singular partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} = 0, \tag{1}$$

where $\nu < -1/2$. If $\nu \neq -1, -2, \cdots$ and interest is focused on solutions of (1) which are regular on the singular line $y = 0$, then separation of variables in polar coordinates $(r, \theta)$ leads to solutions of the form

$$r^n C_n^\nu(\cos\theta), \qquad n = 0, 1, 2, \cdots, \tag{2}$$

where $C_n^\nu$ denotes Gegenbauer's polynomial defined by the generating function

$$(1 - 2r\xi + r^2)^{-\nu} = \sum_{n=0}^{\infty} r^n C_n^\nu(\xi). \tag{3}$$

In view of the representation [8]

$$\lim_{N\to\infty} \sum_{n=0}^{N} (n + \nu) C_n^\nu(\cos\theta) = 0, \qquad \text{uniformly for } \theta \in [0, 2\pi], \tag{4}$$

it is not possible to solve uniquely the Dirichlet problem for the unit disc by a superposition of the solutions given in (2). (We are concerned here with the interior Dirichlet problem. This can be transformed to the exterior problem by means of a generalized Kelvin transformation [3].) The existence of expansions such as (4) leads to the conclusion that Dirichlet's problem for the singular equation (1) defined in domains containing a portion of the singular line $y = 0$ in its interior is in fact an improperly posed problem. Equation (1) (known as the generalized axially symmetric potential equation [7]) is far from being simply a pathological example. The case when $2\nu$ is a negative integer describes axially symmetric Stokes flow in $n = -2\nu + 2$ dimensions, whereas from a mathematical viewpoint, this equation is the simplest example of an elliptic equation with meromorphic coefficients. These remarks serve as motivation for a closer examination of representations of zero by series of Gegenbauer polynomials. The purpose of this paper is to initiate such an investigation through the utilization of some recent

* Received by the editors March 18, 1969, and in revised form May 14, 1969.

† Department of Mathematics, McGill University, Montréal, Québec, Canada, and Department of Mathematics, Indiana University, Bloomington, Indiana 47401.

developments in the analytic theory of partial differential equations. In particular if $2\nu \neq -1, -3, \cdots$, conditions will be given to assure that no nontrivial representation of zero exists, whereas if $2\nu = -1, -3, \cdots$, an upper bound to the number of representations of zero will be given. These results enable one to determine when a solution of the above mentioned Dirichlet problem is unique.

**2. A basic lemma and its application.** In the analysis that follows it is assumed that $\nu < -\frac{1}{2}$ since for $\nu \geqq -\frac{1}{2}$ the Dirichlet problem for (1) is well-posed [6] and no representation of zero of the form of equation (4) can exist; it is further assumed that the coefficients $a_n$ of the representation $\sum_{n=0}^{\infty} a_n C_n^{\nu}(\cos\theta) = 0$ are all real. We first require a few preliminary definitions.

DEFINITION 1. The $m$ nontrivial representations of zero on the interval $[0, 2\pi]$, $\sum_{n=0}^{\infty} a_{nj} C_n^{\nu}(\cos\theta)$, $j = 1, 2, \cdots, m$, are said to be *independent* if there exist constants $C_1, \cdots, C_m$ independent of $n$ such that $C_1 a_{n_1} + \cdots + C_m a_{nm} = 0$ for all $n$. Representations which are not independent are dependent.

DEFINITION 2. If $\sum_{n=0}^{\infty} a_n C_n^{\nu}(\cos\theta)$ is a nontrivial representation of zero on the interval $[0, 2\pi]$ then the series $\sum_{n=0}^{\infty} a_n C_n^{\nu}(1) z^n$ is called the associated power series of the representation.

Since $\sum_{n=0}^{\infty} a_n C_n^{\nu}(1)$ is convergent the associated power series will converge absolutely and uniformly on compact subsets of the disc $|z| < 1$ in the complex $z$-plane. In view of the fact that $C_n^{\nu}(1)$ does not equal zero for $2\nu \neq -1, -2, -3, \cdots$ (this follows from (3),) it is clear that if $2\nu \neq -1, -2, -3, \cdots$, then $m$ nontrivial representations of zero are dependent if and only if their associated power series converge to functions which are linearly dependent on the real interval $(-1, +1)$.

In the use of Gegenbauer series to investigate improperly posed problems for singular partial differential equations interest is focused primarily on those representations of zero which converge uniformly for $\theta \in [0, 2\pi]$. This is due to the fact that the solutions of the differential equation being considered are usually required to be continuous in the closure of their domain of definition (cf. [8]). The fact that the Gegenbauer polynomials satisfy

$$|C_n^{\nu}(\cos\theta)| = O(n^{\nu-1}), \qquad \text{uniformly for } \theta \in [0, 2\pi], \tag{5}$$

$$\frac{\partial}{\partial\theta} C_n^{\nu}(\cos\theta) = \sin\theta\, C_{n-1}^{\nu+1}(\cos\theta), \qquad n \geqq 1, \tag{6}$$

leads in a natural manner to the following definition.

DEFINITION 3. A nontrivial representation of zero on the interval $[0, 2\pi]$, $\sum_{n=0}^{\infty} a_n C_n^{\nu}(\cos\theta)$, is said to be of *class* $C^m$ if $\sum_{n=0}^{\infty} a_n n^{\nu+m-1}$ is absolutely convergent.

We observe that a nontrivial representation of zero of class $C^m$ where $m \geqq [-\nu + \frac{1}{2}]$ does not exist since in this case it would be possible to differentiate the series termwise and make use of (6) to conclude the existence of a nontrivial representation of zero of class $C^0$ for a value of $\nu$ greater than $-\frac{1}{2}$. As was previously mentioned, this is not possible. We are now in a position to prove our basic lemma.

BASIC LEMMA. *Assume* $2\nu \neq -1, -2, -3, \cdots$ *and let* $\sum_{n=0}^{\infty} a_n C_n^{\nu}(\cos\theta)$ *be a nontrivial representation of zero on the interval* $[0, 2\pi]$ *which is of class* $C^1$. *Then the* *associated power series* $\sum_{n=0}^{\infty} a_n C_n^\nu(1) z^n$ *is singular at either* $z = +1, z = -1$, *or both, and nowhere else on the circle* $|z| = 1$.

*Proof.* Consider the function

$$u(r, \theta) = \sum_{n=0}^{\infty} a_n r^n C_n^\nu(\cos\theta). \tag{7}$$

Since $\sum_{n=0}^{\infty} a_n n^\nu$ is absolutely convergent it is seen [6] that the series (7) converges uniformly on compact subsets of the unit disc to a solution of (1). Equations (5) and (6) furthermore show that the first partial derivatives of $u(r, \theta)$ are uniformly continuous in the closed disc $r \leqq 1$, $0 \leqq \theta \leqq 2\pi$. Since $u(1, \theta) = 0$ it is possible [3] to analytically continue $u(r, \theta)$ across the unit circle $r = 1$ provided $\theta \neq 0, \pi$, i.e., for all points on the unit circle not lying on the singular line $y = 0$. It is known from [1] and [2] that for $2\nu \neq -1, -2, -3, \cdots$, the associated power series $\sum_{n=0}^{\infty} a_n C_n^\nu(1) z^n$ is singular at $z = e^{i\theta}$ if and only if the solution of (1) defined by (7) is singular at $(1, \theta)$. Since (7) is analytic at all points $(1, \theta) \neq (1, 0)$ or $(1, \pi)$, it is possible to conclude that the only possible singular points of the associated power series are at $z = \pm 1$. If neither of these points is a singular point then the associated power series has no singularities on the unit circle in the complex $z$-plane and hence converges for $|z| < 1 + \delta$ where $\delta > 0$. This implies $\lim_{n\to\infty} |a_n|^{1/n} < 1$, i.e., $\sum_{n=0}^{\infty} a_n C_n^\nu(\cos\theta)$ is a nontrivial representation of zero of class $C^m$ where $m > [-\nu + \frac{1}{2}]$. As was observed previously this is impossible and hence the associated power series must be singular at either $z = +1, z = -1$, or both.

From the classical results on the relationship between the coefficients of a power series and the location of singular points on its circle of convergence, many theorems can now be given. Two typical examples of such results are given below.

THEOREM 1. *Assume that* $2\nu \neq -1, -2, -3, \cdots$. *Then there exists no nontrivial representation of zero which is of class* $C^1$ *and of the form*

$$\sum_{n=0}^{\infty} a_n C_n^\nu(\cos\theta) = 0, \qquad \theta \in [0, 2\pi],$$

*where* $a_n = 0$ *except when* $n$ *belongs to a sequence* $n_k$ *such that* $n_{k+1} > (1 + \delta) n_k$, $\delta > 0$.

*Proof.* Hadamard's gap theorem shows that the circle $|z| = 1$ is a natural boundary for the associated power series and the result follows by the basic lemma.

THEOREM 2. *Assume that* $2\nu \neq -1, -2, -3, \cdots$; *then there exists no nontrivial representation of zero which is of class* $C^1$ *and of the form*

$$\sum_{n=0}^{\infty} a_n C_{mn}^\nu(\cos\theta) = 0, \qquad \theta \in [0, 2\pi],$$

*where* $m$ *is an integer greater than or equal to three.*

*Proof.* The basic lemma and the fact that if the power series $\sum_{n=0}^{\infty} a_n C_{mn}^\nu(1) z^{mn}$ has a singularity at $z = +1$ or $z = -1$, then a singularity will also exist at $z = e^{2\pi i/m}$ or $z = e^{\pi i/m}$.

**3. The case when** $2\nu = -1, -2, -3, \cdots$. As was pointed out in the introduction, the case when $2\nu$ is a negative integer is of particular interest since (1) then describes axially symmetric Stokes flow in $n = (-2\nu + 2)$-dimensional space.

The existence of nontrivial representations of zero by Gegenbauer polynomials leads to the conclusion that the Dirichlet problem for the unit disc is an improperly posed problem. If $\nu = -1, -2, \cdots$ and interest is focused on solutions of (1) which are analytic functions of $x$ and $y^2$ in a region containing the singular line $y = 0$, then separation of variables in polar coordinates leads to solutions of the form

$$r^n P_n^{(\nu - 1/2, \nu - 1/2)}(\cos\theta), \qquad n = 0, 1, \cdots, \tag{8}$$

where $P_n^{(\alpha,\beta)}$ denotes Jacobi's polynomial. For $\alpha = \beta = \nu - \frac{1}{2}$ these are essentially renormalized Gegenbauer polynomials (note that from (3), for $\nu = -1, -2, -3, \cdots, C_n^\nu(\xi) \equiv 0$ for $n > -2\nu$) and by using similar techniques theorems analogous to those obtained in § 2 can be derived for these polynomials. If, however, instead of requiring solutions to be *even* analytic functions with respect to $y$, it is asked that they be *odd*, then it can be shown [7] that any solution $u(x, y)$ of (1) which is analytic in a neighborhood of the singular line $y = 0$ must be of the form

$$u(x, y) = y^{1-2\nu} u^+(x, y), \tag{9}$$

where $u^+(x, y)$ is a solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2 - 2\nu}{y}\frac{\partial u}{\partial y} = 0. \tag{10}$$

Hence if $u(x, y)$ vanishes on the boundary of a domain $D$ containing a portion of the singular line in its interior, then $u(x, y)$ vanishes on the boundary of $D \cap \{(x, y)| y > 0\}$ and hence from the maximum principle for elliptic partial differential equations [3], $u(x, y)$ is identically zero if $u(x, y) \in C^2(D) \cap C^0(\bar{D})$. Using the results of Parter [6] it can be shown that there exists a solution $u(x, y)$ to (1) such that $u(x, y) = y^{1-2\nu} f(x, y)$ on the boundary of a domain $D$ symmetric with respect to the axis $y = 0$, where $f(x, y) = f(x, -y)$ is a prescribed function continuous in the closure $\bar{D}$ of $D$. Thus Dirichlet's problem for (1) can be made well posed in the case $\nu = -1, -2, -3, \cdots$, and for domains $D$ containing a portion of the singular line in its interior. We therefore turn our attention to the case when $2\nu$ is a negative odd integer.

THEOREM 3. *Assume $2\nu$ is a negative odd integer. Then there exist at most $-2\nu - 1$ independent nontrivial representations of zero which are of class $C^0$.*

*Proof.* Suppose there exist $-2\nu$ independent nontrivial representations of zero

$$\sum_{n=0}^{\infty} a_{nj} C_n^\nu(\cos\theta), \qquad j = 1, 2, \cdots, -2\nu, \tag{11}$$

and consider the following corresponding solutions of (1) in the unit disc, $\Omega = \{(x, y)| x^2 + y^2 < 1\}$:

$$\sum_{n=0}^{\infty} a_{nj} r^n C_n^\nu(\cos\theta), \qquad j = 1, 2, \cdots, -2\nu. \tag{12}$$

A linear combination of these solutions gives a solution $u(r, \theta)$ to (1) of the form

$$u(r, \theta) = \sum_{n=-2\nu-1}^{\infty} b_n r^n C_n^\nu(\cos \theta) \tag{13}$$

such that $u(1, \theta) = 0$ for $\theta \in [0, 2\pi]$ and $u(r, \theta) \in C^0(\overline{\Omega}) \cap C^2(\Omega)$. Since for $2\nu = -1, -3, \cdots, C_n^\nu(1) = 0$ for $n \geqq -2\nu + 1$, whereas $C_n^\nu(1) \neq 0$ for $n < -2\nu + 1$ (this follows from (3)) we have

$$u(1, 0) = 0 = b_{-2\nu-1}C^\nu_{-2\nu-1}(1) + b_{-2\nu}C^\nu_{-2\nu}(1), \tag{14}$$

$$\begin{aligned} u(1, \pi) = 0 &= b_{-2\nu-1}C^\nu_{-2\nu-1}(-1) + b_{-2\nu}C^\nu_{-2\nu}(-1), \\ &= b_{-2\nu-1}C^\nu_{-2\nu-1}(1) - b_{-2\nu}C^\nu_{-2\nu}(1). \end{aligned} \tag{15}$$

Equations (14) and (15) now imply that $b_{-2\nu-1} = b_{-2\nu} = 0$, i.e., along the singular line $y = 0$, $u(r, \theta) = 0$. Hence $u(r, \theta)$ is a solution of (1) in $\Omega^+ = \Omega \cap \{(x, y)|y > 0\}$, vanishes on the boundary of $\Omega^+$, and $u(r, \theta) \in C^0(\overline{\Omega^+}) \cap C^2(\Omega^+)$. By the maximum principle for elliptic partial differential equations it is seen that $u(r, \theta) \equiv 0$ in $\Omega^+$ and hence in $\Omega$. By noting that, for fixed $\theta$, (13) is a power series in $r$ and that $C_n^\nu(\cos \theta)$ is a polynomial of degree $n$ in $\cos \theta$ it is possible to conclude that $b_n = 0$ for $n = 0, 1, 2, \cdots$. Hence the representations given in (11) are dependent and there cannot exist more than $-2\nu - 1$ nontrivial representations of zero of class $C^0$.

The methods used in Theorem 3 can be immediately adapted to show that if $2\nu$ is a negative odd integer, then for a given domain $D$ containing a portion of the singular line in its interior there exist at most $-2\nu - 1$ solutions of (1) which are linearly independent in $D$ and vanish on the boundary of $D$.

By using the methods developed in [1] to examine the analytic theory of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} + \frac{2\mu}{x}\frac{\partial u}{\partial x} = 0 \tag{16}$$

it is possible to derive results analogous to those obtained in §§ 2 and 3 for series of Jacobi polynomials which converge to zero.

For $\nu > 0$ the relationship between the singularities of (7) and the associated power series was given in [4] and [5]. For such values of $\nu$ however there do not exist any nontrivial representations of zero of class $C^0$ (see [6]) and hence such results are not applicable to our investigation.

## REFERENCES

[1] D. Colton, *On the analytic theory of a class of singular partial differential equations*, Proc. Symposium on Analytic Methods in Mathematical Physics, R. P. Gilbert, R. G. Newton, eds., Gordon and Breach, New York, to appear.

[2] A. Erdélyi, *Singularities of generalized axially symmetric potentials*, Comm. Pure Appl. Math., 9 (1956), pp. 403–414.

[3] P. Garabedian, *Partial Differential Equations*, John Wiley, New York, 1964.

[4] R. P. Gilbert, *On the singularities of generalized axially symmetric potentials*, Arch. Rational Mech. Anal., 6 (1960), pp. 171–176.

[5] P. Henrici, *On the domain of regularity of generalized axially symmetric potentials*, Proc. Amer. Math. Soc., 8 (1957), pp. 29–31.

[6] S. PARTER, *On the existence and uniqueness of symmetric axially symmetric potentials*, Arch. Rational Mech. Anal., 20 (1965), pp. 279–286.

[7] A. WEINSTEIN, *Generalized axially symmetric potential theory*, Bull. Amer. Math. Soc., 59 (1953), pp. 20–38.

[8] J. WIMP AND D. COLTON, *Jacobi series which converges to zero, with applications to a class of singular partial differential equations*, Proc. Cambridge Philos. Soc., 65 (1969), pp. 101–106.

SIAM J. MATH. ANAL.
Vol. 1, No. 1, February 1970

# THE CONSTRUCTION OF SOLUTIONS FOR BOUNDARY VALUE PROBLEMS BY FUNCTION THEORETIC METHODS*

R. P. GILBERT†

**1. Introduction.** In this paper we develop a *method of ascent* by which one may obtain a general representation formula for solutions of the differential equation of $n$ variables

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + a(r^2) \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} + c(r^2)u = 0, \tag{1.1}$$

with $r^2 = x_1^2 + \cdots + x_n^2$, in terms of a representation formula for solutions of the differential equation of 2 variables

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + a(r^2)\left(x_1 \frac{\partial u}{\partial x_1} + x_2 \frac{\partial u}{\partial x_2}\right) + c(r^2)u = 0. \tag{1.2}$$

Indeed, we find that all regular solutions of (1.1) (about the origin) may be represented in the form

$$u(\mathbf{r}) = h(\mathbf{r}) + \int_0^1 \sigma^{n-1} G(r; 1 - \sigma^2) h(\mathbf{r}\sigma^2)\, d\sigma; \tag{1.3}$$

here $h(\mathbf{r})$ is an arbitrary harmonic function, and

$$G(r, 1 - \sigma^2) \equiv -rR_1(r\sigma^2, 0; r, r), \tag{1.4}$$

where $R(z, z^*; \zeta, \zeta^*)$ is the Riemann function for (1.2), with $z = x_1 + ix_2$, $z^* = x_1 - ix_2$.

The formula (1.3) is a natural extension of the integral formulas of S. Bergman and I. N. Vekua for $n = 2$ variables. Indeed, for $n = 2$ the $G$-function is an integral transform of Bergman's $E$-function. Also, by certain manipulations with Vekua's representations one may obtain our formula (1.3) when $n = 2$. However, our (1.3) is actually new even for the case of two variables. We present numerous examples to illustrate its use. In addition, a reduction of the Dirichlet problem to a corresponding Fredholm integral equation is given via (1.3) by equations (4.41), (4.42). It is assumed here that $c(r^2) \leqq 0$ for $\mathbf{r}$ in the closure of the particular domain at hand.

**2. Elliptic equations with analytic coefficients of two variables.** As a first step in obtaining an approximate method for solving boundary value problems associated with the real, analytic, partial differential equation

$$\mathbf{e}[u] \equiv \Delta u + \alpha(x, y)u_x + \beta(x, y)u_y + \gamma(x, y)u = 0, \tag{2.1}$$

we first seek suitable integral representations of a fairly wide class of solutions.

* Received by the editors April 25, 1969.

† Department of Mathematics, Indiana University, Bloomington, Indiana 47401. This work was supported in part by the Air Force Office of Scientific Research through AFOSR Grant 1206-67.

Two apparently different approaches to obtaining such representation formulas may be found in the function theoretic methods of S. Bergman [1] and I. N. Vekua [10]. However, as we shall point out in this section, their representation formulas are essentially the same when viewed in a somewhat more general context. Indeed, it is this observation which permits us to make a generalization of their results to higher dimensional problems in a later section.

We assume, as do Bergman and Vekua, that (2.1) has coefficients which permit an analytic extension to a bi-cylinder $D \times D^*$ in terms of the *independent* complex variables $z = x + iy$, $z^* = x - iy$, where $x$ and $y$ are complex. Equation (2.1) then takes on a formally hyperbolic appearance, namely,

$$U_{zz^*} + a(z, z^*)U_z + b(z, z^*)U_{z^*} + c(z, z^*)U = 0, \tag{2.2}$$

where $b = \bar{a}$, $c = \gamma/4$, $a = (\alpha - i\beta)/4$. Vekua introduces the idea of a complex, Riemann function[1] for (2.2), $R(\zeta, \zeta^*; z, z^*)$, and obtains in terms of this function the following representation for the class of real solutions of (2.1) [10, p. 123, (25.2)], which are real analytic in $D$:

$$u(x, y) = \operatorname{Re}\left\{H_0(z, \bar{z})\varphi(z) + \int_0 H(z, \bar{z}, t)\varphi(t)\, dt\right\}; \tag{2.3}$$

here $H_0(z, \bar{z}) \equiv R(z, 0; z, \bar{z})$, $H(z, \bar{z}, t) \equiv -\partial R(t, 0; z, \bar{z})/\partial t + b(t, 0)R(t, 0; z, \bar{z})$, $\varphi(z)$ is an arbitrary holomorphic function in $D$, and $\bar{z}$ is the restriction of $z^*$ to real values of $x$ and $y$. Furthermore, $\varphi(z)$ may be normalized by setting $\varphi(0) = \overline{\varphi(0)}$.

Bergman, on the other hand, also has given a representation for the family of real analytic solutions in $D$, [1, p. 23]. His representation for these solutions of (2.1) has the form

$$\begin{aligned} u(x, y) = \operatorname{Re}\Bigg\{ &\exp\left(-\int_0^{\bar{z}} a(z, t)\, dt + h(z)\right) \\ &\cdot\left(\varphi(z) + \sum_{n \geq 1} \frac{Q^{(n)}(z, \bar{z})}{2^{2n}B(n, n + 1)} \int_0^z (z - \xi)^{n-1}\varphi(\xi)\, d\xi\right)\Bigg\}, \end{aligned} \tag{2.4}$$

where he defines the coefficients $Q^{(n)}(z, z^*) \equiv \int_0^{z^*} P^{(2n)}(z, t)\, dt$, by the recursion formulas

$$\begin{aligned} P^{(2)} &= -2F(z, z^*) \equiv 2(a_z + ab - c), \\ (2n + 1)P^{(2n+2)} &= -2\left[P_z^{(2n)} + DP^{(2n)} + F\int^{z^*} P^{(2n)}\, dz^*\right]; \end{aligned} \tag{2.5}$$

[1] See also Garabedian [4] concerning this.

$D = h'(z) - \int_0^{z^*} a_z\, dz^* + b$, $h(z)$ is an arbitrary analytic function in $D$, and $B(p, q)$ is the beta function.[2]

It is an interesting fact that by an appropriate choice of $h(z)$ both these representations can be seen to associate the same real solution $u(x, y)$ with a given but arbitrary analytic function $\varphi(z)$. However, this identity evidently has not been observed and hence has not been exploited to obtain any new representation formulas for solution. It is this very observation which we use in this paper to generalize and extend the work of Bergman and Vekua. Indeed, this observation gives us the clue for the extension to dimension greater than two.

LEMMA 1. *If $h(z) \equiv 0$, then the Bergman* (2.4) *and the Vekua* (2.3) *representations associate the same real analytic solution with a given holomorphic function $\varphi(z)$.*

*Proof*. The complex Riemann function satisfies the characteristic conditions, $R(\zeta, \zeta^*; \zeta, \zeta^*) = 1$,

$$(2.6)\quad R(z, \zeta^*; z, z^*) = \exp\left(\int_{z^*}^{\zeta^*} a(z, t)\, dt\right),\quad R(\zeta, z^*; z, z^*) = \exp\left(\int_{z}^{\zeta} b(t, z^*)\, dt\right);$$

hence, $H_0(z, \bar{z}) = \exp\left(-\int_0^{\bar{z}} a(z, t)\, dt\right)$. Next, since Bergman's complex solution $U(z, z^*)$, satisfies the Goursat data

$$(2.7)\quad \begin{aligned} U(z, 0) &= \varphi(z), \\ U(0, z^*) &= \varphi(0) \exp\left(-\int_0^{z^*} a(0, t)\, dt\right), \end{aligned}$$

we may represent this solution as (see [5, p. 129]; see also [1] and [10])

$$(2.8)\quad U(z, z^*) = R(0, 0; z, z^*)\varphi(0) + \int_0^z R(t, 0; z, z^*)[\varphi'(t) + \varphi(t)b(t, 0)]\, dt,$$

which if we integrate by parts may be recognized as the complex equivalent of (2.3).

The previous identity suggests a new method for approximating solutions for the interior Dirichlet problem for a domain bounded by a closed simple curve $L$

---

[2] Bergman [1, p. 10] also gives an equivalent representation in terms of his $E$-function, namely

$$u(x, y) = \operatorname{Re}\left\{\exp\left(-\int_0^{\bar{z}} a(z, t)\, dt + h(z)\right)\int_{-1}^{+1} E(z, \bar{z}, t) f\left(\frac{z}{2}[1 - t^2]\right)\frac{dt}{\sqrt{1 - t^2}}\right\},$$

where $E(z, z^*, t)$ satisfies the differential equation

$$0 = (1 - t^2)E_{z^*t} - \frac{1}{t}E_{z^*} + 2tz(E_{zz^*} + DE_{z^*} + FE),$$

and a characteristic condition. Here we have also that

$$\varphi(z) = \int_{-1}^{+1} f\left(\frac{z}{2}[1 - t^2]\right)\frac{dt}{\sqrt{1 - t^2}}.$$

(whose tangent is Hölder continuous); i.e., we seek a solution $u(x, y)$ to (2.1), such that

$$u \in \mathscr{C}^{(2)}(D) \cap \mathscr{C}^{(0)}(D + L), \qquad u^-(t) = f(t), \quad t \in L \equiv \partial D, \tag{2.9}$$

where $f(t) \in \mathscr{C}^{(0)}(L)$ is real-valued and $u^-(t)$ is taken to mean the value of $u(x, y)$ as $z = x + iy \to t$ from the inside of $D$. We recall that Vekua [10, p. 124], has reduced this boundary value problem to a singular integral equation for the Hölder continuous function $\mu(t)$,

$$A(t_0)\mu(t_0) + \int_L K(t_0, t)\mu(t)\, ds = f(t_0), \tag{2.10}$$

where $t, t_0 \in L$, $ds$ is an arc length differential

$$A(t_0) \equiv \operatorname{Re}\left[i\pi t_0 \bar{t}'_0 H_0(t_0)\right], \qquad K(t_0, t) \equiv \operatorname{Re}\left[\frac{tH_0(t_0)}{t - t_0} - tH(t_0, t)\log\left(1 - \frac{t_0}{t}\right) + H^*(t_0, t)\right], \tag{2.11}$$

with

$$H^*(z, t) \equiv \int_0^z \frac{t[H(z, t_1) - H(z, t)]}{t - t_1}\, dt_1. \tag{2.12}$$

Here $t'_0 = dt/ds$ is evaluated at $t_0$, $H_0(t_0) \equiv H_0(t_0, \bar{t}_0)$ and $H(z, t) \equiv H(z, \bar{z}, t)$.

The singular integral equation (2.10) has index zero and may be reduced to a Fredholm type equation by applying to it the singular operator

$$A(t_0)(\cdot) - \frac{B(t_0)}{\pi i}\int_L \frac{(\cdot)\, dt}{t - t_0}, \tag{2.13}$$

where $B(t_0) \equiv i\pi \operatorname{Re}\left[t_0 \bar{t}'_0 H_0(t_0)\right]$. One obtains by an application of this operator and the Poincaré–Bertrand theorem that the reduced equation is of the form [10, p. 129]

$$\mu(t_0) + \int_L K^*(t_0, t)\mu(t)\, ds = f^*(t_0), \tag{2.14}$$

where

$$K^*(t_0, t) = \frac{1}{\pi^2 (t_0)^2 (H_0(t_0))^2} \cdot \left[A(t_0)K(t_0, t) - \frac{A(t_0)B(t_0)}{\pi i(t - t_0)} - \frac{B(t_0)}{\pi i}\int_L \frac{K(t_1, t)\, dt_1}{t_1 - t_0}\right],$$
$$f^*(t_0) = \frac{1}{\pi^2 |t_0|^2 |H_0(t_0)|^2}\left[A(t_0)f(t_0) - \frac{B(t_0)}{\pi i}\int_L \frac{f(t)\, dt}{t - t_0}\right]. \tag{2.15}$$

THEOREM 1. *The solution of the Dirichlet problem* (2.1), (2.9) *may be approximated uniformly by the solution of an associated Fredholm integral equation from* *the sequence of integral equations*

$$(2.16)\qquad (\mathbf{I} + \mathbf{K}_l^*)\mu \equiv \mu(t_0) + \int_L K_l^*(t_0, t)\mu(t)\, ds = f^*(t_0).$$

*Here $K_l^*(t_0, t)$ is obtained from $K^*(t_0, t)$ by replacing $H(z, t) \equiv H(z, \bar{z}, t)$ by its uniform approximation*

$$(2.17)\qquad H_l(z, \bar{z}, t) \equiv H_0(z, \bar{z}) \sum_{j=1}^{l} \frac{Q^j(z, \bar{z})(z - t)^j}{2^{2j}B(j, j + 1)}.$$

*Proof.* To see $H_l(z, t) \Rightarrow H(z, t)$ on $L \times L$, or on $D \times D^* \times D$ for that matter, see the paper of the author with Lo [7]. The functions $H_l(z, t)$ permit us to approximate uniformly the coefficients of the weak singular terms of the singular kernel $K(t_0, t)$ given in (2.11). This leads to the sequence of singular kernels

$$(2.18)\qquad \begin{aligned} K_l(t_0, t) &\equiv \operatorname{Re}\left\{\frac{tH_0(t_0)}{t - t_0} - tH_l(t_0, t)\log\left(1 - \frac{t_0}{t}\right) + H_l^*(t_0, t)\right\} \\ &\equiv \operatorname{Re}\left\{\frac{tH_0(t_0)}{t - t_0} + \int_0^z \frac{tH_l(z, s)}{t - s}\, ds\right\}. \end{aligned}$$

The corresponding singular equations are of index zero and are reduced to a Fredholm equation by the *same* singular operator (2.13). These Fredholm kernels may be seen to uniformly approximate the function $|t - t_0|^\varepsilon K^*(t_0, t)$ on $L \times L$ for $\varepsilon > 0$ arbitrarily small.

*Remark* 1. By direct computation, the truncated kernel $K_l(t_0, t)$ may be seen to be represented in the form (see also in this regard [7])

$$\begin{aligned} K_l(t_0, t) = \operatorname{Re}\Bigg\{ &\frac{tH_0(t_0)}{t - t_0} - tH_0(t_0)\ln\left(1 - \frac{t_0}{t}\right)\sum_{n=1}^{l}\frac{Q^{(n)}(t_0, \bar{t}_0)(t_0 - t)^{n-1}}{2^{2n}B(n, n + 1)} \\ &+ tH_0(t_0)\sum_{n=1}^{l}\frac{Q^{(n)}(t_0, \bar{t}_0)}{2^{2n}B(n, n + 1)}p_{n-1}(t_0, t)\Bigg\}, \end{aligned}$$

where $p_n(z, t) = \sum_{k=1}^n z^k(z - t)^{n-k}/k$.

**3. The radial case for elliptic equations in two dimensions.** In this section we consider the special case of the elliptic equation (2.1), where the coefficients are analytic functions of the radius squared, $r = (x^2 + y^2)^{1/2}$, i.e., the equation

$$(3.1)\qquad \Delta w + a(r^2)r\frac{\partial w}{\partial r} + c(r^2)w = 0.$$

This equation may be simplified by the substitution, $u(r, \theta) = w(r, \theta) \cdot \exp\left(-\frac{1}{2}\int_0^r a(r^2)r\, dr\right)$, to the form

$$(3.2)\qquad \Delta u + F(r^2)u = 0, \qquad F(r^2) \equiv -\frac{r}{2}a_r - a - \frac{r^2}{4}a^2 + c;$$

consequently, we investigate this equation, where $F(r^2)$ is analytic about the origin in the $r^2$-plane.

LEMMA 2. *For the elliptic differential equation* (3.2) *the associated partial differential equation for the E-function is given by*

$$(3.3)\qquad (1 - t^2)E_{rt} - t^{-1}(t^2 + 1)E_r + tr\left(E_{rr} + \frac{2}{r}E_r + FE\right) = 0.$$

*Proof.* Direct substitution in formula (2) of [1, p. 10].

LEMMA 3. *Let* $h(\mathbf{r})$ *be an arbitrary harmonic function defined in a disk centered at the origin. Then the function defined by*

$$(3.4)\qquad \varphi(\mathbf{r}) = h(\mathbf{r}) + \sum_{n\geqq 1} \frac{2e_n(r^2)}{B(n, 1/2)} \int_0^1 \sigma(1 - \sigma^2)^{n-1} h(\sigma^2\mathbf{r})\, d\sigma$$

*is a solution of* (3.2) *if the* $e_n(r^2)$ *are coefficients for the expansion of the solution of* (3.3) *with the form*

$$(3.5)\qquad E(r, t) = 1 + \sum_{n\geqq 1} e_n(r^2)t^{2n}.$$

*Furthermore, each* $\mathscr{C}^\infty$ *solution of* (3.2) *has a representation of the form* (3.4).

*Proof.* First it has been shown in [1, pp. 27–28] that such solutions (3.5) of (3.3) exist. Also every real, $\mathscr{C}^\infty$ solution[3] of (3.2) may be represented in the form (see [1, p. 28])

$$(3.6)\qquad u(r, \theta) = \mathrm{Re}\left\{\int_{-1}^{+1} E(r, t)f(z[1 - t^2])\frac{dt}{\sqrt{1 - t^2}}\right\},$$

where $f(z)$ is analytic for $z \in D$. Since $E(r, t)$ is real for $r$ and $t$ real, one obtains

$$(3.7)\qquad u(\mathbf{r}) \equiv u(r, \theta) = \int_{-1}^{+1} E(r, t)\, \mathrm{Re}\, \{f(z[1 - t^2])\}\frac{dt}{\sqrt{1 - t^2}}.$$

If we define

$$(3.8)\qquad h(\mathbf{r}) \equiv \int_{-1}^{+1} \mathrm{Re}\, \{f(z[1 - t^2])\}\frac{dt}{\sqrt{1 - t^2}},$$

with $\mathrm{Re}\, \{f(z)\} = \sum_{n\geqq 0} a_n r^n Y_n(\theta)$, where the $Y_n(\theta)$ are circular harmonics, then

$$(3.9)\qquad h(\mathbf{r}) = \sum_{n\geqq 0} \frac{\Gamma(\frac{1}{2})\Gamma(n + \frac{1}{2})}{\Gamma(n + 1)} a_n r^n Y_n(\theta).$$

It is clear that the harmonic function (3.9) converges uniformly in each disk $|z| \leqq a$ in which the MacLaurin series for $f(z)$ does. Hence, the indicated integration in (3.4) may be performed termwise to the series (3.9) for $h(\mathbf{r})$. Our result then follows directly by comparing terms in the expansions for (3.7) and (3.4).

*Remark* 2. Our representation (3.7) differs from Bergman's in that we have replaced $z/2$ by $z$. This is of importance to us later, when posing boundary value problems.

*Remark* 3. If $f(z) = \sum_{n\geqq 0} a_n z^n$ and we define $G(z)$ as

$$G(z) \equiv \sum_{n\geqq 0} a_n z^n B(n, \tfrac{1}{2}),$$

[3] Each real $\mathscr{C}^\infty$ solution is also real analytic for the case considered.

then

$$G(z) = \int_{-1}^{+1} f(z[1 - t^2]) \frac{dt}{\sqrt{1 - t^2}},$$

and then a complex solution of (3.2) is

$$V(z, z^*) = G(z) + \sum_{n \geq 1} \frac{e_n(zz^*)z^{-n}}{2^{2n}B(n, n - 1)} \int_0^z (z - t)^{n-1} G(t)\, dt, \tag{3.10}$$

reminiscent of the Bergman formula [1, p. 15, (4a)]; however here $G(z)$ is related differently to $f(z)$ than his function $g(z)$. See also (2.4).

THEOREM 2. *For the elliptic equation* (3.2) *the coefficients* $e_n(r^2)$ *defined in* (3.5) *may be found directly from the Riemann function for* (3.2) *by the formula*

$$e_n(zz^*) = \frac{\sqrt{\pi}}{\Gamma(n + 1/2)}(-z)^n \left[\frac{\partial^n}{\partial \xi^n}(R(\xi, 0; z, z^*))\right]_{\xi = z}, \tag{3.11}$$

*where* $e_n(r^2) = e_n(zz^*)$, *with* $z^*$ *restricted to* $\bar{z}$.

*Proof.* From (2.3) and (2.4) we have in general for the analytic equation (2.4), that

$$\begin{aligned} \sum_{n \geq 1} \frac{e_n(z, z^*)z^{-n}(z - \xi)^{n-1}}{2^{2n}B(n, n + 1)} &= \frac{H(z, \xi)}{H_0(z)} \\ &\equiv \frac{-R_\xi(\xi, 0; z, z^*) + B(\xi, 0)R(\xi, 0; z, z^*)}{R(z, 0; z, z^*)}, \end{aligned} \tag{3.12}$$

where $e_n(z, z^*) \equiv z^n Q^{(n)}(z, z^*)$. For the self adjoint case (3.2), this reduces to

$$\sum_{n \geq 1} \frac{e_n(zz^*)z^{-n}(z - \xi)^{n-1}}{2^{2n}B(n, n + 1)} = \frac{-R_\xi(\xi, 0; z, z^*)}{R(z, 0; z, z^*)}, \tag{3.13}$$

from which we obtain our result by expanding $R(\xi, 0; z, z^*)$ as a Taylor series in $\xi$ about center $z$, and noting that $R(z, 0; z, z^*) \equiv 1$.

THEOREM 3. *For the elliptic equation* (3.2) *the integral representation for a solution* $\varphi(\mathbf{r})$, *in terms of an arbitrary harmonic function* $h(\mathbf{r})$, *has the following equivalent representation in terms of a real integral, involving a real Riemann function*:

$$\varphi(\mathbf{r}) = h(\mathbf{r}) - 2z \int_0^1 \sigma R_1(z\sigma^2, 0; z, \bar{z}) h(\sigma^2 \mathbf{r})\, d\sigma. \tag{3.14}$$

(The subscript indicates differentiation with respect to the first argument.)

*Proof.* Putting (3.11) into the expansion (3.4), we obtain,

$$\varphi(\mathbf{r}) = h(\mathbf{r}) + 2 \sum_{n \geq 1} \frac{(-1)^n z^n}{(n - 1)!}\left[\frac{\partial^n}{\partial \xi^n} R(\xi, 0; z, \bar{z})\right]_{\xi = z} \int_0^1 \sigma^1 (1 - \sigma^2)^{n-1} h(\sigma^2 \mathbf{r})\, d\sigma,$$

from which we obtain our result by regrouping terms as the integral of a Taylor series.

*Example* 1.

$$\Delta u + \lambda^2 u = 0. \tag{3.15}$$

This equation appears to be the only one for which the $E$-function is already known, namely $E(r, t) = \cos \lambda rt$, ([1, p. 28], [5, p. 120]). Here we have $e_n(r^2) = (-1)^n(\lambda r)^{2n}/(2n)!$, and hence (3.4) becomes

$$\varphi(\mathbf{r}) = h(\mathbf{r}) + \sum_{n \geq 1} \frac{(-1)^n(r\lambda/2)^{2n}}{\Gamma(n)\Gamma(n+1)} \int_0^1 \sigma(1-\sigma^2)^{n-1} h(\sigma^2 \mathbf{r})\, d\sigma,$$

or equivalently

$$\varphi(\mathbf{r}) = h(\mathbf{r}) - \lambda r \int_0^1 \sigma J_1\left(\lambda r\sqrt{1-\sigma^2}\right) \frac{h(\sigma^2\mathbf{r})\, d\sigma}{\sqrt{1-\sigma^2}}, \tag{3.16}$$

where $J_1(z)$ is Bessel's function of the first kind and order one. Formula (3.15), however, follows from Vekua's representation (2.3) by recognizing that the Riemann function for (3.15) is $J_0(\lambda[(z-\zeta)(z^*-\zeta^*)]^{1/2})$. One obtains in this case,

$$\varphi(\mathbf{r}) = \operatorname{Re}\left\{\varphi(z) - \int_0^z \varphi(\zeta)\frac{\partial}{\partial \zeta} J_0\left(\lambda\sqrt{z^*(z-\zeta)}\right) d\zeta\right\},$$

which we can put into the form (3.15) by integrating along a straight line from 0 to $z$ and using a real integration parameter $\sigma = +\sqrt{\zeta/z}$. Equation (3.15), however, is already known and may be found in [10, p. 58].

*Example* 2. $\Delta u + \lambda^2 r^{2(m-1)} u = 0, m = 2, 3, \cdots$. The solutions of this equation dependent only on the radius are the cylinder functions $Z_0((\lambda/m)r^m)$; hence, we have as the Riemann function

$$R(\zeta, \zeta^*; z, z^*) = J_0\left(\frac{\lambda}{m}[(z-\zeta)(z^*-\zeta^*)]^{m/2}\right). \tag{3.17}$$

Using formula (3.13) one may compute the coefficients

$$e_n(r^2) = m\frac{B(lm, lm+1)}{l!(l-1)!}(-1)^l\left(\frac{\lambda}{2m}\right)^{2l}(4r^2)^{ml}, \qquad n = ml,$$

$$e_n(r^2) = 0, \qquad n \neq ml.$$

We obtain the general representation formula of the type (3.14),

$$\varphi(\mathbf{r}) = h(\mathbf{r}) - \lambda r^m \int_0^1 \sigma[1-\sigma^2]^{m/2-1} J_1\left(\frac{\lambda r^m}{m}[1-\sigma^2]^{m/2}\right) h(\sigma^2\mathbf{r})\, d\sigma,$$

and the following expression for the $E$-function:

$$E(r, t) = 1 + m\sum_{l \geq 1} \frac{\Gamma(lm)\Gamma(lm+1)}{\Gamma(2lm+1)\Gamma(l)} \frac{(-[4r^2]^m\lambda^2/4m^2)^l}{l!}, \tag{3.18}$$

which is a generalized hypergeometric series of the type studied by E. M. Wright [3].

*Example* 3.

$$\Delta u + \frac{4\lambda(\lambda+1)}{(1+r^2)^2} u = 0.$$

The Riemann function for this case is given by Vekua [10, p. 21] to be

$$(3.19)\qquad R(\zeta,\zeta^*;z,z^*) = P_\lambda\left(\frac{(1-zz^*)(1-\zeta\zeta^*)+2z\zeta^*+2z^*\zeta}{(1+zz^*)(1+\zeta\zeta^*)}\right),$$

where $P_\lambda(z)$ is a Legendre function of degree $\lambda$, i.e.,

$$P_\lambda(z) = F(\lambda+1, -\lambda; 1; \tfrac{1}{2}(1-z)),$$

where $F(\alpha,\beta;\gamma;z)$ is a hypergeometric series. We then have

$$R(t,0;z,z^*) = F\left(\lambda+1, -\lambda; 1; \frac{z^*(z-t)}{1+zz^*}\right),$$

and that

$$\begin{aligned}-R_1(\xi,0;z,z^*) &= \frac{-\lambda(\lambda+1)z^*}{1+r^2}\sum_{l\geqq 0}\frac{(\lambda+2)_l(-\lambda+1)_l}{(2)_l l!}\left[\frac{z^*(z-\xi)}{1+r^2}\right]^l\\ &= \frac{-\lambda(\lambda+1)z^*}{1+r^2}F\left(\lambda+2, -\lambda+1; 2; \frac{z^*(z-\xi)}{1+r^2}\right),\end{aligned}$$

from which we have, after using Legendre's duplication formula,

$$e_l(r^2) = \sqrt{\pi}\left(\frac{r^2}{1+r^2}\right)^l\frac{\Gamma(\lambda+l+1)\Gamma(l-\lambda)}{\Gamma(\lambda+1)\Gamma(-\lambda)l!\Gamma(l+\frac{1}{2})},$$

and hence

$$(3.20)\qquad \begin{aligned}E(r,t) &= F\left(\lambda+1, -\lambda; \frac{1}{2}; \frac{r^2t^2}{1+r^2}\right)\\ &= \sqrt{\pi}\left[\frac{rt}{1+r^2-r^2t^2}\right]^{1/4}P_\lambda^{1/2}\left(\frac{1+r^2-2r^2t^2}{1+r^2}\right).\end{aligned}$$

Here, $P_\lambda^\nu$ is taken to be an associated Legendre function. Finally, from Theorem 3 we may obtain a general representation formula

$$(3.21)\quad \varphi(\mathbf{r}) = h(\mathbf{r}) - \frac{2\lambda(\lambda+1)r^2}{1+r^2}\int_0^1 \sigma F\left(\lambda+2, 1-\lambda; 2; \frac{r^2(1-\sigma^2)}{1+r^2}\right)h(\mathbf{r}\sigma^2)\,d\sigma,$$

for solutions of this partial differential equation.

*Example* 4.

$$\Delta u + a_0 r u_r + \left[a_0 + \left(\frac{a_0 r}{2}\right)^2 + \alpha^2\beta^2 r^{2(\beta-1)}\right]u = 0.$$

As before we may find the Riemann function by considering solutions dependent only on $r$; it is

$$\begin{aligned}R(\zeta,\zeta^*;z,z^*) = [(z-\zeta)(z^*-\zeta^*)]^{\alpha/2}\exp\left\{-\frac{a_0^2}{4}(z-\zeta)(z^*-\zeta^*)\right\}\\ \cdot J_0(\alpha[(z-\zeta)(z^*-\zeta^*)]^{\beta/2}),\end{aligned}$$

The generalized representation is given by

$$\varphi(\mathbf{r}) = h(\mathbf{r}) \exp\left(-\frac{a_0 r^2}{4}\right) - \int_0^1 \frac{\partial}{\partial \sigma}\left\{r^\alpha(1-\sigma^2)^{\alpha/2} \exp\left(-\frac{a_0^2}{4} r^{2[1-\sigma^2]}\right) \cdot J_0(\alpha r^\beta[1-\sigma^2]^{\beta/2})\right\} h(\mathbf{r}\sigma^2)\, d\sigma. \tag{3.22}$$

As before, the $E$-function may be computed; however, in this case we must use (3.12). Since Example 4 is not self-adjoint, the computations are somewhat more detailed and hence we omit them.

*Remark* 4. It is apparent from the above illustrations, that the generalized representation (3.14) is more convenient for doing *analytic* computations than representations involving the $E$-function. Indeed, it is the more complicated nature of the analytic computations involving $E$-functions which is partially responsible for a scarcity of known $E$-functions.

The general representation formula (3.14) allows us to obtain for the radial case, a real singular integral equation, which is the real analogue for the singular integral equation formulation for the Dirichlet problem (2.1), (2.9) as provided by Vekua [10]. If the boundary of the domain is sufficiently smooth, however, the real formulation yields immediately a Fredholm equation.

THEOREM 4. *Let $D$ be a simply connected domain which is bounded by a closed curve $L$ having a parametrization $x(s) + iy(s)$, where the functions $x(s)$, $y(s)$ are $\mathscr{C}^4$. Furthermore, let $D$ be star-like with respect to the origin. Then the Dirichlet problem* (3.2), (2.9) *may be reformulated as a real Fredholm integral equation. Furthermore, if $F < 0$ then there exists a unique solution to the integral equation.*

*Proof*. We begin by rewriting the general representation (3.14) in the more convenient form

$$\varphi(x, y) = h(x, y) - \int_0^1 \hat{R}_\sigma\left(\frac{1}{2}z\sigma^2, \frac{1}{2i}z\sigma^2; x, y\right) h(x\sigma^2, y\sigma^2)\, d\sigma,$$

where $\hat{R}(\xi, \eta; x, y) \equiv R(\zeta, \zeta^*; z, z^*)$ with $z = x + iy$, $\zeta = \xi + i\eta$, etc. Suppose $h(x, y)$ may be given in terms of a double-layer potential of a continuously differentiable distribution, $\mu(s)$, i.e.,

$$h(x, y) = -\int_L \mu(s) \frac{\partial}{\partial \nu_s} \log \sqrt{[x - \xi(s)]^2 + [y - \eta(s)]^2}\, ds,$$

where $s$ is the arc length parameter and $\xi(s) + i\eta(s) \in L$. It is known from the jump discontinuity theorem concerning such integrals, [2, vol. II, p. 300], that as $(x, y) \to (x(t), y(t)) \in L$, one has

$$h(t) \equiv h(x(t), y(t)) = -\pi\mu(t) - \int_L \mu(s) \frac{\partial}{\partial \nu_s} \log \sqrt{[x(t) - \xi(s)]^2 + [y(t) - \eta(s)]^2}\, ds.$$

Now, since $D$ is star-like with respect to the origin, the point $(x(t)\sigma^2, y(t)\sigma^2) \in D$ for $\sigma \in [0, 1]$; hence, one obtains an integral relation to be satisfied for the distributions $\mu(t)$ which correspond to solutions of the Dirichlet problem (2.9):

$$\mu(t) = -\frac{1}{\pi} f(t) - \int_L \mu(s) K(t, s)\, ds,$$

where

$$K(t,s) = \frac{1}{\pi}\left\{\frac{\partial}{\partial v_s}\log\sqrt{[x(t)-\xi(s)]^2+[y(t)-\eta(s)]^2}\right.$$

$$(3.23) \qquad - \int_0^1 \hat{R}_\sigma\left(\frac{\sigma^2}{2}z(t), \frac{\sigma^2}{2i}z(t); x(t), y(t)\right)$$

$$\left.\cdot\frac{\partial}{\partial v_s}\log\sqrt{[x(t)\sigma^2-\xi(s)]^2+[y(t)\sigma^2-\eta(s)]^2}\,d\sigma\right\}.$$

Equation (3.17) follows by making a change in the orders of integration which is justified by Fubini's theorem. The first term in $K(t, s)$ is a $\mathscr{C}^2$-function since the normal derivative is proportional to the curvature of $L$. The second term has a weak singularity; hence (3.17) is Fredholm. It is well known that $F(r^2) \leqq 0$ is necessary and sufficient for the Dirichlet problem (2.1), (2.9) (with $F(r^2)$ analytic) to have a unique solution. Since (3.14) may be written as a Volterra equation for $h(\mathbf{r})$, by an elementary change of variables, it is clear that the unique solution of the Dirichlet problem corresponds to the unique solution of the Fredholm equation.

**4. The equation $\Delta u + B(r^2)u = 0$, in $p + 2$ variables.**

THEOREM 5. *Let $E(r, t; p)$, $p \in Z^+$ be a solution of the partial differential equation*

$$(4.1) \qquad (1-t^2)E_{rt} + (p-1)(t^{-1}-t)E_r + rt\left(E_{rr} + \frac{p}{r}E_r + BE\right) = 0,$$

*where $B(r^2)$ is an entire function of $r^2$. Furthermore, let $E(r, t; p)$ satisfy the following boundary conditions on the infinite strip $\{0 \leqq t \leqq 1\} \times \{0 \leqq t < \infty\}$:*

$$(4.2) \qquad \lim_{t\to 0^+}(t^{p-1}E_r)r^{-1} = 0, \qquad \lim_{t\to 1^-}(\sqrt{1-t^2}E_r)r^{-1} = 0,$$
$$\lim_{r\to 0^+} E = 1.$$

*Then, if $H(\mathbf{r})$ is an arbitrary harmonic function of $n = p + 2$ variables, defined in a star-like region with respect to the origin,*

$$(4.3) \qquad \varphi(\mathbf{r}) = \int_0^1 E(r, t; p)t^p H(\mathbf{r}[1-t^2])\frac{dt}{\sqrt{1-t^2}}$$

*is a solution of*

$$(4.4) \qquad \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} + B(r^2)u = 0.$$

*Proof.* This may be verified directly by substitution, and integration by parts.

*Remark* 5. The integral (4.3) is a natural generalization of (3.7) to $p + 2$ variables. An alternate integral representation for solutions of (4.4), with different boundary characterizations (4.2) was found by the author and Howard, [6], [5].

*Example* 5.

$$\Delta u + \lambda^2 u = 0.$$

We seek a solution of (4.1) of the form $E(r, t; p) = \varepsilon(\lambda rt/2)$, and find it must satisfy the ordinary differential equation

$$\left(\frac{\lambda rt}{2}\right)\ddot{\varepsilon} + p\dot{\varepsilon} + 2\lambda rt\varepsilon = 0;$$

hence

$$E(r, t; p) = C_p(\lambda rt)^{-(p-1)/2}J_{(p-1)/2}(\lambda rt), \tag{4.5}$$

where $C_p$ is a constant. If we choose $C_p = 2^{(p-1)/2}\Gamma((p + 1)/2)$, then (4.3), with this $E$-function, may be shown to generalize Vekua's formula (3.16) to $n$ dimensions; we shall do this presently. First we establish several existence theorems.

LEMMA 4. *If $B(r^2)$ is entire in the $r^2$-plane then there exists a solution $E(r, t)$ to* (4.1) *with the series expansion*

$$E(r, t;\ p) = 1 + \sum_{n\geqq 1} e_n(r; p)t^{2n}, \tag{4.6}$$

*which converges uniformly and absolutely in $D \times \{|t| \leqq 1\}$, where $D$ is any compact set in $\mathbb{C}^1$.*

*Proof*. Seeking a solution of (4.6) we obtain the following recursion formulas for the coefficients: $(p + 1)e_1' = -rB$,

$$(2n + p - 1)e_n' = (2n - 3)e_{n-1}' - re_{n-1}'' - rBe_{n-1}, \tag{4.7}$$

$n \geqq 2$, with $e_n(0; p) = 0$, $n \geqq 1$.

We now attempt the usual reduction of a system of this type, (see [1], [6] or [5, p. 89]) by setting $f_n(r^2) = r^{1-2n}e_n'(r; p)$. We obtain

$$(2n + p - 1)f_n(\rho) = -2\frac{df_{n-1}}{d\rho} - \frac{\rho^{1-n}}{2}B(\rho)\int_0^\rho s^{n-1}f_{n-1}(s)\,ds, \tag{4.8}$$

$n \geqq 2$. We next construct a sequence of functions $F_n(\rho)$ which majorize the $f_n(\rho)$, $f_n(\rho) \ll F_n(\rho)$. First, since $B(\rho)$ is entire we may find an $M_0$, for each $\theta > 0$ and arbitrarily small, such that $|B(\rho)| \ll M_0(1 - \rho\theta)^{-1}$ for all $\rho$ of modulus less than $\theta^{-1}$. We choose and fix a value of $\theta$; then we define

$$\begin{gathered} F_1(\rho) \equiv M_0(1 - \theta\rho)^{-1}, \\ (2n + p - 1)F_n(\rho) = 2\frac{dF_{n-1}(\rho)}{d\rho} + \frac{1}{2}\frac{\rho^{1-n}M_0}{1 - \theta\rho}\int_0^\rho s^{n-1}F_{n-1}(s)\,ds, \end{gathered} \tag{4.9}$$

$n \geqq 2$. Using the method of dominants we may now show that

$$F_n(\rho) \ll \frac{\rho^{1-n}}{\Gamma(2n + p + 1)}\left(\frac{2n + (M_0\theta^{-1})/2}{1 - \rho\theta}\right)^n, \qquad n \geqq 1. \tag{4.10}$$

To this end, we first note that the above is true by assumption for $n = 1$. Assuming it is true for $n = k$ we attempt to show this implies its validity for $n = k + 1$. If we call $(2n + \frac{1}{2}M_0\theta^{-1})^n$ the coefficient $M_n$, then we have

$$F_n(\rho) \ll \frac{M_n}{\Gamma(2n + \rho + 1)}\sum_{l\geqq 0}\frac{\Gamma(l + n)}{l!\Gamma(n)}\rho^{l-\eta+1}\theta^l;$$

hence,

$$\frac{dF_n(\rho)}{d\rho} \ll \frac{M_n}{\Gamma(2n+\rho+1)} \sum_{l \geqq 0} \frac{\Gamma(l+n)}{l!\Gamma(n)} |l-n+1| \rho^{l-n} \theta^l. \tag{4.11}$$

It follows from this, that

$$\rho^n \frac{dF_n(\rho)}{d\rho} \ll \frac{nM_n}{\Gamma(2n+\rho+1)} (1-\theta\rho)^{-(n+1)}. \tag{4.12}$$

Next, we must estimate the integral. A direct computation yields the majorant

$$\int_0^\rho s^{n-1} F_n(s)\, ds \ll \rho^n F_n(\rho). \tag{4.13}$$

Combining the estimates (4.12), (4.13), with (4.9)

$$\begin{aligned} \rho^n(2n+p+1)F_{n+1} &\ll \frac{1}{\Gamma(n+2p+1)} \left\{ \frac{2nM_n + \rho M_0 M_n/2}{(1-\theta\rho)^{n+1}} \right\} \\ &< \frac{M_{n+1}}{\Gamma(n+2\rho+1)(1-\theta\rho)^{n+1}}. \end{aligned} \tag{4.14}$$

We realize now that the series for $rE_r(r, t)$ must be majorized as follows

$$\begin{aligned} r \sum_{n \geqq 1} e_n'(r) t^{2n} &= \sum_{n \geqq 1} r^{2n} f_n(r^2) t^{2n} \ll \sum_{n \geqq 1} r^{2n} F_n(r^2) t^{2n} \\ &\ll r^2 \sum_{n \geqq 1} \frac{M_n}{\Gamma(2n+\rho-1)} \left( \frac{t^2}{1-\theta r^2} \right)^n. \end{aligned} \tag{4.15}$$

A short computation with the ratio test yields that the dominant series converges whenever $|t|^2 < e(1-\theta|r|^2)$, and hence $|r|^2 < \theta^{-1}(1-e^{-1})$, which concludes our proof.

THEOREM 6. *Let $B(r^2)$ be entire in the $r^2$-plane, and $h(\mathbf{r}; p)$ an arbitrary harmonic function of $p+2$ variables defined in a spherical neighborhood of the origin, $\mathcal{N}(0)$. Then*

$$\varphi(\mathbf{r}) = h(\mathbf{r}; p) + \sum_{n \geqq 1} C_n(r; p) \int_0^1 \sigma^{p+1}(1-\sigma^2)^{n-1} h(\mathbf{r}\sigma^2; p)\, d\sigma, \tag{4.16}$$

*with*

$$C_n(r; p) \equiv \frac{2e_n(r; p)\Gamma(n + p/2 + 1/2)}{\Gamma(p/2 + 1/2)\Gamma(n)}, \tag{4.17}$$

*is a solution of the differential equation* (4.4).

*Proof.* Let $H(n, m_1, \cdots, \pm m_p; x_1, \cdots, x_{p+2}) \equiv H(m_k; \pm; \mathbf{r}) = r^n Y(m_k; \theta; \varphi)$, where $\theta$ stands for $\theta_1, \theta_2, \cdots, \theta_p$, be a surface harmonic of degree $n$, [3, vol. 2, p. 240]. Then, if $H(\mathbf{r})$ is a harmonic function, regular on a spherical neighborhood of the origin, it has an expansion of the form

$$H(\mathbf{r}) = \sum_{n, m_k} a_n(m_k; \pm) H(m_k; \pm; \mathbf{r}),$$

which is Abel-summable on the surface of this sphere [8]. On any set, relatively compact in this sphere, the convergence is uniform, and hence the following termwise integration is permissible:

$$h(\mathbf{r};p) \equiv \int_0^{+1} t^p H(\mathbf{r}[1-t^2]) \frac{dt}{\sqrt{1-t^2}}$$

(4.18)
$$= \sum_{n,m_k} a_n(m_k;\pm) r^n Y(m_k;\theta;\varphi) \int_0^{+1} t^p[1-t^2]^{n-1/2}\, dt$$

$$= \frac{1}{2}\sum_{n,m_k} a_n(m_k;\pm)\frac{\Gamma((p+1)/2)\Gamma(n+\frac{1}{2})}{\Gamma(n+p/2+1)} r^n Y(m_k;\theta;\varphi).$$

Since,

$$\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+p/2-1)} \approx n^{-(p+1)/2}\left(1+O\left(\frac{1}{n}\right)\right) \qquad \text{as } n \to \infty,$$

it is obvious that $h(\mathbf{r};p)$ converges in a ball at least as large as $H(\mathbf{r})$ does. Hence, in this ball we may perform a termwise integration of the harmonic series for $h(\mathbf{r}\sigma;p)$ in the integral

$$\int_0^1 \sigma^{p+1}(1-\sigma^2)^{n-1} h(\mathbf{r}\sigma;p)\, d\sigma.$$

Putting these terms into (4.16), and comparing this formal series with the representation obtained by termwise integrating (4.3) with $E(r,t)$ given by (4.6), yields (4.17). The termwise integration of (4.3) is valid because of Lemma 4. This concludes our proof.

*Example* 6. We return to the expression of the $E$-function given for the Helmholtz equation, namely,

(4.19)
$$E(r,t;p) = \Gamma\left(\frac{p+1}{2}\right)\left(\frac{\lambda rt}{2}\right)^{-(p-1)/2} J_{(p-1)/2}(\lambda rt),$$

and consider the corresponding expression (4.16). Since

$$E(r,t;p) = \Gamma\left(\frac{p+1}{2}\right)\sum_{n\geq 0}\frac{(-1)^n}{n!\Gamma(n+(p+1)/2)}\left(\frac{\lambda rt}{2}\right)^{2n},$$

we have by (4.6) that $e_n(r;p) = (-1)^n(\lambda r/2)^{2n}[((p+1)/2)_n n!]^{-1}$, and hence the integral representation (4.16) takes on the rather simple form

(4.20)
$$\varphi(\mathbf{r}) = h(\mathbf{r};p) - \lambda r \int_0^1 \sigma^{p+1} J_1(\lambda r\sqrt{1-\sigma^2}) h(\mathbf{r}\sigma^2) \frac{d\sigma}{\sqrt{1-\sigma^2}}.$$

An elementary change of integration parameter shows that this integral formula is identical to Vekua's extension of (3.16) to higher dimensions [10, p. 59, (13.14)]. Hence, Theorem 4.3 provides a natural generalization of Vekua's method to the higher dimensional equations of type (4.3). We proceed to give several other examples to underscore the use of the representation (4.16), (4.17).

*Example* 7.

$$\Delta_{p+2}u + \frac{\lambda(\lambda+1)}{(1+r^2)^2}u = 0.$$

We seek a solution of (4.1) of the form $E(r, t; p) = \mathscr{E}(X)$, with $X = r^2t^2/(1 + r^2)$ and find $\mathscr{E}(X)$ must satisfy the hypergeometric differential equation

$$X(1 - X)\ddot{\mathscr{E}} + \left(\frac{p+1}{2} - 2X\right)\dot{\mathscr{E}} + \lambda(\lambda+1)\mathscr{E} = 0.$$

Hence we have

$$\begin{aligned} E(r,t;p) &= F\left(\lambda+1, -\lambda; \frac{p+1}{2}\cdot\frac{r^2t^2}{1+r^2}\right) \\ &= \Gamma\left(\frac{p+1}{2}\right)\left[\frac{rt}{1+r^2(1-t^2)}\right]^{(1-p)/4} P_\lambda^{(1-p)/2}\left(\frac{1+r^2-2r^2t^2}{1+r^2}\right), \end{aligned} \tag{4.21}$$

which reduces to the two-dimensional case for $p = 0$. The coefficients of (4.17) are given by

$$C_l(r;p) = 2\left(\frac{r^2}{1+r^2}\right)^l \frac{(\lambda+1)_l(-\lambda)_l}{l!(l-1)!},$$

and the representation (4.16) becomes

$$\begin{aligned} \varphi(\mathbf{r}) &= h(\mathbf{r};p) \\ &\quad - \frac{2\lambda(\lambda+1)r^2}{1+r^2}\int_0^1 \sigma^{p+1}F\left(\lambda+2, 1-\lambda; 2; \frac{r^2(1-\sigma^2)}{1+r^2}\right)h(\mathbf{r}\sigma^2;p)\,d\sigma. \end{aligned} \tag{4.22}$$

Hence, we have found an extension of the representation formula (3.21) to $(p + 2)$ variables.

We next turn to an alternate method of computing the $E$-function. We notice that the function $K(r, t) \equiv t^p(1 - t^2)^{-1/2}E(r, t)$ satisfies the differential equation [6], [5, p. 87]

$$(1 - t^2)K_{rt} - t^{-1}(t^2 + 1)K_r + rt\left(K_{rr} + \frac{p+1}{r}K_r + BK\right) = 0. \tag{4.23}$$

If we define the transformed function $k(r, \alpha)$ by

$$k(r,\alpha) \equiv \int_0^{+1} (1 - t^2)^\alpha K(r,t)\,dt, \qquad \alpha \geqq 0, \tag{4.24}$$

then it may be seen that $k(r, \alpha)$ satisfies the ordinary differential equation in the variable $r$,

$$\ddot{k} + \frac{1}{r}(p + 2\alpha + 1)\dot{k} + B(r^2)k = 0,$$

or in terms of $y = r^{p/2+\alpha}k$, the equation

$$r^2\ddot{y} + r\dot{y} + \left[r^2B(r^2) - \left(\alpha + \frac{p}{2}\right)^2\right]y = 0.$$

We notice that by a change of integration parameter $k(r, \alpha)$ may be represented as a Laplace transformation,

$$k(r, \alpha) = \mathscr{L}_u\left\{\frac{K(r, [1 - e^{-u}]^{1/2})}{[e^u - 1]^{1/2}}e^{-u/2}\right\}, \tag{4.25}$$

where $\mathscr{L}_u\{f\} \equiv \int_0^\infty e^{-u\alpha}f(u)\,du.$

Since $K(r, t) = t^p[1 - t^2]^{-1/2}E(r, t)$, we have the two initial conditions,

$$k(0, \alpha) = \frac{\Gamma((p + 1)/2)\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha + p/2 + 1)}, \tag{4.26}$$

and

$$\dot{k}(0, \alpha) = 0, \tag{4.27}$$

recalling that the $e_n'(0; p) = 0$ for all $n \geqq 1$. Turning to our example $B(r) \equiv \lambda^2$ again, we see that

$$k(r; \alpha) = \Gamma\left(\frac{p + 1}{2}\right)\Gamma\left(\alpha + \frac{1}{2}\right)\left(\frac{\lambda r}{2}\right)^{-(p/2+\alpha)} J_{p/2+\alpha}(\lambda r),$$

and that

$$K(r, t) = [(e^u - 1)^{1/2}\mathscr{L}_\alpha^{-1}\{k(r, \alpha - \tfrac{1}{2})\}]_{u=-\ln(1-t^2)}. \tag{4.28}$$

Using the convolution theorem for the Laplace transform, we obtain,

$$K(r, t) = \frac{\Gamma((p + 1)/2)}{\sqrt{\pi}\Gamma(p/2)}\frac{t}{\sqrt{1 - t^2}}\left\{\int_0^u [1 - \exp(-(u - s))]^{p/2-1}\frac{\exp(-(p - 1)s/2)}{[e^s - 1]^{1/2}} \cdot \cos(\lambda r\sqrt{1 - e^{-s}})\,ds\right\}n$$

$$= -\ln(1 - t^2)$$

$$= \frac{\Gamma((p + 1)/2)}{\sqrt{\pi}\Gamma(p/2)}\frac{2t}{\sqrt{1 - t^2}}\int_0^t (t^2 - \sigma^2)^{p/2-1}\cos(\lambda r\sigma)\,d\sigma$$

$$= \frac{2t^p\Gamma((p + 1)/2)}{\sqrt{\pi}\Gamma(p/2)\sqrt{1 - t^2}}\int_0^{\pi/2}\cos^{p-1}\varphi\cos(\lambda rt\sin\varphi)\,d\varphi$$

$$= \frac{t^p\Gamma((p + 1)/2)}{\sqrt{1 - t^2}}\left(\frac{\lambda rt}{2}\right)^{-(p-1)/2} J_{(p-1)/2}(\lambda rt),$$

which agrees with (4.19).

In order to develop an approach similar to Vekua's for elliptic equations in two independent variables, for the partial differential equation

$$\frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} + B(r^2)u = 0, \tag{4.4}$$

it is of interest to represent (4.16) in the form

$$\varphi(\mathbf{r}) = h(\mathbf{r}; p) + \int_0^1 \sigma^{p+1} G(r, 1 - \sigma^2) h(\mathbf{r}\sigma^2; p)\, d\sigma, \tag{4.29}$$

and to obtain simple methods for determining the function $G(r, \tau)$. To this end we establish the following lemma and theorem.

LEMMA 5. *The function $G(r, \tau)$ which appears in* (4.29) *and is defined by the formal series*

$$G(r, \tau) \equiv \sum_{n \geqq 1} C_n(r; p)\tau^{n-1} \tag{4.30}$$

*satisfies the partial differential equation,*

$$2(1 - \tau)G_{r\tau} - G_r + r(G_{rr} + B(r^2)G) = 0. \tag{4.31}$$

*Furthermore, $G(r, \tau)$ is independent of the parameter $p = n - 2$.*

*Proof.* Direct substitution of the $G$-function into (4.30) and use of the recursion formulas (4.7) and the definition of the coefficients $C_n(r; p)$ by (4.17) yields the first part of the lemma. That $G(r, \tau)$ is independent of $p$ follows from the differential equation (4.31) and the initial-boundary data, $G(0, \tau) = 0$, and

$$G(r, 0) = C_1(r; p) \equiv -\int_0^r rB(r^2)\, dr.$$

THEOREM 7 (A method of ascent). *The G-function may be represented in terms of the Riemann function by means of the formula*

$$G(r, 1 - \sigma^2) = -2rR_1(r\sigma^2, 0; r, r). \tag{4.32}$$

*Proof.* For the case of $n = 2$ variables (4.29) becomes (3.14); hence we may form the identification

$$G(r, 1 - \sigma^2) \equiv -2zR_1(z\sigma^2, 0; z, \bar{z}). \tag{4.33}$$

However, since $G$ is a real function of $r^2 = z\bar{z}$ each $\bar{z}$ appearing in the right-hand side of (4.33) must combine with a $z$ to appear only as an $r^2$. This permits us to extract (4.32).

*Example* 8. For $B(r^2) = \lambda^2$, one finds that

$$G(r, 1 - \sigma^2) = \frac{-r\lambda}{\sqrt{1 - \sigma^2}} J_1(\lambda r \sqrt{1 - \sigma^2})$$

satisfies (4.31) and the initial data. This also agrees with (4.20) and (3.16).

*Remark* 6. The method of ascent given above permits a representation of $(p + 2)$-dimensional solutions in the form

$$\varphi(\mathbf{r}) = h(\mathbf{r}; p) - 2r \int_0^1 \sigma^{p+1} R_1(r\sigma^2, 0; r, r) h(\mathbf{r}\sigma^2; p)\, d\sigma. \tag{4.34}$$

THEOREM 8. *Let* $S(\rho) \equiv \{\|\mathbf{r}\| \leqq \rho\}$, *and let* $\varphi(\mathbf{r})$ *be any solution of* (4.4) *defined in* $S^0(\rho)$ *and continuous on* $\partial S(\rho)$. *Furthermore, let* $\varphi(\mathbf{r})|_{r=\rho} \equiv \varphi(\rho;\theta;\varphi)$ *be expandible in terms of hyperspherical harmonics as*

$$\varphi(\rho;\theta;\varphi) = \sum a_n(m_k;\pm)Y(m_k;\theta;\varphi). \tag{4.35}$$

*Then* $\varphi(\mathbf{r})$ *may be expanded in the following series which is uniformly convergent for all* $D \subset\subset S(\rho)$:

$$\varphi(\mathbf{r}) = \sum_{n,m_k} \left(\frac{r}{\rho}\right)^n \frac{J_n(r;p)}{J_n(\rho;p)} a_n(m_k;\pm)Y(m_k;\theta;\varphi), \tag{4.36}$$

*where* $J_n(r;p) \equiv \int_0^1 t^p E(r,t;p)(1-t^2)^{n-1/2}\,dt$.

*Proof.* Following the arguments used by Bergman ([1, p. 66]) for the three-dimensional case we can show if there exists a positive, continuous function $A(\mathbf{r})$ defined in $S^0(\rho)$, such that $\varphi(\mathbf{r})$ satisfies the inequality

$$|\varphi(\mathbf{r})| \leqq A(\mathbf{r}) \max_{\|\mathbf{r}\|=\rho} |\varphi(\mathbf{r})|,$$

then $\varphi(\mathbf{r})$ may be expanded as in (4.36) where the convergence is uniform. Clearly, the existence of such a function $A(\mathbf{r})$, with the above properties, follows from the existence of a Green's function for $\Delta u + Bu = 0$ for the sphere. That this is indeed the case, when $B(r^2)$ is entire, is known to be true; see for example [4, Chap. IX, § 1].

*Remark* 7. The equation (4.34) may be rewritten as the Volterra integral equation

$$\Phi(r;\theta;\varphi) = H(r;\theta;\varphi) - \int_0^r R_1(\rho,0;r,r)H(\rho;\theta;\varphi)\,d\rho, \tag{4.37}$$

with $\Phi(r,\theta;\varphi) = r^{p/2}\varphi(\mathbf{r})$, $H(r;\theta;\varphi) = r^{p/2}h(\mathbf{r})$, and $\theta \equiv (\theta_1,\theta_2,\cdots,\theta_p)$. Hence (4.34) considered as an integral operator on the class of harmonic functions, regular in a region star-like with respect to the origin, has an inverse, namely,

$$h(r;\theta;\varphi) = \varphi(r;\theta;\varphi) + r^{-p/2}\int_0^r \Gamma(\rho,r)\rho^{p/2}\varphi(\rho;\theta;\varphi)\,d\rho, \tag{4.38}$$

where

$$\Gamma(\rho,r) \equiv \sum_{l\geqq 1} K^{(l)}(\rho,r),$$

with

$$K^{(l+1)}(\rho,r) \equiv \int_\rho^r K^{(1)}(t,r)K^{(l)}(\rho,t)\,dt \tag{4.39}$$

and $K^{(1)}(\rho,r) \equiv R_1(\rho,0;r,r)$.

The inverse kernel $\Gamma(\rho,r)$ may be seen to converge uniformly for all finite $r$ when $B(r^2)$ is entire [5, Chap. III].

LEMMA 6. *If* $\varphi(\mathbf{r})$ *is a regular solution of* (4.4) *in a domain* $D$, *star-like with respect to the origin, then the harmonic function* $h(\mathbf{r};p)$ *which generates* $\varphi(\mathbf{r})$ *by* *means of the linear operator equation* (4.34) *is also regular in D, and vice versa.*

*Proof.* This follows immediately from (4.37) and (4.38).

THEOREM 9. *Let D be star-like with respect to the origin and $B(r^2)$ an entire function, such that $B(r^2) < 0$ in D. Furthermore, let $\partial D$ be a Lyapunov boundary and $f(\mathbf{r})$ be a continuous function of $\mathbf{r} \in \partial D$. Then, there exists a unique solution of* (4.4) *such that for $\mathbf{r} \in \partial D$, $\varphi(\mathbf{r}) = f(\mathbf{r})$, which may be represented as*

$$\varphi(\mathbf{x}) = h(\mathbf{x}) - 2r\int_0^1 \sigma^{p+1}R_1(r\sigma^2, 0; r, r)h(\mathbf{x}\sigma^2)\,d\sigma, \qquad r = \|\mathbf{x}\|, \quad \mathbf{x} \in D, \tag{4.40}$$

*with*

$$h(\mathbf{x}) = \frac{\Gamma(n/2)}{\pi^{n/2}}\int_{\partial D} \mu(\mathbf{y})\frac{\partial}{\partial \nu_y}\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|^{n-2}}\right) d\omega_y, \tag{4.41}$$

*and where the double-layer density $\mu(\mathbf{y})t$ is a solution of the Fredholm integral equation*

$$\begin{aligned} f(\mathbf{x}) = \mu(\mathbf{x}) + \frac{\Gamma(n/2)}{\pi^{n/2}}\int_{\partial D} \mu(\mathbf{y})\Bigg\{&\frac{\partial}{\partial \nu_y}\left(\frac{1}{\|\mathbf{x}-\mathbf{y}\|^{n-2}}\right) \\ &- 2r\int_0^1 \sigma^{p+1}R_1(r\sigma^2, 0; r, r)\frac{\partial}{\partial \nu_y}\left(\frac{1}{\|\mathbf{x}\sigma^2 - y\|^{n-2}}\right)\Bigg\}\, d\omega_y, \\ &\mathbf{x} \in \partial D. \end{aligned} \tag{4.42}$$

*Proof.* It is well known that for $B(r^2) < 0$ and sufficiently differentiable in $D$ the Dirichlet problem is well-posed (see for instance [4, Chap. 8]); hence, there exists a unique solution satisfying the continuous boundary data of our hypothesis. The condition that the boundary be Lyapunov is sufficient that the first boundary value problem for Laplace's equation be reducible to a Fredholm integral equation by means of the double layer potential representation[4] (4.41). Equation (4.42) arises by putting (4.41) into (4.40) and computing the residue as $\mathbf{x}$ tends to a boundary point from the inside.

## REFERENCES


[1] STEFAN BERGMAN, *Integral Operators in the Theory of Linear Partial Differential Equations*, Springer-Verlag, Berlin, 1961.

[2] RICHARD COURANT AND DAVID HILBERT, *Methods of Mathematical Physics*, vol. II, John Wiley, New York, 1962.

[3] ARTHUR ERDÉLYI, *Higher Transcendental Functions*, vols. I, II, III, McGraw-Hill, New York, 1953–1955.

[4] PAUL R. GARABEDIAN, *Partial Differential Equations*, John Wiley, New York, 1964.

[5] ROBERT P. GILBERT, *Function Theoretic Methods in Partial Differential Equations*, Academic Press, New York, 1969.

[6] ROBERT P. GILBERT AND HENRY C. HOWARD, *On a class of elliptic partial differential equations*, Portugal. Math., 26 (1967), no. 3, pp. 353–373.

[7] ROBERT P. GILBERT AND CHI YEUNG LO, *On the approximation of solutions of elliptic partial differential equations in two and three dimensions*, to appear.

[8] CLAUS MÜLLER, *Spherical Harmonics*, Lecture Notes in Mathematics, vol. 17, Springer-Verlag, New York, 1966.

[9] S. L. SOBOLEV, *Partial Differential Equations of Mathematical Physics*, Pergamon Press, New York, 1964.

[10] ILYA N. VEKUA, *New Methods for Solving Elliptic Equations*, John Wiley, New York, 1967.


[4] See Sobolev [8, Lecture 15].

SIAM J. MATH. ANAL.
Vol. 1, No. 1, February 1970

# NOTE ON CONTOUR INTEGRAL REPRESENTATIONS FOR PRODUCTS OF AIRY FUNCTIONS*

N. A. LOGAN AND K. S. YEE†

J. C. P. Miller [1] has shown that when $z$ is real the differential equation

$$\frac{d^3y}{dz^3} - 4z\frac{dy}{dz} - 2y = 0 \tag{A}$$

is satisfied by products of solutions of the Airy differential equation

$$\frac{d^2y}{dz^2} - zy = 0. \tag{1}$$

In this note we derive integral representations of solutions of this equation by the Laplace transformation method. These solutions have "rotational" symmetry properties which are hard to see from other representations.

Properties of the solutions of (1) are well known. Three solutions as definite contour integrals have been given by Jeffreys and Jeffreys [2] and others:

$$Ai(z) = \frac{1}{2\pi i}\int_{L_{31}} \exp\left(\tfrac{1}{3}t^3 - zt\right) dt,$$

$$Bi(z) + iAi(z) = \frac{1}{\pi}\int_{L_{21}} \exp\left(\tfrac{1}{3}t^3 - zt\right) dt,$$

$$Bi(z) - iAi(z) = \frac{1}{\pi}\int_{L_{23}} \exp\left(\tfrac{1}{3}t^3 - zt\right) dt,$$

where the various contours are shown in Fig. 1.

Power series for $Ai(z)$ and $Bi(z)$ are given by

$$Ai(z) = \alpha g_1(z) - \beta g_2(z), \qquad Bi(z) = \sqrt{3}(\alpha g_1(z) - \beta g_2(z)), \tag{2}$$

where

$$\alpha = 3^{-2/3}/\Gamma(\tfrac{2}{3}), \qquad \beta = 3^{-1/3}/\Gamma(\tfrac{1}{3}),$$

and

$$g_1(z) = 1 + \frac{1}{3!}z^3 + \frac{1\cdot 4}{6!}z^6 + \frac{1\cdot 4\cdot 7}{9!}z^9 + \cdots,$$

$$g_2(z) = z + \frac{2}{4!}z^4 + \frac{2\cdot 5}{7!}z^7 + \cdots.$$

Integral representations of the solutions of (A) can be derived by means of the Laplace transformation method [3]. This yields the following three linearly

* Received by the editors May 20, 1969, and in revised form June 27, 1969.

† Department of Mathematics, Kansas State University, Manhattan, Kansas 66504.

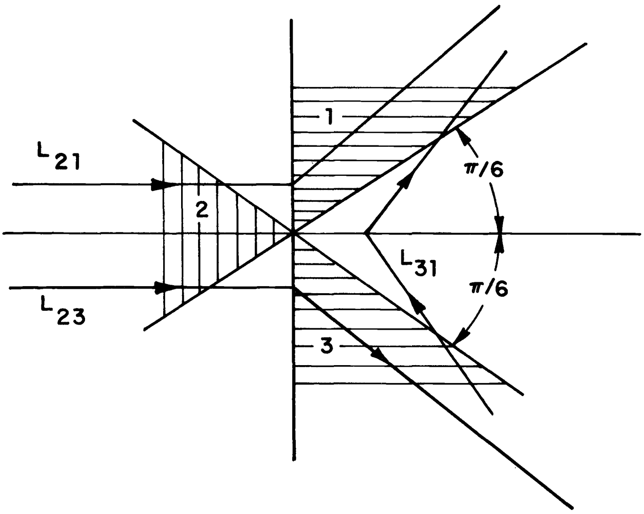


FIG. 1. *Complex t-plane*

independent solutions:

(3a) $$y_1(z) = \int_{L_{21}} \frac{1}{\sqrt{t}} \exp\left(-tz + \frac{1}{12}t^3\right) dt,$$

(3b) $$y_2(z) = \int_{L_{23}} \frac{1}{\sqrt{t}} \exp\left(-tz + \frac{1}{12}t^3\right) dt,$$

(3c) $$y_3(z) = \int_{L_{31}} \frac{1}{\sqrt{t}} \exp\left(-tz + \frac{1}{12}t^3\right) dt.$$

A branch cut is placed in the negative real axis to insure single-valuedness of the integrand; thus $-\pi < \arg t < \pi$. We first show that $y_3(z)$ is a constant multiple of $Ai^2(z)$.

From (3c) we find

(4a) $$y_3(0) = \left\{\int_{\infty e^{-\pi i/3}}^{0} + \int_{0}^{\infty e^{\pi i/3}}\right\} \frac{1}{\sqrt{t}} \exp\left(\frac{t^3}{12}\right) dt = i\int_0^\infty \frac{1}{\sqrt{x}} \exp\left(-\frac{x^3}{12}\right) dx$$

$$= i\,4(12)^{-5/6}\Gamma\left(\frac{1}{6}\right),$$

(4b) $$y_3'(0) = -\left\{\int_{\infty e^{-\pi i/3}}^{0} + \int_{0}^{\infty e^{\pi i/3}}\right\} \sqrt{t} \exp\left(\frac{t^3}{12}\right) dt = -i\,4 \cdot 3^{-1/2}\Gamma\left(\frac{1}{2}\right),$$

(4c) $$y_3''(0) = \left\{\int_{\infty e^{-\pi i/3}}^{0} + \int_{0}^{\infty e^{\pi i/3}}\right\} t^{3/2} \exp\left(\frac{t^3}{12}\right) dt = i\,4(12)^{-1/6}\Gamma\left(\frac{5}{6}\right).$$

And from the power series (2) we obtain

$$\text{(5a)} \qquad Ai^2(0) = \frac{1}{3^{4/3}\Gamma(2/3)},$$

$$\text{(5b)} \qquad \frac{d}{dx}[Ai^2(x)]_{x=0} = -\frac{2}{3^{7/6}\Gamma(1/3)\Gamma(2/3)},$$

$$\text{(5c)} \qquad \frac{d^2}{dx^2}[Ai^2(x)]_{x=0} = \frac{2}{3^{2/3}\{\Gamma(1/3)\}^2}.$$

It is easily verified that the ratio of the right sides of (4a) and (5a) is $4i\pi^{3/2}$, and that the same is true of (4b) and (5b), and also of (4c) and (5c). Hence we obtain the desired result

$$\text{(6)} \qquad y_3(z) = \int_{L_{31}} \frac{1}{\sqrt{t}} \exp(-tz + t^3/12)\, dt = i4\pi^{3/2}Ai^2(z),$$

where $L_{31}$ is the path of integration shown in Fig. 1.

The corresponding results for $y_1(z)$ and $y_2(z)$ are easily obtained by rotation of the paths of integration. This gives

$$\text{(7)} \qquad y_1(z) = e^{-2\pi i/3}y_3(ze^{2\pi i/3}).$$

Using the known relation

$$\text{(8)} \qquad Bi(z) + iAi(z) = 2e^{\pi i/6}Ai(e^{2\pi i/3}z),$$

we find

$$\text{(9)} \qquad y_1(z) = -2i\pi^{3/2}\{Bi(z) + iAi(z)\}^2;$$

similarly

$$\text{(10)} \qquad y_2(z) = 2i\pi^{3/2}\{Bi(z) - iAi(z)\}^2.$$

**Acknowledgments.** The authors wish to thank Dr. F. W. J. Olver for the improvement of the presentation of this paper. They also express their sincere appreciation to Mildred Rundquist for her skillful typing of the manuscript.

## REFERENCES

[1] J. C. P. MILLER, *The Airy Integral, Giving Tables of Solutions of Differential Equation $y'' = xy$*, Cambridge University Press, Cambridge, 1946.

[2] H. JEFFREYS AND B. JEFFREYS, *Methods of Mathematical Physics*, Cambridge University Press, Cambridge, 1950, pp. 508–511.

[3] E. L. INCE, *Ordinary Differential Equations*, Dover, New York, 1944, pp. 186–189.

SIAM J. MATH. ANAL.
Vol. 1, No. 1, February 1970

# ASYMPTOTIC EXPANSION OF LAPLACE TRANSFORMS NEAR THE ORIGIN*

RICHARD A. HANDELSMAN AND JOHN S. LEW†

**1. Introduction.** Let $f$ be a locally integrable function on $[0, \infty)$ and let $Lf$ be its Laplace transform, when this exists. Then for functions $f$ having a finite number of well-defined moments

$$\mu_m = \int_0^\infty t^m f(t)\,dt, \qquad m = 0, 1, \cdots, n, \tag{1.1}$$

it is well known that

$$[Lf](s) = \sum_{m=0}^{n} \mu_m(-s)^m/m! + o(s^n) \tag{1.2}$$

for small $s$ with $\operatorname{Re}(s) > 0$.[1] Thus for functions $f$ having all their nonnegative moments well-defined, one can obtain the standard expansion of $Lf$ by moments, which is in general an asymptotic, but not necessarily convergent, power series. (Consider, for example, $f(t) = \exp(-2t^{1/2})$.)

We point out, however, that the moment $\mu_m$ is simply the value at $m + 1$ of the Mellin transform $Mf$. Indeed, in this paper, for functions $f$ with a rather general asymptotic form near $+\infty$, we shall relate $Lf$ to $Mf$, and thereby generalize (1.2) in two ways. (a) When the integral for $Mf$ converges in some vertical strip, we will show in Theorem 5 that a function meromorphic in the right half-plane can be obtained as the continuation of $Mf$, and, using this, that an infinite expansion, not limited by any fixed $o(s^n)$, can be given near zero for $[Lf](s)$. Usually this expansion will consist of a regular part, generalizing the expansion by moments, in which the $\mu_m$ are replaced by the values of the continued $Mf$, and also a singular part, reflecting the poles of this continued $Mf$, in which the terms are obtained from the detailed form near $+\infty$ of $f$. (b) Even when the integral for $Mf$ converges nowhere, so that no positive nor negative moments of $f$ are defined at all, we will nevertheless show in Corollary 5.1 that, remarkably, a function $Mf$ can still be defined by analytic continuation, and an expansion of $Lf$ in terms of this function can be obtained as before.

Results like ours, which obtain the limiting behavior of $Lf$ from that of $f$, are called Abelian theorems, while converse results, which recover properties of $f$ from $Lf$, are called Tauberian theorems. The usual Abelian theorems give only a leading term and an error estimate [10, pp. 180–183], [4, Volume I, pp. 455–460], and only a few special results yield full series expansions [4, Volume II, pp. 97–100], [2]. We

* Received by the editors March 18, 1969.

† Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912. This work was supported in part by the Office of Naval Research under Contract NONR 562(36) and by the Department of Defense Advanced Projects Agency under Contract SD-86.

[1] Since we shall be concerned only with values of $s$ which are small, we shall use the symbol $O(s^k)$ throughout this paper (and $o(s^k)$ similarly) to mean $O(s^k)$ as $s \to 0$ for $\operatorname{Re}(s) > 0$.

obtain, on the other hand, an infinite asymptotic expansion of $Lf$, through assumptions on $f$ more restrictive than for some other Abelian theorems, yet sufficiently broad to include essentially all special functions with well-defined Mellin transforms.

We recognize that in the theory of generalized functions [6, especially Chap. 1, Section 3] there are developments which parallel some of our work and might therefore extend some of our results. However, from this theory we would probably obtain statements involving, say, the finite part of an integral or the regularization of a function, and for an explicit expansion we would then need to compute these entities, which are defined originally through analytic continuation. Thus we have chosen to use more elementary methods in our main discussion, which anticipate the final computation by explicit reliance on analytic continuation, but in concluding we have added, as Corollary 5.3, a result for a limited class of generalized functions.

**2. Fundamental results.** For any locally integrable complex-valued function $f$ on $[0, \infty)$, let $Lf$ and $Mf$ denote respectively, when they exist, the Laplace and Mellin transforms of $f$. That is, for the complex variables $s$ and $z = x + iy$, let

$$[Lf](s) = \int_0^\infty \exp(-st) f(t)\, dt, \tag{2.1}$$

$$[Mf](z) = \int_0^\infty t^{z-1} f(t)\, dt \tag{2.2}$$

whenever these defining integrals converge. For future use, it is convenient also to note that $Mf$ can be expressed as a bilateral Laplace transform, since the change of variable $t = \exp(u)$ in (2.2) yields

$$[Mf](z) = \int_{-\infty}^\infty \exp(zu) f[\exp(u)]\, du. \tag{2.3}$$

Of course, the integrals (2.1) and (2.2) need not converge anywhere, and thus need not converge absolutely; but whenever they do converge, it is well known, [10, pp. 46–48, 240–241] that $Lf$ and $Mf$ converge absolutely in regions of form $s_0 < \operatorname{Re}(s)$ and $a < \operatorname{Re}(z) < b$ respectively, where $a, b$ and $s_0$ are either real numbers or $\pm\infty$.

We shall now review, in Lemma 1, some properties of the Mellin transform $Mf$ that will be used both implicitly and explicitly below. We shall then derive in Theorem 1 and exploit in Theorem 2 a fundamental relation between the two transforms $Lf$ and $Mf$. Theorem 2 may be considered the main result of this section.

LEMMA 1. *Let $[Mf](z)$ be absolutely convergent for $a < \operatorname{Re}(z) < b$, and let $I$ be any compact interval in $(a, b)$. Then $[Mf](z)$ is holomorphic for $a < \operatorname{Re}(z) < b$ with its derivative uniformly bounded for $\operatorname{Re}(z)$ in $I$, and*

$$N(f, I; y) = \sup\{|[Mf](x + iy)| : x \in I\} \tag{2.4}$$

*is continuous for $-\infty < y < +\infty$ with $\lim_{y \to \pm\infty} N(f, I; y) = 0$.*

*Proof.* For $a < \mathrm{Re}(z) < b$ it is well known [10, pp. 240–241] that $[Mf](z)$ is holomorphic and we will not reproduce here a proof of this fact. It is also clear that

$$[Mf]'(z) = \frac{d[Mf](z)}{dz} = \int_0^\infty t^{z-1} \log t \cdot f(t)\, dt. \tag{2.5}$$

Now if $g(t) = |f(t) \log t|$ then $[Mg](z)$ is absolutely convergent and $|[Mf]'(x + iy)| \leqq [Mg](x)$ in this same strip, so that

$$|[Mf]'(z)| \leqq K = \sup \{[Mg](x) : x \in I\} \tag{2.6}$$

for $\mathrm{Re}(z)$ in $I$. However $Mg$ is continuous and $I$ is compact, so that $K$ is finite.

For any $z_1, z_2$ with $\mathrm{Re}(z_i)$ in $I$ this implies by the mean value theorem that

$$|[Mf](z_1) - [Mf](z_2)| \leqq K|z_1 - z_2|, \tag{2.7}$$

which, in turn, implies that

$$|N(f, I; y_1) - N(f, I; y_2)| \leqq K|y_1 - y_2|. \tag{2.8}$$

Moreover $[Mf](x + iy)$, for each fixed $x$ in $I$, is by (2.3) the Fourier transform of an absolutely integrable function, so that $\lim_{y \to \pm\infty} [Mf](x + iy) = 0$ pointwise by the Riemann–Lebesgue lemma. However, if $F_y(x) = [Mf](x + iy)$ for $x$ in $I$ then the family $\{F_y : -\infty < y < \infty\}$ is equicontinuous by (2.7), so that uniform and pointwise limits in $y$ coincide [7, Theorem 7.15].

THEOREM 1. *Let* $[Mf](z)$ *be absolutely convergent for* $a < \mathrm{Re}(z) < b$ *with* $a < 1$. *Then*

$$[Lf](s) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} [Mf](z)\Gamma(1 - z)s^{z-1}\, dz \tag{2.9}$$

*for any s with* $\mathrm{Re}(s) > 0$ *and any c with* $a < c < \min(1, b)$.

*Proof.* If we put $g(t) = \exp(-st)$ with $\mathrm{Re}(s) > 0$ then we find the transform, $[Mg](z) = \Gamma(z)s^{-z}$, absolutely convergent for $\mathrm{Re}(z) > 0$, so that $[Mg](1 - z)$ is holomorphic for $\mathrm{Re}(z) < 1$. Also from Stirling's approximation for $\Gamma(z)$ we can obtain the estimate [1, (6.1.45)]

$$|\Gamma(x + iy)| \sim (2\pi)^{1/2}|y|^{x-1/2} \exp(-\tfrac{1}{2}\pi|y|), \qquad |y| \to \infty, \tag{2.10}$$

so that $[Mg](1 - c - iy)$ is absolutely integrable on $-\infty < y < \infty$. But $t^{c-1}f(t)$ is absolutely integrable on $0 \leqq t < \infty$ since by hypothesis $a < c < b$, and thus

$$[Lf](s) = \int_0^\infty f(t)g(t)\, dt = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} [Mf](z)[Mg](1 - z)\, dz \tag{2.11}$$

by the Parseval theorem for the Mellin transform [9, Theorem 42]. Moreover, this identity can be continued to all $s$ for which both sides of (2.11) are well-defined and analytic.

THEOREM 2. *Let* $\mathrm{Re}(s) > 0$ *and* $\theta = \arg(s)$. *Let* $[Mf](z)$ *be absolutely convergent for* $a < \mathrm{Re}(z) < b$ *with* $a < 1$, *and be continuable to a meromorphic function in* $a < \mathrm{Re}(z)$ *such that for each compact interval I in* $(a, \infty)$ *there exists an* $\varepsilon > 0$ *for* *which* $N(f, I; y) \exp[(\varepsilon + |\theta| - \pi/2)|y|]$ *is bounded as* $|y| \to \infty$.[2] *Then for any real* $k > a$ *and* $\delta > 0$ *we have* (see footnote 1)

$$[Lf](s) = \sum_{a<\mathrm{Re}(z)<k} \mathrm{Res}\{-[Mf](z)\Gamma(1-z)s^{z-1}\} + O(|s|^{k-\delta-1}). \tag{2.12}$$

In (2.12) and hereafter, we use $Mf$ to denote not merely the given absolutely convergent Mellin transform but also the assumed meromorphic continuation, when it exists.

In other words, the series obtained when we let $k \to \infty$ is by definition [5, pp. 11–12] an asymptotic expansion of $Lf$ near $s = 0$ for $\mathrm{Re}(s) > 0$. Moreover, this series is uniformly valid in each sector $|\theta| \leqq \theta_0$ with $\theta_0 < (1/2)\pi$. (The occurrence of $\delta$ in (2.12) is to allow for possible logarithmic terms in the series.)

*Proof.* For any $k > a$ we can choose $\delta$ arbitrarily small and such that $[Mf](z)\Gamma(1 - z)$ has no poles with $k - \delta \leqq \mathrm{Re}(z) < k$. Then under the stated assumptions we can shift the contour to the right and obtain

$$\begin{aligned}[Lf](s) = &\sum_{a<\mathrm{Re}(z)<k-\delta} \mathrm{Res}\{-[Mf](z)\Gamma(1-z)s^{z-1}\} \\ &+ (2\pi i)^{-1}\int_{k-\delta-i\infty}^{k-\delta+i\infty} [Mf](z)\Gamma(1-z)s^{z-1}\,dz.\end{aligned} \tag{2.13}$$

However an upper bound for the last term is clearly

$$(2\pi)^{-1}|s|^{k-\delta-1}\int_{k-\delta-i\infty}^{k-\delta+i\infty} |[Mf](z)\Gamma(1-z)|\cdot\exp(-\theta y)|dz| \tag{2.14}$$

in which the integral is absolutely convergent and bounded on each sector $|\theta| \leqq \theta_0$ by (2.10) and our hypotheses.

COROLLARY 2.1. *If* $[Mf](z)\Gamma(1 - z)$ *has a pole at* $z_0$ *with* $a < \mathrm{Re}(z_0) < b$, *and if its Laurent expansion at* $z_0$ *has singular part* $\sum_{m=0}^{n} a_m(z - z_0)^{-m-1}$, *then to our expansion of* $[Lf](s)$ *the point* $z_0$ *contributes the terms*

$$-s^{z_0-1}\sum_{m=0}^{n} a_m(\log s)^m/m!. \tag{2.15}$$

*Proof.* The expression $a_m s^{z_0-1}(\log s)^m/m!$ is precisely the residue at $z = z_0$ of

$$a_m(z - z_0)^{-m-1}s^{z-1} = s^{z_0-1}a_m(z - z_0)^{-m-1}\exp[(z - z_0)\log s]. \tag{2.16}$$

COROLLARY 2.2. *If no positive integer* $m$ *is a pole of* $[Mf](z)$, *then each such* $m$ *is a simple pole of* $[Mf](z)\Gamma(1 - z)$, *and to our expansion of* $[Lf](s)$ *contributes the term* $[Mf](m)(-s)^{m-1}/(m - 1)!$. *If* $z_0(\neq 1, 2, \cdots)$ *is a simple pole of* $[Mf](z)$, *then it is a simple pole of* $[Mf](z)\Gamma(1 - z)$, *and contributes a term* $-\mathrm{Res}\{[Mf](z); z_0\} \cdot\Gamma(1 - z_0)s^{z_0-1}$.

*Proof.* The points $z = m = 1, 2, \cdots$ are the singular points of $\Gamma(1 - z)$, and are simple poles with residues $(-1)^m/\Gamma(m)$.

---

[2] In the next section we will show that these seemingly artificial restrictions on $[Mf](z)$ are in fact satisfied for a large class of functions $f(t)$, including all those with asymptotic form near $\infty$:

$$f(t) \sim \exp(-ct^p)\sum_{m,n=0}^{\infty} c_{mn}t^{r_m}(\log t)^n.$$

*Example* 2.1. For any complex number $r$ let

$$f(t) = \begin{cases} 0 & \text{on } [0, 1), \\ t^{r-1} & \text{on } [1, \infty). \end{cases} \tag{2.17}$$

Then $[Mf](z)$ is absolutely convergent to $-1/(z + r - 1)$ on the left half-plane $\operatorname{Re}(z + r) < 1$, so that $[Mf)(z)$ is trivially continuable into the entire complex plane as a function with a pole only at $z = 1 - r$. Thus, by Corollary 2.2 for $r \neq 0, -1, -2, \cdots$, we have

$$[\mathrm{Lf}](s) \sim \Gamma(r)s^{-r} - \sum_{n=0}^{\infty} (n + r)^{-1}(-s)^n/n!; \tag{2.18}$$

and by Corollary 2.1 for $r = -m = 0, -1, -2, \cdots$, we have

$$[\mathrm{Lf}](s) \sim [\log s - \psi(m + 1)](-s)^m/m! - \sum_{m \neq n = 0}^{\infty} (n - m)^{-1}(-s)^n/n!, \tag{2.19}$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$.

Now $[Lf](s) = s^{-r}\Gamma(r, s)$, where $\Gamma(r, s)$ is an incomplete gamma function. If $r \neq 0, -1, -2, \cdots$, then $[Lf](s)$ can be written

$$[Lf](s) = s^{-r}\Gamma(r) - r^{-1}M(r, r + 1; -s), \tag{2.20}$$

where $M(a, c; z)$ is Kummer's solution of the confluent hypergeometric equation. If $r = 0, -1, -2, \cdots$, then $[Lf](s)$ can be expressed in terms of the exponential integral

$$E_1(z) = \int_z^{\infty} t^{-1} \exp(-t)\, dt. \tag{2.21}$$

However, in both cases, the standard series for these special functions [6, (5.1.11) and (13.1.2)], yield agreement with the expansions (2.18) and (2.19).

We finally note that Theorem 2 is a nontrivial extension of the expansion of $[Lf](s)$ by moments, since this result holds even for large positive $r$, in which case no moments of $f(t)$ are defined.

**3. Sufficient conditions for expansion.** For those locally integrable functions $f$ which satisfy the hypotheses of Theorem 2, we have just obtained an expansion near $s = 0$ of the Laplace transform $[Lf](s)$. For $\operatorname{Re}(s) > 0$ and $s \to 0$, the resulting series has been shown to be asymptotic, and through a later example it will be found not always convergent. The object of this section is to provide more constructive and explicit conditions on $f$ than those assumed in Theorem 2 for this expansion to be valid. In order to accomplish this, it will be convenient to split the function $f$ at $t = 1$, and to put

$$\begin{aligned} f_1(t) &= f(t) \quad \text{on } [0, 1), \quad 0 \text{ on } [1, \infty), \\ f_2(t) &= 0 \qquad \text{on } [0, 1), \quad f(t) \text{ on } [1, \infty). \end{aligned} \tag{3.1}$$

If the domains of definition for $Mf_1$ and $Mf_2$ overlap, then clearly $Mf = Mf_1 + Mf_2$ on the common domain. We, of course, are concerned here with the expansions for small $s$ of $Lf_1$ and $Lf_2$. It is well known that the expansion by moments of $[Lf_1](s)$ holds although the assumptions of Theorem 2 are not quite satisfied. We will, however, recover this expansion of $[Lf_1](s)$ through the use of Theorem 2, so that combination with corresponding theorems on $[Lf_2](s)$ can then be easily achieved in the following section.

LEMMA 2. *If $f(t)$ is locally integrable on $[0, \infty)$, then $[Mf_1](z)$ is absolutely convergent and hence holomorphic for $\operatorname{Re}(z) > 1$, though not necessarily for $\operatorname{Re}(z) \geqq 1$. If $f(t)$ is also continuous at $t = 0$ then $[Mf_1](z)$ is absolutely convergent for $\operatorname{Re}(z) > 0$.*

*Proof.* The representation (2.3) exhibits $[Mf_1](z)$ as a one-sided Laplace transform, for which these results are well known.

THEOREM 3. *Let $f(t)$ be locally integrable on $[0, \infty)$, and let $k$ be any positive real number. Then for $\operatorname{Re}(s) > 0$*

$$(3.2)\qquad \begin{aligned}[Lf_1](s) &= [Mf_1](1) + \sum_{1 < \operatorname{Re}(z) < k+1} \operatorname{Res}\{-[Mf_1](z)\Gamma(1-z)s^{z-1}\} + O(|s|^k) \\ &= \sum_{n=0}^{[k]} [Mf_1](n+1)(-s)^n/n! + O(|s|^k).\end{aligned}$$

Here $[k]$ denotes the greatest integer less than $k$.

*Proof.* The assumptions of Theorem 2 are not quite satisfied, since $[Mf_1](z)$, by Lemma 2, may converge at no points with $\operatorname{Re}(z) < 1$. However for $\varepsilon > 0$, by Lebesgue's theorem

$$(3.3)\qquad [Lf_1](s) = \lim_{\varepsilon \to 0} \int_0^1 \exp(-st)t^\varepsilon f(t)\,dt$$

and hence for $1 - \varepsilon < \operatorname{Re}(z)$, by Lemma 2, the Mellin transform of $t^\varepsilon f(t)$ converges absolutely to $Mf_1(z + \varepsilon)$. Thus we can let $1 - \varepsilon < c < 1 < c' < 2$ in Theorem 1 and obtain

$$(3.4)\qquad \begin{aligned}[Lf_1](s) &= \lim_{\varepsilon \to 0} (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} [Mf_1](z+\varepsilon)\Gamma(1-z)s^{z-1}\,dz \\ &= \lim_{\varepsilon \to 0} \left\{[Mf_1](z+\varepsilon) + (2\pi i)^{-1} \int_{c'-i\infty}^{c'+i\infty} [Mf_1](z+\varepsilon)\Gamma(1-z)s^{z-1}\,dz\right\}.\end{aligned}$$

The preceding shift in contour is permissible by Lemma 1, and we may now continue as in Theorem 2. Also $[Mf_1](z)$ is continuous on $[1, \infty)$, and hence we can put $\varepsilon = 0$ in (3.4). However the integrand in (3.4) has poles only at positive integers, so that if $k \neq 1, 2, \cdots$, then the $\delta > 0$ of Theorem 2 is unnecessary and our error estimate is $O(|s|^k)$. Moreover, the integrand has only simple poles, so that if $k = 1, 2, \cdots$, then the contour may be shifted to $k + \delta + 1$, and our error estimate is $[Mf_1](k+1)(-s)^k/k! + O(|s|^{k+\delta})$.

Theorem 3 is valid for arbitrary $f_1$, but the transform $Mf_2$ need not even exist for arbitrary $f_2$. However, if $f(t) = O(t^b)$ as $t \to \infty$ for some real $b$, then $[Mf_2](z)$ will converge absolutely for $\operatorname{Re}(z) < b$; but without further restrictions $[Mf_2](z)$ may still not be continuable as assumed in Theorem 2. Thus we require in the next two lemmas that as $t \to \infty$,

$$(3.5)\qquad f(t) \sim \exp(-ct^p) \sum_{m,n=0}^{\infty} c_{mn} t^{r_m} (\log t)^n.$$

Here $p > 0$, $\operatorname{Re}(c) \geqq 0$, $\operatorname{Re} r_m \downarrow -\infty$ as $m \to \infty$, and the set $\{n : c_{mn} \neq 0\}$ is finite for each $m$. For any real $k$ we also define

$$g(k, t) = \begin{cases} 0, & 0 \leqq t < 1, \\ \displaystyle\sum_{\operatorname{Re}(r_m + k) > 0} \{\exp(-ct^p) c_{mn} t^{r_m} (\log t)^n\}, & 1 \leqq t < \infty, \end{cases} \tag{3.6}$$

and

$$f^*(k, t) = f(t) - g(k, t). \tag{3.7}$$

LEMMA 3. *If* $\operatorname{Re}(c) > 0$, *then* $[Mf_2](z)$ *is absolutely convergent and hence holomorphic on the entire z-plane. If* $\operatorname{Re}(c) = 0$ *then* $[Mf_2](z)$ *is absolutely convergent at least for* $\operatorname{Re}(z + r_0) < 0$.

*Proof.* By assumption, for any $\delta > 0$ there exists $K > 0$ such that

$$|f_2(t)| \leqq K \exp[-\operatorname{Re}(c) t^p] t^{\operatorname{Re}(r_0) + \delta}, \qquad 0 \leqq t < \infty. \tag{3.8}$$

The result then follows from Lemma 1 and the estimate

$$\int_0^\infty |t^{z-1} f_2(t)| \, dt \leqq K \int_1^\infty \exp[-\operatorname{Re}(c) t^p] t^{\operatorname{Re}(z + r_0) + \delta - 1} \, dt. \tag{3.9}$$

LEMMA 4. *If* $c = 0$ *then* $[Mf_2](z)$ *can be continued to a meromorphic function on the entire z-plane, with poles at* $z = -r_m$, *and for any compact interval* $I$ *in* $(-\infty, \infty)$

$$\lim_{y \to \pm\infty} N(f, I; y) = 0, \tag{3.10}$$

*where* $N(f, I; y)$ *is defined by* (2.4). *If* $c = i\gamma$ *for some real* $\gamma \neq 0$, *then* $[Mf_2](z)$ *can be continued to a holomorphic function on the entire z-plane; and for any compact interval* $I$ *in* $(-\infty, \infty)$ *there exists an integer* $n(I)$, *depending only on the upper limit of* $I$, *such that*

$$N(f, I; y) = O(|y|^{n(I)}) \quad \text{as } |y| \to \infty. \tag{3.11}$$

*Proof.* For any $k > -\operatorname{Re}(r_0)$ we will show that $[Mf_2](z)$ has the asserted properties in the region $\operatorname{Re}(z) < k$. In both of the cases to be treated we write $f(t) = g(k, t) + f^*(k, t)$ and recall that $g(k, t)$ is a finite linear combination of terms of the form $\exp(-ct^p) t^{r_m} (\log t)^n$ with $\operatorname{Re}(r_m + k) > 0$. It then follows from (3.5), (3.7) and Lemma 3 that $[Mf^*](z)$ is absolutely convergent for $\operatorname{Re}(z) < k$. Also for any compact interval $I$ in $(-\infty, k)$ we see by Lemma 1 that

$$\lim_{y \to \pm\infty} N(f^*, I; g) = 0, \tag{3.12}$$

and hence the asserted properties of $Mf_2$ depend only on $g$.

If $c = 0$ and $\operatorname{Re}(z + r_m) < 0$ then we compute

$$\int_1^\infty t^{z + r_m - 1} (\log t)^n \, dt = \frac{\Gamma(n + 1)}{(-z - r_m)^{n+1}}. \tag{3.13}$$

Thus $[Mg](z)$, being a finite linear combination of terms of the form (3.13) for $\operatorname{Re}(z)$ sufficiently negative, is continuable to the entire $z$-plane as a meromorphic function with poles at $z = -r_m$ in $\operatorname{Re}(z) < k$. Moreover, $N(f, I; y)$ has the required limit since from (3.13),

(3.14) $$\lim_{y \to \pm\infty} N(g, I; y) = 0.$$

If $c = i\gamma \neq 0$ and $u = t^p$ then we can write

(3.15) $$\int_1^\infty \exp(-ct^p) t^{z+r_m-1} (\log t)^n \, dt = p^{-n-1} G(z + r_m, n),$$

(3.16) $$G(w, n) = \int_1^\infty \exp(-cu) u^{(w/p)-1} (\log u)^n \, du$$

and observe that $G(w, n)$ converges absolutely for $\operatorname{Re}(w) < 0$. But, upon integrating by parts in (3.16), we obtain

(3.17) $$G(w,n) = c^{-1}[\exp(-c)\delta_{0n} + p^{-1}(w - p)G(w - p, n) + nG(w - p, n - 1)],$$

where $\delta_{0n}$ is the Kronecker delta. Then, upon using (3.17) we can write (3.15), which converges absolutely for $\operatorname{Re}(z + r_m) < 0$, as a sum of integrals which converge absolutely for $\operatorname{Re}(z + r_m) < p$. Thus by induction we can continue $[Mg](z)$ in a finite number of steps, say $l$ steps, to a function holomorphic for $\operatorname{Re}(z) < k$. However, at each step, we multiply certain terms in the continuation by linear factors in $z$. Hence the form obtained after $l$ steps, with $\operatorname{Re}(z)$ in $I$, will be a linear combination of integrals which are bounded by Lemma 1 and whose coefficients are polynomials in $z$ of degree $\leqq l$. Therefore, we have that $N(g, I, y) = O(|y|^l)$, $|y| \to \infty$.

THEOREM 4. *Let $f(t)$ be locally integrable on $[0, \infty)$ and as $t \to \infty$ let*

(3.18) $$f(t) \sim \exp\left[-c \sum_{l=0}^{\infty} a_l t^{p-lq}\right] \sum_{m,n=0}^{\infty} c_{mn} t^{r_m} (\log t)^n,$$

*where $a_0 = 1$, $\operatorname{Re}(c) \geqq 0$, $p$ and $q > 0$, $\operatorname{Re}(r_m) \downarrow -\infty$ as $m \to \infty$ and the set $\{n : c_{mn} \neq 0\}$ is finite for each $m$. Then $[Mf_2](z)$ is absolutely convergent at least for* $\operatorname{Re}(z + r_0) < 0$ *and is continuable to the entire $z$-plane as required in Theorem* 2. *Thus for* $\operatorname{Re}(s) > 0$ *and for any $k > -\operatorname{Re}(r_0)$, $\delta > 0$,*

(3.19) $$[Lf_2](s) = \sum_{-\operatorname{Re}(r_0) \leqq \operatorname{Re}(z) < k} \operatorname{Res}\{-[Mf_2](z)\Gamma(1 - z)s^{z-1}\} + O(|s|^{k-\delta-1}).$$

*Proof.* If $t(u)$ is a strictly increasing $C^\infty$ function from $[1, \infty)$ onto $[1, \infty)$ then $[Mf_2](z)$ can be rewritten

(3.20) $$[Mf_2](z) = \int_1^\infty [t(u)]^{z-1} f[t(u)] (dt/du) \, du.$$

We can construct [5, pp. 22–24] a function $t(u)$ such that as $u \to \infty$, $dt/du$ has an expansion given by

(3.21) $$dt/du \sim \sum_{j=0}^{\infty} b_j u^{-jq}, \qquad b_0 = 1,$$

and such that the change of variable $t \to t(u)$ takes $\sum_{l=0}^\infty a_l t^{p-lq}$ into $u^p$. Then by substitution $f[t(u)]\, dt/du$ has an asymptotic expansion of the form (3.5), and $h(u) = [t(u)/u]^{z-1}$ has an asymptotic power series in $u^{-q}$ whose coefficients are polynomials in $z$. For any $k > -\operatorname{Re}(r_0)$ we can truncate the series for $h(u)$ after so many terms that the part of (3.20) which contains the remainder is absolutely convergent for $\operatorname{Re}(z) < k$. The preceding terms in this series all yield integrals of the type considered in Lemma 4 each multiplied by a polynomial in $z$. Thus the integral (3.20) satisfies the conclusions of Lemma 4.

In (3.18) the terms $a_l t^{p-lq}$ with $p \leqq lq$ are redundant, since in

$$\exp\left[-c \sum_{[(p/q)+1]}^{\infty} a_l t^{p-lq}\right] \sum_{m,n=0}^{\infty} c_{mn} t^{r_m} (\log t)^n$$

formal series expansion and multiplication yield a result like the second sum alone. Thus in (3.18) the first sum can always be assumed finite with positive exponents only. We point out that for a linear differential equation with analytic coefficients, the solutions near an irregular singular point at $\infty$ are known [3, pp. 168–169] to have asymptotic expansions of the form (3.18) with this restriction. Thus Theorem 4 includes all such $f$, and hence effectively all standard functions $f$, for which $[Mf_2](z)$ exists.

Our concluding results, which recombine $Lf_1$ and $Lf_2$, will be presented in the next section, together with two illustrative examples.

**4. Conclusions.** We shall now assemble the results of the previous sections and state the immediate conclusion as Theorem 5. We shall then add several corollaries which extend this theorem to certain functions having no Mellin transform, and to certain generalized functions. Also two examples will be given to illustrate these results. The first of these will show that, in general, our expansion of $[Lf](s)$ is no more than asymptotic. Throughout this section we shall use the functions $f_1$ and $f_2$ defined by (3.1).

THEOREM 5. *Let $f(t)$ be a finite linear combination of functions each locally integrable on* $[0, \infty)$ *with the asymptotic form* (3.18), *and let* $[Mf](z)$ *be absolutely convergent for $a < \operatorname{Re}(z) < b$. If $l > 1$, $\delta > 0$, and* $\operatorname{Re}(s) > 0$, *then*

$$[Lf](s) = [Mf](1) + \sum_{b \leqq \operatorname{Re}(z) \leqq 1} \operatorname{Res}\{-[Mf_2](z)\Gamma(1-z)s^{z-1}\} \tag{4.1}$$
$$+ \sum_{1 < \operatorname{Re}(z) < l} \operatorname{Res}\{-[Mf](z)\Gamma(1-z)s^{z-1}\} + O(|s|^{l-\delta-1}).$$

*Proof.* By hypothesis $[Mf_1](z)$ and $[Mf_2](z)$ are absolutely convergent and thus holomorphic for $a < \operatorname{Re}(z)$ and $\operatorname{Re}(z) < b$, respectively. However we can continue $Mf_2$ by Theorem 4, and can thus continue $Mf = Mf_1 + Mf_2$, to a meromorphic function on $a < \operatorname{Re}(z)$ which satisfies the conditions of Theorem 2. Moreover by Lemma 2 we may select $a \leqq 1$. Hence $Mf$ is defined for $1 < \operatorname{Re}(z) < l$ so that the right-hand side of (4.1) is also defined. Now $Mf_2$ has no poles with $\operatorname{Re}(z) < b$, and we may choose $c'$ with $1 < c' < 2$ so that $Mf_2$ has no poles with $1 < \operatorname{Re}(z) \leqq c'$. It then follows from Theorem 2 that

$$[Lf_2](s) = \sum_{b \leqq \operatorname{Re}(z) < c'} \operatorname{Res}\{-[Mf_2](z)\Gamma(1-z)s^{z-1}\} \tag{4.2}$$
$$+ (2\pi i)^{-1} \int_{c'-i\infty}^{c'+i\infty} [Mf_2](z)\Gamma(1-z)s^{z-1}\, dz.$$

Upon adding (3.4) to (4.2) with the same $c'$ we find

$$[Lf](s) = [Mf](1) + \sum_{b \leqq \operatorname{Re}(z) \leqq 1} \operatorname{Res}\{-[Mf_2](z)\Gamma(1-z)s^{z-1}\} + (2\pi i)^{-1} \int_{c'-i\infty}^{c'+i\infty} [Mf](z)\Gamma(1-z)s^{z-1}\,dz, \tag{4.3}$$

from which we get (4.1) by shifting the contour as in Theorem 2.

*Example* 4.1. For any $\beta > -\frac{1}{4}$ and $\operatorname{Re}(\alpha) > -\beta$ let

$$f(t) = t^{\alpha-1}J_{2\beta}(2t^{1/2}). \tag{4.4}$$

Then by the change of variable $t = u^2$ and an identity in Bessel functions [1, (11.4.16)],

$$[Mf](z) = \Gamma(\alpha + \beta + z - 1)/\Gamma(\beta - \alpha - z + 2), \tag{4.5}$$

for $1 - \beta < \operatorname{Re}(z + \alpha) < 5/4$. However, as predicted by Lemma 4, and as seen from (4.5), $Mf$ can be continued to a holomorphic function on $(1 - \beta) < \operatorname{Re}(z + \alpha)$ with a zero at $z = \beta - \alpha + 2$. Thus by Theorem 5 we have

$$[Lf](s) \sim \sum_{n=0}^{\infty} \Gamma(\alpha + \beta + n)(-s)^n/\Gamma(\beta - \alpha - n + 1)\Gamma(n + 1). \tag{4.6}$$

On the other hand we find directly [1, (11.4.28)] that

$$[Lf]\,(s) = [\Gamma(\alpha + \beta)/\Gamma(2\beta + 1)]s^{-\alpha-\beta}M(\alpha + \beta, 2\beta + 1; +1/s), \tag{4.7}$$

where $M(a, c; z)$ is again Kummer's solution of the confluent hypergeometric equation. As is well known, $M(a, c; z)$ has no convergent expansion near $z = \infty$, but only asymptotic expansions in various sectors. If we substitute into (4.7) the asymptotic expansion of $M(a, c; z)$ valid for large negative $z$ [1, (13.5.1)], then we find

$$[Lf](s) \sim \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + \beta + n)(-s)^n}{\Gamma(\beta - \alpha - n + 1)\Gamma(n + 1)} + \frac{\exp[-i\pi(\alpha + \beta) - s^{-1}]}{\Gamma(2\beta + 1)} \cdot \sum_{n=0}^{\infty} \frac{\Gamma(1 - \alpha + \beta + n)\Gamma(1 - \alpha - \beta + n)}{\Gamma(1 - \alpha + \beta)\Gamma(1 - \alpha - \beta)} \frac{(-s)^{n+1-2\alpha}}{\Gamma(n + 1)}. \tag{4.8}$$

Upon comparing (4.5) and (4.8) we can conclude that our procedure in this instance yields an asymptotic expansion of $[Lf](s)$ and, as indicated by its error estimates, does not recover terms which are exponentially small.

We now point out that the conclusions of Theorem 5 will often hold even when the integral for $[Mf](z)$ converges nowhere. Indeed by Lemma 2 we know that $[Mf_1](z)$ converges absolutely in the region $a < \operatorname{Re}(z)$ for some $a \leqq 1$. Thus if $[Mf_2](z)$ can be continued to the entire $z$-plane, then we can define

$$[Mf](z) = [Mf_1](z) + [Mf_2](z), \qquad a < \operatorname{Re}(z), \tag{4.9}$$

even when (2.1), the integral for $[Mf](z)$, does not converge. By using this generalized definition of the Mellin transform we shall now obtain Corollary 5.1 as an extension of Theorem 5, and Corollary 5.2 as an important special case.

COROLLARY 5.1. *Let $f(t)$ be a finite linear combination of functions each locally integrable on $[0, \infty)$ with asymptotic form* (3.18), *and let $[Mf](z)$ be defined by* (4.9) *whether or not the integral for $[Mf](z)$ converges anywhere. If $l > 1$, $\delta > 0$ and* $\operatorname{Re}(s) > 0$, *then* (4.1) *holds for any $b$ such that $f(t) = O(t^{-b})$ as $t \to \infty$.*

*Proof.* For $\operatorname{Re}(z) < b$, $Mf_2$ is absolutely convergent and hence holomorphic. We may then proceed as in the proof of Theorem 5 to obtain (4.1) since nothing in that proof required $a < b$.

COROLLARY 5.2. *Let $f(t)$ be a locally integrable function on $[0, \infty)$ such that*

$$f(t) \sim \sum_{m=0}^{\infty} c_m t^{r_m} \quad \text{as } t \to \infty, \tag{4.10}$$

*where* $\operatorname{Re}(r_m) \downarrow -\infty$. *If $Mf$ is defined by* (4.9) *and if no $r_m = -1, -2, \cdots$, then*

$$[Lf](s) \sim \sum_{m=0}^{\infty} c_m \Gamma(r_m + 1) s^{-r_m - 1} + \sum_{n=0}^{\infty} [Mf](n + 1)(-s)^n/n!. \tag{4.11}$$

*Proof.* By Lemma 4 we find that $Mf_2$ has singular points only at $z = -r_m$ for $m = 0, 1, 2, \cdots$, and that these are all simple poles with residues $-c_m$. Thus the expansion follows by Corollary 5.1 as in Corollary 2.2.

*Example* 4.2. For any real $\nu$ let

$$f(t) = (1 + t^2)^{\nu - 1/2} \tag{4.12}$$

and note that Corollary 5.2 applies whenever $2\nu \neq$ integer, since

$$f(t) = t^{2\nu - 1} \sum_{m=0}^{\infty} \binom{\nu - \frac{1}{2}}{m} t^{-2m}, \qquad t \to \infty. \tag{4.13}$$

Then by definition (4.9), $Mf$ can be shown meromorphic in $z$ for $0 < \operatorname{Re}(z)$, with poles at $z = 2m + 1 - 2\nu$, $m = 0, 1, 2, \cdots$, and $Mf$ can be shown meromorphic in $\nu$ for complex $\nu$, with poles at $\nu = m + (1 - z)/2$ for each fixed $z$. However if $\operatorname{Re}(\nu - \frac{1}{2}) < 0 < \operatorname{Re}(z) < \operatorname{Re}(1 - 2\nu)$ then the integral for $[Mf](z)$ converges to yield [1, (6.2.1)]

$$[Mf](z) = \Gamma(\tfrac{1}{2}z)\Gamma(\tfrac{1}{2} - \tfrac{1}{2}z - \nu)/\Gamma(\tfrac{1}{2} - \nu), \tag{4.14}$$

so that for all other $\nu$ (4.14) holds by analytic continuation except at the singular points. Thus by Corollary 5.2, for $2\nu \neq$ integer,

$$\begin{aligned} [Lf](s) \sim & \sum_{m=0}^{\infty} \binom{\nu - \frac{1}{2}}{m} \Gamma(2\nu - 2m) s^{2m - 2\nu} \\ & + \sum_{n=0}^{\infty} \Gamma(\tfrac{1}{2}n + \tfrac{1}{2})\Gamma(-\tfrac{1}{2}n - \nu)(-s)^n/\Gamma(n + 1)\Gamma(\tfrac{1}{2} - \nu). \end{aligned} \tag{4.15}$$

In particular (4.15) holds for $\nu > \frac{1}{2}$ although $[Mf](z)$ then converges nowhere. However [1, (12.1.8)], for all $\nu > -\frac{1}{2}$,

$$[Lf](s) = \tfrac{1}{2}\Gamma(\tfrac{1}{2})\Gamma(\nu + \tfrac{1}{2})(s/2)^{-\nu}[H_\nu(s) - Y_\nu(s)], \tag{4.16}$$

where $H_\nu(s)$ is the Struve function and $Y_\nu(s)$ the Bessel function of second kind.

Finally, when the standard series for these functions [1, (12.1.3)], are inserted into (4.16), we obtain agreement with the expansion (4.15).

The form of our conclusions suggests that they might also hold for suitable generalized functions, but an optimal result of this kind would require a corresponding generalized theory of Mellin transforms [11, Chap. 4]. That is, we would need to introduce suitable Mellin test functions on $[0, \infty)$, and to obtain a space of generalized functions $f$, growing slowly at 0 and $\infty$, for which $Mf$ could be conveniently defined. Such a theory will not be developed or utilized here; but for a limited class of generalized functions we can obtain directly the following extension of Corollary 5.1.

COROLLARY 5.3. *Let $f(t)$ be a generalized function on $[0, \infty)$ which is locally integrable outside a compact interval $[\alpha, \beta]$ in $(0, \infty)$ and is a finite linear combination of functions with asymptotic form* (3.18). *Then $Mf$ satisfies the assumptions of Theorem* 2, *whence $Lf$ satisfies the conclusions of Theorem* 5.

*Proof.* By hypothesis $f = g + h$, where $g$ is a function of the kind treated in Corollary 5.1 and $h$ is a generalized function with support contained in $[\alpha, \beta]$. Thus, by linearity, we need only prove that the expansion by moments is valid for $h$. However, a well-known theorem [8, p. 91], states that $h$ is the $n$th derivative of a continuous function $H$ for some nonnegative integer $n$, where $H$ may be assumed the zero function on $[0, \alpha)$ and a polynomial of degree $n - 1$ on $[\beta, \infty)$. Thus

$$\begin{aligned} [Mh](z) &= \int_0^\infty t^{z-1} h(t)\, dt \\ &= (1 - z)(2 - z) \cdots (n - z)\left[\int_\alpha^\beta + \int_\beta^\infty\right] t^{z-n-1} H(t)\, dt. \end{aligned} \tag{4.17}$$

Now the integral on $[\alpha, \beta]$ is absolutely convergent for any $z$, while the integral on $[\beta, \infty)$ yields a rational function of $z$ with poles at $z = 1, \cdots, n$, all of which are canceled by the factor $(1 - z) \cdots (n - z)$. Hence $Mh$ is holomorphic on the entire $z$-plane, and satisfies the assumptions of Theorem 2. Finally, Theorem 1 holds for $h$, since the Parseval theorem may be used to define $Mh$ [6, pp. 166–168], [8, p. 250], so that the expansion of $h$ may be obtained from Theorem 2.

In Corollary 5.3 we could prove by a different argument that the expansion of $Lh$ by moments is actually convergent, but this must then be combined with an expansion of Lg which, we have seen, may be no more than asymptotic. Obvious examples for Corollary 5.3 are the finite derivatives of any delta function or any $H(t)$, but these have few novel features and will not be considered here.

**Acknowledgments.** The authors wish to acknowledge with thanks a number of helpful conversations with Professor M. L. Cartwright concerning the results of this paper and their possible extensions.

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions*, U.S. Government Printing Office, Washington, 1964.

[2] W. A. BEYER AND L. HELLER, *Analytical continuation of Laplace transforms by means of asymptotic series*, J. Math. Phys., 8 (1967), pp. 1004–1018.

[3] E. A. CODDINGTON AND N. LEVINSON, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[4] G. DOETSCH, *Handbuch der Laplace-Transformation*, vols. I, II, III, Birkhäuser Verlag, Basel und Stuttgart, 1950, vol. I; 1955, vol. II; 1956, vol. III.
[5] A. ERDÉLYI, *Asymptotic Expansions*, Dover, New York, 1956.
[6] I. M. GELFAND AND G. E. SHILOV, *Properties and Operations*, Generalized Functions, vol. 1, Academic Press, New York, 1964.
[7] J. L. KELLEY, *General Topology*, Van Nostrand, New York, 1955.
[8] L. SCHWARTZ, *Théorie des distributions*, 2nd ed., Hermann, Paris, 1966.
[9] E. C. TITCHMARSH, *Introduction to the Theory of Fourier Integrals*, 2nd ed., Oxford University Press, London, 1948.
[10] D. V. WIDDER, *The Laplace Transform*, Princeton University Press, Princeton, 1941.
[11] A. H. ZEMANIAN, *Generalized Integral Transformations*, Interscience, New York, 1968.

SIAM J. MATH. ANAL.
Vol. 1, No. 1, February 1970

# ON THE HAHN POLYNOMIALS*

M. WAYNE WILSON†

**Abstract.** The Hahn Polynomials are discrete analogues of the Jacobi polynomials. Here we try to ascertain the depth of the analogy, by examining the relation between these two sets. We also obtain bounds on the integral of the Hahn polynomial which corresponds to the Legendre polynomial. The paper contains certain implications for least squares curve fitting applications.

**1. Introduction.** The Hahn polynomials

$$Q_n(x;\alpha,\beta,N+1) \equiv {}_3F_2(-n,-x,n+\alpha+\beta+1;\alpha+1,-N;1),$$

for $0 \leqq n \leqq N$ are a discrete analogue of the Jacobi polynomials,

$$P_n(x;\alpha,\beta) \equiv {}_2F_1(-n,n+\alpha+\beta+1;\alpha+1;(1-x)/2).$$

(In our normalization, $P_n(1,\alpha,\beta) = 1$. The conventional polynomial $P_n^{(\alpha,\beta)}(x)$ is $\binom{n+\alpha}{n}P_n(x;\alpha,\beta)$.) Karlin and McGregor [4], whose notation we will generally follow, give several properties of these polynomials, including the relation,

$$\lim_{N\to\infty} Q_n(Nt;\alpha,\beta,N+1) = P_n(1-2t;\alpha,\beta), \tag{1.1}$$

uniformly on compact sets of the complex plane. For computing applications, the discrete orthogonality relation of the Hahn polynomials,

$$\sum_{x=0}^{N} Q_n(x;\alpha,\beta,N+1)Q_m(x;\alpha,\beta,N+1)\rho(x) = 0, \qquad m \neq n, \tag{1.2}$$

where

$$\rho(x) = \rho(x;\alpha,\beta,N+1) = \frac{\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}}{\binom{N+\alpha+\beta+1}{N}}$$

is preferable to the continuous relation,

$$\int_0^1 t^\alpha(1-t)^\beta P_n(1-2t;\alpha,\beta)P_m(1-2t;\alpha,\beta)\,dt = 0, \qquad m \neq n, \tag{1.3}$$

of the Jacobi polynomials. We note also that the polynomials, $Q_n(x;0,0,N) \equiv Q_n(x;N)$, are the classical Chebyshev or Gram or "orthogonal" polynomials used in least squares approximation of data over a set of $N$ equidistant points; see Jordan [3], for example.

---

* Received by the editors May 22, 1969.

† IBM Thomas J. Watson Research Center, Yorktown Heights, New York. Now at IBM Scientific Center, Houston, Texas 77025. This work was supported in part by the Office of Naval Research under Contract NONR-652 (36) at Brown University.

In § 2, we examine the relation between the two sets of polynomials more closely than is expressed by (1.1). We develop and examine the asymptotic (in $N$) series,

$$Q_n(Nt;\alpha,\beta,N+1) = P_n(1-2t;\alpha,\beta) + \sum_{k=1}^{\infty} \frac{\varphi_k(t,n,\alpha,\beta)}{N^k}, \tag{1.4}$$

and use this to examine how well the Hahn polynomials satisfy the orthogonality relation of the Jacobi polynomials.

There is an immediate application of this result to data analysis. Let $f(t)$ be continuous on $[0, 1]$, and consider its Fourier series development in the natural inner products suggested by (1.2) and (1.3), given by

$$f_n(t) = \sum_{j=0}^{n} a_j P_j(1-2t;\alpha,\beta), \tag{1.5}$$

$$F_n(t) = \sum_{j=0}^{n} A_j Q_j(Nt;\alpha,\beta,N+1), \qquad N \geqq n. \tag{1.6}$$

It is quite evident from the analysis shown here, that if $F_n(t)$ *is expected to mimic* the properties of $f_n(t)$, as is reasonable to suppose from (1.1), one should take $N$ at least $n^2/2$.

In § 3, we obtain bounds on the integral

$$I_n(N+1) = \int_0^N Q_n(x, N+1)\,dx, \tag{1.7}$$

which, in addition to reinforcing the above conclusions, are used in Wilson [7], where it is shown that a sufficient condition that there exists a quadrature formula, of degree $n$, involving $N$ equidistant points, is that $N \geqq C_1 n^2$, $C_1$ a constant. For a different constant, the same condition is also shown to be necessary.

The author gratefully acknowledges the constructive criticism of R. Askey of an earlier version of this paper. In particular, he suggested the use of asymptotic series (1.4), stating the formula,

$$\begin{aligned} &Q_n(Nt;\alpha,\beta,N+1) \\ &\quad = P_n(1-2t;\alpha,\beta) + \frac{t(t-1)}{2N}\frac{d^2}{dt^2}P_n(1-2t;\alpha,\beta) + O(N^{-2}). \end{aligned} \tag{1.8}$$

**2. Asymptotic (in $N$) expansions.** Our first task is to develop the series (1.4). Since many classes of orthogonal polynomials on discrete sets are similarly related to known classical polynomials, the methods may be applied elsewhere.

Let

$$T_k(n,\alpha,\beta) = \frac{(-n)_k(n+\alpha+\beta+1)_k}{(1+\alpha)_k k!}, \tag{2.1}$$

where $(a)_0 = 1$, $(a)_k = a(a+1)\cdots(a+k-1)$.

Then, from the definitions as hypergeometric series,

$$P_n(1-2t;\alpha,\beta) = \sum_{k=0}^{n} T_k(n,\alpha,\beta)t^k \tag{2.2}$$

and

$$Q_n(Nt;\alpha,\beta,N+1)=\sum_{k=0}^{n}T_k(n,\alpha,\beta)\frac{(-Nt)_k}{(-N)_k}. \tag{2.3}$$

LEMMA 2.1. *If* $(-Nt)_k/(-N)_k=\sum_{\alpha=0}^{\infty}N^{-\alpha}\psi_\alpha(t,k)$, *then*

$$\psi_\alpha(t,k)=\sum_{j=0}^{\min(\alpha,k)}S_k^{(k-j)}\mathscr{S}_{k+\alpha-j-1}^{(k-1)}t^{k-j}, \tag{2.4}$$

*where* $S_k^{(m)}$ *and* $\mathscr{S}_m^{(k)}$ *are respectively the Stirling numbers of the first and second kind.*

*Proof.* First,

$$\frac{(-Nt)_k}{(-N)_k}\equiv\frac{(Nt)(Nt-1)\cdots(Nt-k+1)}{N(N-1)\cdots(N-k+1)}.$$

From Abramowitz [1, pp. 824–825] we have

$$x(x-1)\cdots(x-k+1)=\sum_{j=0}^{k}S_k^{(j)}x^j$$

and

$$\frac{1}{(1-x)(1-2x)\cdots(1-mx)}=\sum_{i=m}^{\infty}\mathscr{S}_i^{(m)}x^{i-m},\qquad |x|\leqq\frac{1}{m}.$$

Substitution and rearrangement gives the desired result.

LEMMA 2.2.

(i) $\psi_0(t,k)=t^k$,

(ii) $\psi_1(t,k)=\dfrac{-t(1-t)}{2}D^2t^k$,

(iii) $\psi_2(t,k)=t(1-t)\left\{\dfrac{t(1-t)D^4}{8}+\dfrac{(1-2t)D^3}{3}-\dfrac{D^2}{2}\right\}t^k$,

*where* $D\equiv d/dt$.

*Proof.* From Jordan [3, Chap. 4], we have

$$S_k^k=1,\qquad \mathscr{S}_{k-1}^{(k-1)}=1,$$

$$S_k^{(k-1)}=-\binom{k}{2},\qquad \mathscr{S}_k^{(k-1)}=\binom{k}{2},$$

$$S_k^{(k-2)}=3\binom{k}{4}+2\binom{k}{3},\qquad \mathscr{S}_{(k+1)}^{(k-1)}=3\binom{k+1}{4}+\binom{k+1}{3}.$$

Substituting into (2.4), and using "operator" formulas like

$$\binom{k}{r}t^\rho=\left\{\frac{t^{\rho+r-k}}{r!}D^r\right\}t^k,$$

we obtain the above expressions.

COROLLARY 2.2.1.

$$
\begin{aligned}
Q_n(Nt;\alpha,\beta,N+1) &= P_n(1-2t;\alpha,\beta) - \frac{t(1-t)D^2}{2N}P_n(1-2t;\alpha,\beta) \\
&\quad + \frac{t(1-t)}{N^2}\left\{\frac{t(1-t)D^4}{8} + \frac{(1-2t)D^3}{3} - \frac{D^2}{2}\right\}P_n(1-2t,\alpha,\beta) \qquad (2.5)\\
&\quad + O(N^{-3}).
\end{aligned}
$$

*Proof*. The proof follows immediately from (2.2), (2.3), (2.4), and the lemma.

Further terms could be developed, but even the third term requires a large amount of calculation. However, it is worth pointing out that further terms could be developed using a computer and a symbol manipulation routine.

From (2.5), it is clear that, for $N$ large, $Q_n(Nt;\alpha,\beta,N+1) \backsimeq P_n(1-2t;\alpha,\beta)$. But $Q_n(Nt;\alpha,\beta,N+1)$ is defined, for given $n,\alpha,\beta$, for $N \in [n,\infty)$. If we let $\alpha=\beta$, and $n=2m$, then $Q_n(\frac{1}{2}N;\alpha,\alpha,N+1)$ is, by symmetry conditions, a local extremum of $Q_n(Nt;\alpha,\alpha,N+1)$. Evaluating (2.5), we obtain

$$
Q_n(\tfrac{1}{2}N;\alpha,\alpha,N+1) = P_n(0;\alpha,\alpha)\left\{1 + \frac{n(n+2\alpha+1)}{2N} \cdot \left[1 + \frac{(n-2)(n+2\alpha+1)-4}{4N}\right]\right\} + O(N^{-3})
$$

where

$$
P_n(0,\alpha,\alpha) = \frac{(-1)^m n!}{2^n m!}\,\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha+m)}.
$$

For $N$ just larger than $n$, the higher order terms swamp the first term. In fact, $N$ must be approximately $n(n+2\alpha+1)/2$, in order that the second term be of the same order of magnitude as the first term. Thus, it would appear that in practical applications, *if we want $Q_n$ to resemble $P_n$*, we should take $N$ at least $n^2/2$, or larger.

The above is a one point comparison. We can obtain a global estimate of how well $Q_n$ approximates $P_n$ by considering the integral

$$
I_n(\alpha,\beta,N+1) \equiv \int_0^1 t^\alpha(1-t)^\beta Q_n(Nt,\alpha,\beta,N+1)\,dt,
$$

recalling that, for $n \geqq 1$, the corresponding integral of $P_n(1-2t;\alpha,\beta)$ vanishes, by orthogonality.

For this computation, we require the following formulas: Szego [6, (9.4.1)],

$$
\int_{-1}^1 (1-x)^\alpha(1+x)^\beta P_n^{(\alpha+\sigma,\beta)}(x)\,dx = 2^{\alpha+\beta+1}\frac{\Gamma(1+\alpha)\Gamma(1+\beta+n)\Gamma(n+\sigma)}{\Gamma(2+\alpha+\beta+n)\Gamma(\sigma)n!};
$$

Erdélyi [2, § 10.8, (32)],

$$
xP_n^{(\alpha+1,\beta)}(x) = P_n^{(\alpha+1,\beta)}(x) + \frac{2}{(2+\alpha+\beta+n)}[(n+1)P_{n+1}^{(\alpha,\beta)}(x) - (1+\alpha+n)P_n^{(\alpha,\beta)}(x)];
$$

Erdélyi [2, § 10.8, (17)],

$$\frac{d^m}{dx^m} P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha+\beta+n)_m}{2^m} P_{n-m}^{(\alpha+m,\beta+m)}(x);$$

Erdélyi [2, § 10.8, (37)],

$$P_n^{(\alpha,\beta)}(x) = P_{n+1}^{(\alpha,\beta-1)}(x) - P_{n+1}^{(\alpha+1,\beta)}(x).$$

We define the operator $R_n\{f(\alpha,\beta)\} \equiv f(\alpha,\beta) + (-1)^n f(\beta,\alpha)$.

Integrating (2.5), we obtain

$$I_n(\alpha,\beta,N+1) = \frac{J_n(\alpha,\beta)}{N} + \frac{K_n(\alpha,\beta)}{N^2} + O\left(\frac{1}{N^3}\right), \qquad n \geqq 1,$$

where

$$J_n(\alpha,\beta) = \frac{-n!\Gamma(1+\alpha)}{2\Gamma(1+\alpha+n)\Gamma(1+\alpha+\beta+n)} R_n\{\Gamma(2+\alpha)\Gamma(1+\beta+n)\},$$

$$K_n(\alpha,\beta) = \frac{n!\Gamma(1+\alpha)}{\Gamma(1+\alpha+n)\Gamma(1+\alpha+\beta+n)} R_n\Bigg\{\Gamma(2+\alpha)\Gamma(1+\beta+n) \cdot\left[\frac{(n-2)(3+\alpha+\beta+n)(3\alpha-2)}{24} - \frac{1}{2} + \frac{(\alpha-\beta)(\alpha+2)}{8}\right]\Bigg\}.$$

In particular, we have, for $n$ even, $n \geqq 2$,

$$I_n(\alpha,\alpha,N+1) = \frac{-n!(1+\alpha)[\Gamma(1+\alpha)]^2}{\Gamma(1+2\alpha+n)N}\left[1 + \frac{(2-3\alpha)n(n+2\alpha+1)}{12N} + O\left(\frac{1}{N^2}\right)\right]$$

and

$$I_n(0,0,N+1) = -\frac{1}{N}\left[1 + \frac{n(n+1)}{6N} + O\left(\frac{1}{N^2}\right)\right].$$

By symmetry, $I_n(\alpha,\alpha,N+1) = 0$, for $n$ odd.

Again, we see that $N$, the number of points considered (less 1) should be much larger than the highest degree polynomial considered, in order that $Q_n$ be like $P_n$.

As a final check, we compute the inner product of $Q_m$ and $Q_n$ for $\alpha = \beta = 0$. In this case

$$\int_0^1 Q_m(Nt; N+1) Q_n(Nt; N+1)\, dt$$

$$= \begin{cases} \dfrac{1}{2n+1}\left[1 + \dfrac{n(n-1)}{N} + O\left(\dfrac{1}{N^2}\right)\right], & n = m, \\ -\left[\dfrac{1}{N} + \dfrac{n(n+1)+m(m+1)}{6N^2} + O\left(\dfrac{1}{N^3}\right)\right], n \neq m, & n+m \text{ even}, \\ 0, & n+m \text{ odd}, \end{cases}$$

for comparison against

$$\int_0^1 P_m(1-2t)P_n(1-2t)\, dt = \frac{\delta_{n,m}}{2n+1}.$$

For $m = n - 2$, where $n$ is the highest degree being considered, $[n(n + 1) + (m)(m + 1]/6 = (n^2 - n + 1)/3$, so that again, we would need $N \geqq n^2/3$ in a practical problem, if we expect the polynomials $Q_n$ to imitate the behavior of the $P_n$ set.

**3. Bounds on the integral of $Q_n(x; N + 1)$.** In this section, we bound the integral

$$I_n(N + 1) = \int_0^N Q_n(x; N + 1)\,dx = N\int_0^1 Q_n(Nt; N + 1)\,dt.$$

By symmetry, for $n$ odd, $I_n(N + 1) = 0$. For even $n$, $n \geqq 2$, we show that

$$1 + \frac{(n - 1)(n + 2)}{6(N + 1)} \leqq -I_n(N + 1) \leqq \frac{N + 1}{N}\left[1 + \frac{(n - 1)(n + 2)}{6(N - 1)}\frac{1}{1 - \alpha}\right], \tag{3.1}$$

where, on the right, $1 + N \geqq (n + 1)^2/3\alpha$, $\alpha \in (0, 1)$. We already have the asymptotic behavior

$$-I_n(N + 1) = 1 + \frac{n(n + 1)}{6N} + O\left(\frac{1}{N^2}\right).$$

For $\alpha = \beta = 0$, $T_k(n, \alpha, \beta) = (-1)^k\binom{n}{k}\binom{n + k}{k}$ and we rewrite (2.2) and (2.3) as

$$P_n(1 - 2t) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n + k}{k}t^k$$

and

$$Q_n(x, N + 1) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n + k}{k}\frac{(-x)_k}{(-N)_k}. \tag{3.2}$$

Direct integration of (3.2) is not very fruitful; however, we do have the exact values

$$I_0(N + 1) = N, \qquad I_2(N + 1) = -\frac{N}{N - 1},$$

$$I_4(N + 1) = \frac{-N[3N^2 - 8N + 18]}{3(N - 1)(N - 2)(N - 3)}.$$

LEMMA 3.1. *For $n$ even, $n \geqq 2$,*

$$I_n(N + 1) = \frac{-2}{N}\int_{-1}^{0}\psi(x)Q_{n-1}(x; 1, 1, N)\,dx, \tag{3.3}$$

*where* $\psi(x) = (1 + x)(N - x)$.

*Proof.* Levit [5, (43)] gives the recurrence relation

$$\begin{aligned}-NQ_n(x; N + 1) = {} & (1 + x)(x - N)Q_{n-1}(x; 1, 1, N)\\ & - x(x - N - 1)Q_{n-1}(x - 1; 1, 1, N).\end{aligned}$$

Integrating both sides, from 0 to $N$, we have

$$-NI_n(N+1) = -\int_0^N \psi(x)Q_{n-1}(x;1,1,N)\,dx - \int_0^N x(x-N-1)Q_{n-1}(x-1;1,1,N)\,dx.$$

Changing variables in the second integral, $x = x - 1$, we obtain

$$-NI_n(N+1) = -\int_{N-1}^N \psi(x)Q_{n-1}(x;1,1,N)\,dx - \int_{-1}^0 \psi(x)Q_{n-1}(x;1,1,N)\,dx.$$

For $n$ even, using the symmetry of $\psi(x)$, and antisymmetry of $Q_{n-1}(x;1,1,N)$ about $\frac{1}{2}N$, we obtain $-NI_n(N+1) = 2\int_{-1}^0 \psi(x)Q_{n-1}(x;1,1,N)\,dx$, which completes the proof.

We now write

$$A_n(N+1) = Q_{n-1}(-1;1,1,N), \tag{3.4}$$

and

$$B_n(N+1) = -\left[\frac{d}{dx}Q_{n-1}(x;1,1,N)\right]_{x=0}. \tag{3.5}$$

LEMMA 3.2. *For $n$ even, $n \geqq 2$,*

$$1 + \frac{B_n(N+1)}{3} < -I_n(N+1) < \frac{N+1}{N}\left[1 + \frac{A_n(N+1)-1}{3}\right]. \tag{3.6}$$

*Proof.* $Q_{n-1}(x;1,1,N)$ is an orthogonal polynomial with respect to an inner product which is a weighted summation over the integer points $0, 1, \cdots, N-1$. Hence, its zeros lie in the interval $(0, N-1)$. Thus, on $[-1, 0]$, $Q_{n-1}(x;1,1,N)$ is convex, and decreases monotonically from $A_n(N+1)$ at $-1$, to $Q_{n-1}(0;1,1,N) = 1$. Hence we have the bounds $[1 - B_n(N+1)\cdot x] \leqq Q_{n-1}(x;1,1,N) \leqq [1 + (1 - A_n(N+1)\cdot x)]$.

For a constant $K$,

$$\int_{-1}^0 (1 - Kx)\psi(x)\,dx = \frac{3N+1}{6} + K\left[\frac{2N+1}{12}\right],$$

so that

$$\int_{-1}^0 (1 - B_n(N+1)\cdot x)\psi(x)\,dx > \frac{N}{2} + \frac{NB_n(N+1)}{6}$$

and

$$\int_{-1}^0 (1 + (1 - A_n(N+1)x))\psi(x)\,dx < \frac{N+1}{2} + \frac{N+1}{6}[A_n(N+1) - 1].$$

Lemma 3.1. implies the result.

From (2.1) and (2.3), we have

$$(3.7)\quad Q_n(x;1,1,N+1)=\sum_{k=0}^{n} t_k(n,N+1)\frac{(-x)(-x+1)\cdots(-x+k-1)}{k!},$$

where

$$(3.8)\qquad t_k(n,N+1)=\frac{\binom{n}{k}\binom{n+k+2}{k}}{(k+1)\binom{N}{k}}.$$

Immediately, we have

$$(3.9)\qquad A_n(N+1)=\sum_{k=0}^{n-1} t_k(n-1,N).$$

On the other hand,

$$\frac{d}{dx}(-x)(-x+1)\cdots(-x+k-1)\Big|_{x=0}=-(k-1)!$$

so that

$$(3.10)\qquad B_n(N+1)=\sum_{k=1}^{n-1}\frac{t_k(n-1,N)}{k}.$$

From (3.10), we obtain two lower bounds on $B_n(N+1)$, namely, $t_1(n-1,N)$ and $t_{n-1}(n-1,N)/(n-1)$, which, with (3.6), gives

$$(3.11)\qquad -I_n(N+1)>1+\frac{(n-1)(n+2)}{6(N-1)}$$

and

$$(3.12)\qquad -I_n(N+1)>1+\frac{\binom{2n}{n-1}}{3n(n-1)\binom{N-1}{n-1}}.$$

It is clear from (3.12), for $N$ just larger than or equal to $n$, that $-I_n(N+1)$ is very large. For $N \geqq 2n+1$, however, (3.11) is the more appropriate bound. In order to obtain an upper bound, we require an upper bound on $A_n(N+1)$.

LEMMA 3.3. *For* $n \geqq 2$, *and for* $\alpha \in (0,1)$,

$$(3.13)\qquad N+1 \geqq \frac{(n+1)^2}{3\alpha} \quad \textit{implies} \quad A_n(N+1) \leqq \frac{t_1(n-1,N)}{1-\alpha}.$$

*Proof.* From (3.7),

$$A_{n+1}(N+2)=\sum_{k=0}^{n} t_k(n,N+1)$$
$$=1+t_1(n,N+1)\left[1+\sum_{i=1}^{n-1} R_iR_{i-1}R_1\right],$$

where

$$R_k(n, N+1) \equiv \frac{t_{k+1}(n, N+1)}{t_k(n, N+1)} = \frac{(n-k)(n+k+3)}{(k+2)(N-k)}, \qquad k = 1, 2, \cdots, n-1.$$

The ratio

$$\begin{aligned} r_k(n, N+1) &\equiv \frac{R_{k+1}(n, N+1)}{R_k(n, N+1)} \\ &= \frac{(n-k-1)(n+k+4)(k+2)}{(n-k)(n+k+3)(k+3)}\left(1 + \frac{1}{N-k-1}\right), \\ &\qquad\qquad k = 1, 2, \cdots, n-2, \end{aligned}$$

is monotonic decreasing with increasing $N$, for fixed $k, n$. Thus

$$r_k(n, N+1) \leqq r_k(n, n) = \frac{(k+2)(n+k+4)}{(k+3)(n+k+3)} < 1,$$
$$k = 1, 2, \cdots, n-2, \quad n \geqq 0.$$

Hence, $R_1(n, N+1) > R_2(n, N+1) > \cdots > R_{n-1}(n, N+1)$, and

$$A_{n+1}(N+2) \leqq 1 + t_1(n, N+1) \sum_{k=0}^{n-1} [R_1(n, N+1)]^k.$$

Now, for $\alpha \in (0, 1)$, $n \geqq 1$, $N + 2 \geqq (n+2)^2/3\alpha$ implies $R_1(n, N+1) \leqq \alpha$, so that, by summing a geometric progression, using $\alpha < 1$, we have

$$N + 2 \geqq \frac{(n+2)^2}{3\alpha} \quad \text{implies} \quad A_{n+1}(N+2) \leqq 1 + \frac{t_1(n, N+1)}{1-\alpha},$$

which gives the lemma.

The right-hand side of (3.1) is immediately obtained from (3.6) and (3.13). In particular, $N + 1 \geqq (n+1)^2$ implies

$$-I_n(N+1) \leqq \frac{N+1}{N}\left[1 + \frac{(n-1)(n+2)}{4(N-1)}\right]. \tag{3.14}$$

Although we have obtained an upper bound on $-I_n(N+1)$ for $N$ in the range $[n, (n+1)^2/3]$, we will not give it here. In this range, it is the lower bound, indicating the large size of $-I_n(N+1)$, which is important.

## REFERENCES

[1] M. Abromowitz et al., *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, D.C., 1964.

[2] A. Erdélyi et al., *Higher Transcendental Functions*, vol. II, McGraw-Hill, New York, 1953.

[3] C. Jordan, *Calculus of Finite Differences*, Rottig and Romwalter, Supron, Hungary, 1939.

[4] S. Karlin and J. L. McGregor, *The Hahn polynomials, formulas, and an application*, Scripta Math., 26 (1961), pp. 33–46.

[5] R. J. Levit, *The zeros of the Hahn polynomials*, SIAM Rev., 9 (1967), pp. 191–203.

[6] G. Szëgo, *Orthogonal Polynomials*, American Mathematical Society, Providence, 1959.

[7] M. W. Wilson, *Necessary and sufficient conditions for equidistant quadrature formula*, SIAM J. Numer. Anal., to appear.

SIAM J. MATH. ANAL.
Vol. 1, No. 2, May 1970

# SOME EXAMPLES OF SINGULAR PERTURBATION PROBLEMS WITH TURNING POINTS*

FRED W. DORR†

**1. Introduction.** In this paper, we consider the asymptotic behavior as $\varepsilon \to 0^+$ of solutions $y(t) = y(t, \varepsilon)$ to boundary value problems of the form

$$(1)\qquad \begin{aligned} &\varepsilon y''(t) + t\phi(t)y'(t) = 0, \qquad a < t < b,\\ &y(a) = A, \qquad y(b) = B, \end{aligned}$$

where $a < 0 < b$. We assume that $t\phi(t)$ is continuous, $\phi(t)$ is bounded, and either $\phi(t) \leqq \phi_1 < 0$ or $\phi(t) \geqq \phi_2 > 0$ for $a \leqq t \leqq b$. Thus (1) has a simple turning point at $t = 0$ (see [3], [5] and the references in these works). These problems are closely related to recent work of O'Malley [4] and Dorr and Parter [1].

In § 2 we treat the case $\phi(t) < 0$. We will show that, under appropriate conditions,

$$\lim_{\varepsilon \to 0^+} y(t, \varepsilon) = \bar{y}, \qquad a < t < b,$$

and, by choosing $\phi(t)$ properly, $\bar{y}$ can be *any* value between $A$ and $B$.

In § 3 we consider the case $\phi(t) > 0$, and then

$$\lim_{\varepsilon \to 0^+} y(t, \varepsilon) = \begin{cases} A, & a \leqq t < 0,\\ B, & 0 < t \leqq b. \end{cases}$$

This result can also be extended to quasi-linear equations of the form

$$(2)\qquad \begin{aligned} &\varepsilon y''(t) + t^k F(t, y(t))y'(t) = 0, \qquad a < t < b,\\ &y(a) = A, \qquad y(b) = B, \end{aligned}$$

where $0 < F_1 \leqq F(t, y) \leqq F_2$ and $k$ is a positive odd integer.

**2. Linear examples.** In the remainder of the paper we assume, without loss of generality, that $A < B$. If $A > B$ we can apply this analysis to the function $-y(t)$, and if $A = B$ we would have only the trivial solution $y(t, \varepsilon) \equiv A$, which we do not want to consider. For convenience, we will use the function

$$\varphi(t) = \int_a^t x\phi(x)\, dx.$$

The solution to (1) is then given by

$$(3)\qquad y(t, \varepsilon) = A + (B - A)\left(\int_a^t E\varphi(x, \varepsilon)\, dx\right)\left(\int_a^b E\varphi(x, \varepsilon)\, dx\right)^{-1},$$

* Received by the editors July 8, 1969.

† Los Alamos Scientific Laboratory of the University of California, Los Alamos, New Mexico 87544. This research was supported by the United States Atomic Energy Commission under Contract W-7405-ENG-36.

where

$$E\varphi(x,\varepsilon) = \exp\left(-\frac{1}{\varepsilon}\varphi(x)\right).$$

THEOREM 1. *Assume that* $\varphi(t) \in C^1[a,b] \cap C^2[a,0] \cap C^2[0,b]$* *satisfies*:

(i) $\varphi(a) = \varphi(b) = 0$,

(ii) $\varphi(t) > 0$ *if* $a < t < b$,

(iii) $[\varphi'(a)]^2 + [\varphi'(b)]^2 > 0$.

*If* $y(t) = y(t,\varepsilon)$ *is the solution to*

$$\text{(4)} \qquad \begin{aligned} &\varepsilon y''(t) + \varphi'(t)y'(t) = 0, \qquad a < t < b,\\ &y(a) = A, \qquad y(b) = B, \end{aligned}$$

*then*

$$\lim_{\varepsilon \to 0^+} y(t,\varepsilon) = \bar{y}, \qquad a < t < b,$$

*where*

$$\text{(5)} \qquad \bar{y} = \left[\frac{\varphi'(a)}{\varphi'(a) - \varphi'(b)}\right]A + \left[\frac{-\varphi'(b)}{\varphi'(a) - \varphi'(b)}\right]B.$$

*Proof.* If $0 < \delta \leqq \frac{1}{2}(b - a)$, we have

$$0 \leqq \left(\frac{1}{B-A}\right)(y(b-\delta) - y(a+\delta)) = \left(\int_{a+\delta}^{b-\delta} E\varphi(x,\varepsilon)\,dx\right)\left(\int_a^b E\varphi(x,\varepsilon)\,dx\right)^{-1}.$$

Using conditions (i) and (ii), it is easy to show that there exist two constants $C_i = C_i(\delta) > 0$ such that

$$0 \leqq \left(\frac{1}{B-A}\right)(y(b-\delta) - y(a+\delta)) \leqq \frac{C_1}{\varepsilon}\exp\left(-\frac{C_2}{\varepsilon}\right)$$

for $0 < \varepsilon \leqq \varepsilon_0$. Thus

$$\lim_{\varepsilon \to 0^+} (y(b-\delta,\varepsilon) - y(a+\delta,\varepsilon)) = 0.$$

Therefore, we can conclude that there exist a sequence $\varepsilon_n \to 0^+$ and a constant $\bar{y} \in [A,B]$ such that

$$\text{(6)} \qquad \lim_{n \to \infty} y(t,\varepsilon_n) = \bar{y}, \qquad a < t < b.$$

Integrating (4) by parts, we obtain

$$\text{(7)} \qquad \varepsilon(y'(b,\varepsilon) - y'(a,\varepsilon)) = A\varphi'(a) - B\varphi'(b) + \int_a^b \varphi''(x)y(x,\varepsilon)\,dx.$$

Since $\varphi(a) = \varphi(b)$, from (3) we see that $y'(b,\varepsilon) = y'(a,\varepsilon)$. Thus we can let $\varepsilon = \varepsilon_n \to 0$ in (7) to find that

$$0 = A\varphi'(a) - B\varphi'(b) + \bar{y}(\varphi'(b) - \varphi'(a)).$$

* We will use this notation to mean that $\varphi(t) \in C^1[a,b] \cap C^2[a,0) \cap C^2(0,b]$, and both $\lim_{x\to 0^-} \varphi''(x)$ and $\lim_{x \to 0^+} \varphi''(x)$ exist.

From conditions (i) through (iii) we have $\varphi'(b) \neq \varphi'(a)$, so that $\bar{y}$ is given by (5). Since $\bar{y}$ is unique, (6) remains valid for all sequences $\varepsilon_n \to 0^+$, and this completes the proof of Theorem 1.

*Example* 1. By letting $a = -b$ and $\varphi(t) = \frac{1}{2}(b^2 - t^2)$, we find that, if $y(t) = y(t, \varepsilon)$ is the solution to

$$\varepsilon y''(t) - ty'(t) = 0, \qquad -b < t < b,$$
$$y(-b) = A, \quad y(b) = B,$$

then

$$\lim_{\varepsilon \to 0^+} y(t, \varepsilon) = \tfrac{1}{2}(A + B), \qquad -b < t < b.$$

*Example* 2. On a nonsymmetric interval, we can use the function

$$\varphi(t) = \begin{cases} \dfrac{b^2}{a^2}(a^2 - t^2), & a \leqq t \leqq 0, \\ (b^2 - t^2), & 0 \leqq t \leqq b, \end{cases}$$

and then

$$\bar{y} = \left(\frac{1}{b - a}\right)(bA - aB).$$

We now show that the constant $\bar{y}$ in Theorem 1 can be any value in the interval $[A, B]$.

THEOREM 2. *Let $\gamma \in [0, 1]$ be given. Then there exists a piecewise continuous function $\phi(t)$, which is linear on $[a, 0)$ and $[0, b]$ and satisfies $\phi(t) \leqq \phi_1 < 0$ for $a \leqq t \leqq b$, such that*

$$\lim_{\varepsilon \to 0^+} y(t, \varepsilon) = \gamma A + (1 - \gamma)B, \qquad a < t < b,$$

*where $y(t, \varepsilon)$ is the solution to* (1).

*Proof.* Given parameters $\alpha \geqq 0$ and $\beta \geqq 0$ with $\alpha^2 + \beta^2 > 0$, we want to exhibit a function $\varphi(t) \in C^1[a, b] \cap C^2[a, 0] \cap C^2[0, b]$ satisfying

$$\varphi'(t) \begin{cases} >0, & a < t < 0, \\ =0, & t = 0, \\ <0, & 0 < t < b, \end{cases}$$

and the following conditions:

(i) $\varphi(a) = \varphi(b) = 0$,
(ii) $\varphi(t) > 0$ if $a < t < b$,
(iii) $\varphi'(a) = \alpha$,
(iv) $\varphi'(b) = -\beta$.

It is easily verified that a suitable function is:

$$\varphi(t) = \begin{cases} d(t - a)^3 + \left(\dfrac{3da^2 + \alpha}{2a}\right)(t - a)^2 + \alpha(t - a), & a \leqq t \leqq 0, \\ c(t - b)^3 + \left(\dfrac{3cb^2 - \beta}{2b}\right)(t - b)^2 - \beta(t - b), & 0 \leqq t \leqq b. \end{cases}$$

Here $c$ is any constant satisfying

$$c > \max\left(0, -\frac{\beta b + \alpha a}{b^3}\right)$$

and

$$d = \left(\frac{cb^3 + \beta b + \alpha a}{a^3}\right).$$

If we now define

$$\phi(t) = \begin{cases} \dfrac{1}{t}\varphi'(t), & t \neq 0, \\ \lim\limits_{x \to 0^+} \dfrac{1}{x}\varphi'(x), & t = 0, \end{cases}$$

then from Theorem 1 we have

$$\lim_{\varepsilon \to 0^+} y(t, \varepsilon) = \bar{y} = \left(\frac{\alpha}{\alpha + \beta}\right) A + \left(\frac{\beta}{\alpha + \beta}\right) B.$$

Thus we have only to choose $\alpha$ and $\beta$ such that

$$\gamma = \left(\frac{\alpha}{\alpha + \beta}\right).$$

**3. Quasi-linear examples.** Let $y(t) = y(t, \varepsilon)$ be a solution to a quasi-linear boundary value problem of the form of (2). We assume that, for $a \leqq t \leqq b$ and $A \leqq y \leqq B$, we have:

(i) $t^k F(t, y)$ is continuous,

(ii) $|F(t, y)| \leqq F_M < \infty$, and either $F(t, y) \leqq F_- < 0$ or $F(t, y) \geqq F_+ > 0$,

(iii) $k$ is a nonnegative integer.

It is well known [2] that there exist bounding functions $z_1(t, \varepsilon)$ and $z_2(t, \varepsilon)$ such that

$$z_1(t, \varepsilon) \leqq y(t, \varepsilon) \leqq z_2(t, \varepsilon), \qquad a \leqq t \leqq b.$$

Specifically, if $F_1 \leqq F(t, y) \leqq F_2$ with $F_1 F_2 > 0$, we define the following functions:

$k$ even

$$G_1(t) = F_1, \quad a \leqq t \leqq b,$$

$$G_2(t) = F_2, \quad a \leqq t \leqq b,$$

$k$ odd

$$G_1(t) = \begin{cases} F_2, & a \leqq t \leqq 0, \\ F_1, & 0 < t \leqq b, \end{cases}$$

$$G_2(t) = \begin{cases} F_1, & a \leqq t \leqq 0, \\ F_2, & 0 < t \leqq b. \end{cases}$$

Then $z_i(t, \varepsilon)$ can be defined as the solution to the boundary value problem

$$\varepsilon z_i''(t) + t^k G_i(t) z_i'(t) = 0, \qquad a < t < b,$$

$$z_i(a) = A, \qquad z_i(b) = B.$$

We therefore now consider problems of the form

$$(8)\qquad \begin{aligned} &\varepsilon z''(t) + t^k\Psi(t)z'(t) = 0, \qquad a < t < b,\\ &z(a) = A, \qquad z(b) = B, \end{aligned}$$

where

$$\Psi(t) = \begin{cases} \Psi_1, & a \leqq t \leqq 0,\\ \Psi_2, & 0 < t \leqq b, \end{cases}$$

and $\Psi_1\Psi_2 > 0$. If $k = 0$ we require that $\Psi_1 = \Psi_2$, so that the function $t^k\Psi(t)$ is continuous for all values of $k$. In many cases, it is possible to determine the asymptotic behavior of $y(t, \varepsilon)$ because $z_1(t, \varepsilon)$ and $z_2(t, \varepsilon)$ converge to the same limiting function. For example, this technique is applicable when the solution $z(t, \varepsilon)$ to (8) converges to a function that is independent of $\Psi_1$ and $\Psi_2$. The next result gives several examples of this.

THEOREM 3. *Let $z(t) = z(t, \varepsilon)$ be the solution to* (8). *Then we have*:

*Case* 1. *If $k$ is even and $\Psi(t) > 0$,*

$$\lim_{\varepsilon\to 0^+} z(t, \varepsilon) = B, \qquad a < t \leqq b;$$

*Case* 2. *If $k$ is even and $\Psi(t) < 0$,*

$$\lim_{\varepsilon\to 0^+} z(t, \varepsilon) = A, \qquad a \leqq t < b;$$

*Case* 3. *If $k$ is odd and $\Psi(t) > 0$,*

$$\lim_{\varepsilon\to 0^+} z(t, \varepsilon) = \begin{cases} A, & a \leqq t < 0,\\ B, & 0 < t \leqq b. \end{cases}$$

*Remark.* Cases 1 and 2 reflect the fact that, for $k$ even, the point $t = 0$ is not a turning point for the differential equation, since the function $t^k\Psi(t)$ does not change sign at $t = 0$. Case 3 corresponds to the general condition $\phi(t) \geqq \phi_2 > 0$ in (1). Note that this is the opposite of the cases treated in § 2, and the asymptotic behavior is quite different. Also note that the conclusion for Case 3 holds when the *order* of the zero at the turning point is of an arbitrary (odd) degree.

COROLLARY. *Let $y(t) = y(t, \varepsilon)$ be a solution to* (2). *Then we have*:

*Case* 1. *If $k$ is even and $F(t, y) > 0$,*

$$\lim_{\varepsilon\to 0^+} y(t, \varepsilon) = B, \qquad a < t \leqq b;$$

*Case* 2. *If $k$ is even and $F(t, y) < 0$,*

$$\lim_{\varepsilon\to 0^+} y(t, \varepsilon) = A, \qquad a \leqq t < b;$$

*Case* 3. *If $k$ is odd and $F(t, y) > 0$,*

$$(9)\qquad \lim_{\varepsilon\to 0^+} y(t, \varepsilon) = \begin{cases} A, & a \leqq t < 0,\\ B, & 0 < t \leqq b. \end{cases}$$

As a consequence of Theorem 2, we know that the asymptotic behavior of $z(t, \varepsilon)$ in the case $k$ odd, $\Psi(t) < 0$, cannot be independent of $\Psi_1$ and $\Psi_2$. Indeed, we have the following interesting result.

THEOREM 4. *Let $z(t) = z(t, \varepsilon)$ be the solution to* (8) *with $k$ odd and $\Psi(t) < 0$. Then, for $a < t < b$, we have*:

$$\lim_{\varepsilon \to 0^+} z(t, \varepsilon) = \begin{cases} A, & \dfrac{\Psi_1}{\Psi_2} < \left(\dfrac{b}{a}\right)^{k+1}, \\ \left[\left(\dfrac{b}{b-a}\right)A + \left(\dfrac{-a}{b-a}\right)B\right], & \dfrac{\Psi_1}{\Psi_2} = \left(\dfrac{b}{a}\right)^{k+1}, \\ B, & \dfrac{\Psi_1}{\Psi_2} > \left(\dfrac{b}{a}\right)^{k+1}. \end{cases}$$

The proofs of Theorems 3 and 4 follow by direct computation, and hence are omitted.

Two conclusions can be drawn from these results. First, the use of bounding functions is, in general, not applicable to the case $k$ odd and $F(t, y) < 0$. Second, for $k$ odd, there is a fundamental difference between the cases $\Psi(t) > 0$ and $\Psi(t) < 0$: if $\Psi(t) > 0$, the asymptotic behavior is very regular (cf. (9)), whereas if $\Psi(t) < 0$ the asymptotic behavior can be quite arbitrary (cf. Theorem 2).

**Acknowledgment.** I am grateful to Professor Seymour V. Parter for many helpful ideas related to the results in this paper.

REFERENCES

[1] F. W. DORR AND S. V. PARTER, *Singular perturbations of non-linear boundary value problems with turning points*, J. Math. Anal. Appl., to appear.

[2] F. W. DORR AND L. F. SHAMPINE, *The use of bounding functions in singular perturbation problems*, to appear.

[3] R. E. O'MALLEY, JR., *Topics in singular perturbations*, Advances in Math., 2 (1968), pp. 365–470.

[4] ———, *On boundary value problems for a singularly perturbed differential equation with a turning point*, to appear.

[5] W. WASOW, *Asymptotic Expansions for Ordinary Differential Equations*, Interscience, New York, 1965.

SIAM J. MATH. ANAL.
Vol. 1, No. 2, May 1970

# ERROR BOUNDS FOR ASYMPTOTIC APPROXIMATIONS OF ZEROS OF TRANSCENDENTAL FUNCTIONS*

HERBERT W. HETHCOTE†

**1. Introduction.** Precise error bounds on asymptotic approximations are useful both theoretically and numerically. In this paper general theorems are presented which show how error bounds for the asymptotic approximation of both real and complex zeros depend on error bounds for the asymptotic approximation of the functions. There are many possible applications of the general theorems in addition to those in this paper [3].

**2. General theorems.** F. Tricomi [10] pointed out that the asymptotic behavior of the zeros of a transcendental function can be deduced from the asymptotic behavior of the function itself. Tricomi's method uses the Taylor series repeatedly. L. Gatteschi has used the intermediate value form of the Taylor series remainder to obtain error bounds on the asymptotic approximations of real zeros. New applications of the following theorem and corollary derived from his method [2] are given in § 3 and § 4.

THEOREM 1. *In the interval* $[b - \rho, b + \rho]$, *suppose* $f(x) = g(x) + \varepsilon(x)$, *where* $f(x)$ *is continuous*, $g(x)$ *is differentiable*, $g(b) = 0$, $m = \min |g'(x)| > 0$, *and*

$$E = \max |\varepsilon(x)| < \min \{|g(b - \rho)|, |g(b + \rho)|\}.$$

*Then there exists a zero c of* $f(x)$ *in the interval such that* $|c - b| \leqq E/m$.

COROLLARY 1.[1] *In the interval* $[n\pi - \psi - \rho, n\pi - \psi + \rho]$, *where* $\rho < \pi/2$, *suppose* $f(x) = \sin(x + \psi) + \varepsilon(x)$, $f(x)$ *is continuous and* $E = \max |\varepsilon(x)| < \sin \rho$. *Then there exists a zero c of* $f(x)$ *in the interval such that* $|c - (n\pi - \psi)| \leqq E/\cos \rho$.

The next theorem summarizes another result of Gatteschi [1].

THEOREM 2. *In the interval* $[n\pi - \psi - \rho, n\pi - \psi + \rho]$ *where* $\rho < \pi/2$, *suppose*

$$f(x) = [1 + g(x)] \sin(x + \psi) + h(x) \cos(x + \psi) + \varepsilon(x),$$

*where* $|h(x)| < 1$, $g(x) > -1$, *and*

$$\lambda = \frac{\max |\varepsilon(x)| + \max |g(x)h(x)| + \frac{1}{2} \max |h(x)|^3}{1 + \min g(x)} \leqq 1.$$

*If* $c_n$ *denotes a zero of* $f(x)$ *lying in the interval, then*

$$|c_n + h(c_n) - (n\pi - \psi)| \leqq \mu = \lambda + (\tfrac{1}{2}\pi - 1)\lambda^3.$$

* Received by the editors March 18, 1969, and in revised form September 19, 1969.

† Department of Mathematics, University of Iowa, Iowa City, Iowa 52240. The results of this paper constitute part of the author's doctoral thesis written at the University of Michigan. This work was supported in part by the National Science Foundation under Grants GP7445 and GP8355.

[1] F. W. J. Olver has pointed out that Corollary 1 can be improved by using $\rho \leqq \frac{1}{2}\pi$ and

$$|c - (n\pi - \psi)| = |\sin^{-1} \varepsilon(c)| \leqq \sin^{-1} E \leqq E\rho \operatorname{cosec} \rho,$$

the last inequality being a consequence of the fact that $\sin^{-1} x/x$ is an increasing function when $x \in [0, 1]$. This result does not significantly change the error bound in §3.

*Moreover if* $h(c_n) = k/c_n$, *where* $k$ *is a constant and* $c_n > 0$, *then*

$$\left|c_n + \frac{k}{n\pi - \psi} - (n\pi - \psi)\right| \leqq \frac{k^2}{(n\pi - \psi)c_n^2} + \mu\left[1 + \frac{k}{(n\pi - \psi)c_n}\right].$$

Our final general theorem concerns complex zeros and appears to be new.

THEOREM 3. *Suppose* $f(z) = g(z) + \varepsilon(z)$, *where* $f(z)$ *and* $g(z)$ *are holomorphic in the disk* $|z - b| \leqq \rho$ *and* $g(z)$ *has exactly one zero* $b$ *in the disk. Let* $m = \min_\theta |g(b + \rho e^{i\theta})|$, $E = \max_\theta |\varepsilon(b + \rho e^{i\theta})|$, *and* $M = \max_\theta |g'(b + \rho e^{i\theta})|$. *If* $m > E$, *then* $f(z)$ *has exactly one zero* $c$ *in the disk and* $|c - b| \leqq E\rho^2 M/m(m - E)$.

*Proof.* The hypothesis $m > E$ implies $|g(b + z)| > |\varepsilon(b + z)|$ on the circle $|z| = \rho$. By Rouché's theorem, there exists a simple zero $a$ of $f(b + z)$ and $a$ is the only zero of $f(b + z)$ inside the disk $|z| \leqq \rho$.

There exists some $R > \rho$ such that $g(b + z)$ is holomorphic for $|z| < R$ and $g(b + z) \neq 0$ for $0 < |z| < R$. By Rouché's theorem $g(b + t) + \varepsilon(b + a)$ has one simple zero inside $C: |t| = \rho$ and this zero is $a$. Thus $g(b + t) + \varepsilon(b + a) = (t - a)h(t)$, where $h(t)$ is holomorphic and $h(a) \neq 0$. The residue at $t = a$ of

$$\frac{tg'(b + t)}{g(b + t) + \varepsilon(b + a)} = \frac{t}{t - a} + \frac{th'(t)}{h(t)}$$

is $a$. Hence

$$\frac{1}{2\pi i}\int_C \frac{tg'(b + t)\,dt}{g(b + t) + \varepsilon(b + a)} = a.$$

The series

$$\frac{1}{g(b + t) + \varepsilon(b + a)} = \sum_{k=0}^{\infty} [-\varepsilon(b + a)]^k [g(b + t)]^{-k-1}$$

is uniformly convergent with respect to $t$ if $|t| = \rho$ and $|\varepsilon(b + a)| < m(1 - \delta)$, where $\delta > 0$. If we multiply by the bounded function $tg'(b + t)$ and integrate term by term, then

$$a = \sum_{k=0}^{\infty} [-\varepsilon(b + a)]^k \frac{1}{2\pi i}\int_C \frac{tg'(b + t)\,dt}{[g(b + t)]^{k+1}}.$$

Since $g'(b + t)/g(b + t)$ has a pole of order one at $t = 0$, the first term of the above series vanishes. If we let $c = b + a$, then

$$|c - b| = |a| \leqq \sum_{k=1}^{\infty} E^k \frac{\rho^2 M}{m^{k+1}} \leqq \frac{E\rho^2 M}{m(m - E)}.$$

COROLLARY 2.[2] *Suppose* $f(z) = \sin(z + \psi) + \varepsilon(z)$, *where* $f(z)$ *is holomorphic in the disk* $|z - (n\pi - \psi)| \leqq \rho < \pi/2$. *If*

$$E = \max_\theta |\varepsilon(n\pi - \psi + \rho e^{i\theta})| < \sin\rho,$$

[2] F. W. J. Olver has also noted that Corollary 2 can be improved using the method of footnote 1 since $|\sin^{-1} z| \leqq \sin^{-1}|z|$. This result would improve the error bound in §5 by a factor of about two.

*then $f(z)$ has exactly one zero C in the disk and*

$$|c - (n\pi - \psi)| \leqq \frac{E\rho^2 \cosh \rho}{\sin \rho(\sin \rho - E)}.$$

**3. The negative zeros $a_n$ of the Airy function** Ai$(z)$. An asymptotic approximation of the Airy function of negative argument is

$$\pi^{1/2}x^{1/4}\mathrm{Ai}(-x) = \sin(\zeta + \tfrac{1}{4}\pi)P(\zeta) - \cos(\zeta + \tfrac{1}{4}\pi)Q(\zeta), \tag{3.1}$$

where $\zeta = 2x^{3/2}/3$,

$$P(\zeta) \sim 1 - \frac{5 \cdot 7 \cdot 9 \cdot 11}{2!\,216^2\zeta^2} + \frac{9 \cdot 11 \cdot 13 \cdot 15 \cdot 17 \cdot 19 \cdot 21 \cdot 23}{4!\,216^4\zeta^4} - \cdots \tag{3.2}$$

and

$$Q(\zeta) \sim \frac{3 \cdot 5}{1\,!\,216\zeta} - \frac{7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17}{3\,!\,216^3\zeta^3} + \cdots \tag{3.3}$$

[4, p. B17]. It has been shown that the error in truncating the expansions for $P(\zeta)$ and $Q(\zeta)$ is less than the first omitted term [12, p. 206].

If $f(\zeta)$ in Corollary 1 is equal to the left-hand side of (3.1), then

$$|\varepsilon(\zeta)| \leqq 0.0695\zeta^{-1} + 0.0372\zeta^{-2}.$$

In this paper, the last significant figure in decimal numbers is the result of rounding to the nearest digit except for numbers in inequalities, which are rounded to obtain the weakest inequality. Using $\zeta \geqq 2.3$, $\rho = 0.04$ and $\psi = \pi/4$ in Corollary 1, we find $E < \sin \rho$ so that there exists a zero $c_n$ in the interval and

$$|c_n - (n\pi - \tfrac{1}{4}\pi)| \leqq E/\cos \rho \leqq 0.086/(n\pi - \tfrac{1}{4}\pi - 0.04)$$

if $n \geqq 1$. The assumption that $\zeta \geqq 2.3$ is justified since $c_1$ belongs to the interval [2.32, 2.40].

If $a_n$ denotes the $n$th negative zero of Ai$(z)$, then Taylor's formula applied to $a_n = -(3c_n/2)^{2/3}$ yields

$$a_n = -[\tfrac{3}{8}\pi(4n - 1)]^{2/3}(1 + e_n),$$

where

$$|e_n| \leqq 0.130[\tfrac{3}{8}\pi(4n - 1.051)]^{-2}$$

for $n \geqq 1$. This bound on $e_n$ is quite good since 0.130 is only slightly greater than the coefficient 5/48 of the next term in the asymptotic expansion of $a_n$ (see [4, p. B48]).

If we apply Theorem 2 with the left-hand side of (3.1) as $f(\zeta)$, the second term of (3.2) as $g(\zeta)$, and the first term of (3.3) as $-h(\zeta)$, then

$$|\varepsilon(\zeta)| \leqq 0.0577\zeta^{-4} + 0.0380\zeta^{-3}.$$

If we use $\rho = 0.04$ and $\zeta \geqq 2.3$, then $|h(\zeta)| \leqq 1$, $g(\zeta) > -1$ and

$$\lambda \leqq 0.222[\tfrac{3}{8}\pi(4n - 1.051)]^{-3} \leqq 1$$

for $n \geqq 1$. Using the known existence of a zero $c_n$ in the interval and $k = 5/72$ in

Theorem 2, we have

$$\left|c_n - \frac{5}{72(n\pi - \frac{1}{4}\pi)} - (n\pi - \tfrac{1}{4}\pi)\right| \leqq 0.243[\tfrac{3}{8}\pi(4n - 1.051)]^{-3}$$

for $n \geqq 1$.

Taylor's formula applied to $a_n = -(3c_n/2)^{2/3}$ yields

$$a_n = -[\tfrac{3}{8}\pi(4n - 1)]^{2/3}\left[1 + \frac{5/48}{[\frac{3}{8}\pi(4n - 1)]^2} + e_n\right],$$

where

$$|e_n| \leqq 0.256[\tfrac{3}{8}\pi(4n - 1.051)]^{-4}$$

for $n \geqq 1$. The constant 0.256 in the error bound is less than twice the coefficient 5/36 of the next term in the asymptotic expansion of $a_n$ (see [4, p. B48]).

**4. The first positive zero $j_{\nu,1}$ of $J_\nu(z)$.** To obtain error bounds on the uniform asymptotic approximation of the Bessel function $J_\nu(\nu z)$ for large real $\nu$, we use the results of Olver [8, pp. 206–210] except that we apply Theorem 1 of [7] instead of Theorem 1 of [8]. We obtain

$$\left(\frac{1 - z^2}{\zeta}\right)^{1/4} \frac{e^\nu \Gamma(\nu + 1)}{\nu^{\nu + 1/6} 2\pi^{1/2}} J_\nu(\nu z) = \mathrm{Ai}(\nu^{2/3}\zeta) + \varepsilon(\nu^{2/3}\zeta), \tag{4.1}$$

where

$$|\varepsilon(u)| \leqq 0.700[e^{1.430F(u)} - 1]M(u). \tag{4.2}$$

The quantities $z(\zeta)$, $F(u)$ and $M(u)$ are defined in [7, pp. 750–751]. From [6, p. 9] we find $F(\nu^{2/3}\zeta) \leqq 0.2103/\nu$.

Let us now apply Theorem 1 with $\nu^{2/3}\zeta$ as the independent variable, $f(\nu^{2/3}\zeta)$ equal to the left-hand side of (4.1), $g(\nu^{2/3}\zeta) = \mathrm{Ai}(\nu^{2/3}\zeta)$, $b = a_1 = -2.33811$ [4] and $\rho = 0.17$ Now $\nu^{2/3}\zeta \leqq b + \rho$ implies $M(\nu^{2/3}\zeta) \leqq 0.47$ (see [7, p. 752]). These values yield $E \leqq 0.116$ if $\nu \geqq 1$. From tables of Airy functions [4], we find $m \geqq 0.67600$ and

$$\min\{|\mathrm{Ai}(a_1 - \rho)|, |\mathrm{Ai}(a_1 + \rho)|\} \geqq 0.117.$$

Hence $\nu^{2/3}\zeta_1 = a_1 + \delta$, where

$$|\delta| \leqq 0.486[e^{0.302/\nu} - 1] \leqq 0.21/\nu$$

for $\nu \geqq 1$.

Now $j_{\nu,1} = \nu z_1$, where

$$z_1 = 1 - z^{-1/3}\zeta_1 + \tfrac{1}{2}\zeta_1 z''(\theta)$$

with $\theta$ as some intermediate value [5, p. 336]. If we use $\zeta_1 = (a_1 + \delta)\nu^{-2/3}$, then

$$\begin{aligned} j_{\nu,1} &= \nu - \left(\frac{\nu}{2}\right)^{1/3} a_1 + \nu^{-1/3}[-\nu^{2/3}2^{-1/3}\delta + \tfrac{1}{2}(a_1 + \delta)z''(\theta)] \\ &= \nu + 1.85576\nu^{1/3} + \alpha\nu^{-1/3}. \end{aligned}$$

If $\nu \geqq 1$, then $|\nu^{2/3}2^{-1/3}\delta| \leqq 0.167$, $4.70 \leqq (a_1 + \delta)^2 \leqq 6.30$ and $-2.508 \leqq \theta \leqq 0$. It can be shown by analyzing the derivatives of $z''(\zeta)$ that $z''(\zeta)$ is an increasing function of $\zeta$ (see [3, pp. 85–88]) and, consequently, $0.284 \leqq z''(\theta) \leqq 0.378$. Combining these inequalities, we find $0.500 \leqq \alpha \leqq 1.357$ for $\nu \geqq 1$. Similar reasoning yields $0.935 \leqq \alpha \leqq 1.105$ for $\nu \geqq 10$ (see [3, p. 21]). These bounds on $\alpha$ are reasonable since the coefficient of the $\nu^{-1/3}$ term in the asymptotic expansion of $j_{\nu,1}$ is 1.03315 (see [11]).

**5. The complex zeros $j_{\nu,n}$ of $J_\nu(z)$.** An asymptotic approximation for the Bessel function $J_\nu(z)$ is

$$(\pi z/2)^{1/2}J_\nu(z) = \sin(z - \tfrac{1}{2}\nu\pi + \tfrac{1}{4}\pi) + \varepsilon(z), \tag{5.1}$$

where

$$|\varepsilon(z)| \leqq [|e^{iz - i\nu\pi/2}| + \tfrac{1}{2}\pi|e^{-iz + i\nu\pi/2}|]\frac{|4\nu^2 - 1|}{8|z|}\exp\left(\frac{\pi|4\nu^2 - 1|}{8|z|}\right) \tag{5.2}$$

for $|\arg z| \leqq \pi/2$ (see [9, p. 179]). In this application of Corollary 2, let $f(z)$ be the left-hand side of (5.1) and $\psi = -\frac{1}{2}\nu\pi + \frac{1}{4}\pi$. If $\mathrm{Re}(z) \geqq 0$ and $z \neq 0$, then $f(z)$ is holomorphic. If we choose $\rho = 0.2$, then $|z - (n\pi - \psi)| \leqq \rho$ implies

$$|z| \geqq \pi(n - 0.314 + \tfrac{1}{2}\mathrm{Re}\,\nu)$$

and

$$E \leqq \frac{1.00|4\nu^2 - 1|}{8(n - 0.314 + \frac{1}{2}\mathrm{Re}\,\nu)}\exp\left[\frac{|4\nu^2 - 1|}{8(n - 0.314 + \frac{1}{2}\mathrm{Re}\,\nu)}\right]. \tag{5.3}$$

Let us find how large $n$ must be to imply that the right-hand side of (5.3) is less than 0.1. Now $y \leqq k/(1 + k)$ implies $y \exp y \leqq y/(1 - y) \leqq k$. Thus we need

$$y = \frac{|4\nu^2 - 1|}{8(n - 0.314 + \frac{1}{2}\mathrm{Re}\,\nu)} \leqq \frac{0.1}{(1.1)(1.00)}$$

which is equivalent to

$$n \geqq 0.314 - \tfrac{1}{2}\mathrm{Re}\,\nu + 1.38|4\nu^2 - 1|. \tag{5.4}$$

Now $|z - (n\pi - \psi)| \leqq \rho$ and (5.4) justify our assumption that $\mathrm{Re}(z) \geqq 0$. If (5.4) holds, then $E < \sin\rho$ so that $f(z)$ has exactly one zero $c$ in the disk. Since $\mathrm{Re}(z) \geqq 0$ and $z \neq 0$, $c$ is a zero of $J_\nu(z)$, which we denote by $j_{\nu,n}$. Hence (5.4) implies

$$|j_{\nu,n} - (n\pi - \nu\pi/2 - \pi/4)| \leqq \frac{0.90|4\nu^2 - 1|}{\pi(n - 0.314 + \frac{1}{2}\mathrm{Re}\,\nu)}.$$

The constant 0.90 in the error bound is considerably greater than the coefficient 1/8 of the next term in McMahon's asymptotic expansion for $j_{\nu,n}$ (see [12, p. 507]). An error bound on the two term McMahon asymptotic approximation of $j_{\nu,n}$ is given in [3, pp. 34–38].

**Acknowledgment.** The author is grateful to Professor N. D. Kazarinoff for guidance during the development of this research.

## REFERENCES


[1] L. GATTESCHI, *On the zeros of certain functions with applications to Bessel functions*, Nederl. Akad. Wetensch. Proc. Ser. A, 55 (1952), pp. 224–229.

[2] ———, *Limitazione degli errori nelle formule asintotiche per le funzioni speciali*, Univ. e Politec. Torino Rend. Sem. Mat., 16 (1956–57), pp. 83–97.

[3] H. W. HETHCOTE, *Asymptotic approximations with error bounds for zeros of Airy and cylindrical functions*, Doctoral thesis, University of Michigan, Ann Arbor, 1968.

[4] J. C. P. MILLER, *The Airy integral*, British Association Mathematical Tables, vol. B, Cambridge University Press, Cambridge, 1946.

[5] F. W. OLVER, *The asymptotic expansion of Bessel functions of large order*, Philos. Trans. Roy. Soc. London Ser. A, 247 (1954), pp. 328–368.

[6] ———, *Tables of Bessel functions of moderate or large orders*, National Physical Laboratory Mathematical Tables, vol. 6, Department of Scientific and Industrial Research, Her Majesty's Stationery Office, London, 1962.

[7] ———, *Error bounds for first approximations in turning point problems*, J. Soc. Indust. Appl. Math., 11 (1963), pp. 748–772.

[8] ———, *Error bounds for asymptotic expansions in turning point problems*, Ibid., 12 (1964), pp. 200–214.

[9] ———, *Error bounds for asymptotic expansions with an application to cylindrical functions of large argument*, Asymptotic Solutions of Differential Equations, C. H. Wilcox, ed., John Wiley, New York, 1964, pp. 163–184.

[10] F. G. TRICOMI, *Sugli zeri delle funzioni di cui si conosce una rappresentazione asintotica*, Ann. Mat. Pura Appl., 26 (1947), pp. 283–300.

[11] ———, *Sulle funzioni di Bessel di ordine e argomento pressochè uguali*, Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur., 83 (1949), pp. 3–20.

[12] G. N. WATSON, *Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1944.

SIAM J. MATH. ANAL.
Vol. 1, No. 2, May 1970

# SIMPLE TURNING-POINT PROBLEMS IN UNBOUNDED DOMAINS*

WOLFGANG WASOW†

**Abstract.** Differential equations of the form $\varepsilon^2 u'' = p(x)u$, where $p(x)$ is a polynomial, are reduced to Airy's equation by linear transformations of the dependent variable that admit asymptotic expansions in powers of $\varepsilon$. It is shown that such transformations may either be required to be valid in bounded regions containing one simple zero of $p(x)$ (Condition L) or to be doubly asymptotic, as $\varepsilon \to 0$ or $x \to \infty$, in certain unbounded domains (Condition F). These conditions are shown to be, in general, incompatible. Asymptotic connection formulas between the distinct fundamental solutions obtained by these two methods are derived which permit the continuation into the zeros of $p(x)$ of the asymptotic evaluation for solutions based on Condition F.

**1. Introduction.** The best known of the so-called *turning-point problems* is concerned with the properties, as $\varepsilon \to 0+$, of differential equations of the form

$$\varepsilon^2 \frac{d^2u}{dx^2} - p(x, \varepsilon)u = 0 \tag{1.1}$$

in regions of the complex $x$-plane where $p(x, 0)$ has zeros. Many of the essential features of this problem are present even in the simple case that $p(x, \varepsilon) = p(x)$ is a *polynomial*, independent of $\varepsilon$. To avoid secondary, primarily technical difficulties this assumption will here be adopted.

I shall be primarily interested in regions of the plane that contain exactly one turning point, say $x = x_1$. A decisive restrictive condition will be imposed: this turning point is to be simple, i.e.,

$$p(x_1) = 0, \qquad p'(x_1) \neq 0. \tag{1.2}$$

Under this assumption it is possible to solve the differential equation

$$\varepsilon^2 \frac{d^2u}{dx^2} - p(x)u = 0, \tag{1.3}$$

and considerably more general ones, by a combination of Airy functions and asymptotic series in powers of $\varepsilon$.

In many applications one is interested in the asymptotic nature of the solutions of the differential equations (1.1) or (1.3) with respect to the passage to the limit as $x \to \infty$, as well as with the passage to the limit as $\varepsilon \to 0$. These two questions are distinct but closely related, and a theory that pays attention to both simultaneously is especially valuable. In other words, one should aim at approximate solutions such that the errors are small when either $x$ is large or $\varepsilon$ is small.

It is well known that there are particular solutions of the differential equation (1.3)—or (1.1)—which tend exponentially to zero, as $x$ tends to infinity in certain sectors. They are called subdominant solutions in such a sector, and they are uniquely characterized, to within an arbitrary constant factor, which may depend on $\varepsilon$, by this asymptotic property. Fedoryuk [2], [3] has shown that for an appro-

* Received by the editors June 5, 1969, and in revised form October 12, 1969.

† Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706.

priate choice of the constant factor the resulting subdominant solution admits a series expansion which is *doubly* asymptotic in the sense that the error terms become small of increasingly high order of magnitude both in $\varepsilon$ and in $x$, as more terms of the series are included. Theorem 4.3 below, which is proved in [2], gives a precise statement. We shall refer to such solutions as being of Fedoryuk's type. Fedoryuk and Evgrafov have developed this result into a global theory of the asymptotic behavior of these solutions, which is probably the most complete such analysis in existence. It does not include, however, the neighborhoods of the turning points.

The existence of solutions of Fedoryuk's type implies the possibility of transforming the given equation (1.3) into Airy's equation by means of functions that also have doubly asymptotic expansions as $\varepsilon \to 0$ or $x \to \infty$. This is proved in Theorem 4.2 below. A formally very similar transformation was used by Langer ([4] and many other papers) and generalized by Wasow [8], [9] to transform the given equation into Airy's equation in *bounded* domains that contain a turning point. The power series involved are asymptotic as $\varepsilon \to 0$. Olver ([5], [6], etc.) has shown that these expansions can be chosen so as to be uniformly valid even in *certain unbounded* domains. The solutions so obtained do not, however, have the convenient doubly asymptotic expansion. As all these constructions contain arbitrary constants, it is natural to ask if they can be chosen so as to be valid in unbounded domains containing a turning point and, at the same time, to have the doubly asymptotic property described. Theorem 3.1 below answers this question essentially in the negative and thus dashes the hope that the "matching" of different solutions could be obviated in this way. By treating the two requirements, viz., validity at the turning point and doubly asymptotic character at infinity, in analogous ways, the present paper attempts to make these matters more transparent. I also believe that the matching procedure explained in § 5 and illustrated in § 6 is comparatively simple.

The matching could also have been based on the work of Cherry in [1], which is based on transformations of the *independent* variable, but the computations would be more involved. There is hope that some of the methods of the present paper can be extended to differential equations of higher order. Transformations of the independent variable are probably too special for that purpose.

In [7] Olver develops a theory for the approximate solution of linear second order differential equations which does not introduce a small parameter explicitly. Instead, solutions are constructed which differ from known solutions of simpler equations, such as Airy's equation, by quantities for which explicit estimates are derived. Olver then shows how this approach can be developed so as to give approximate lateral connection formulas for various turning-point problems. For many applications, particularly those where the small parameter is only a mathematical device to express precisely the idea of a "slowly varying" function, this is a most appropriate procedure. For more precise requirements, such as approximations to any degree of accuracy at and away from a turning point in problems with a physically important parameter, Olver's technique probably involves as much additional work as went into Fedoryuk's theory. Moreover, as asymptotic series in powers of $x^{-1}$ or of $\varepsilon$ are in general use, a comparative study of different such expansions as attempted in this paper may be of interest.

**2. Preliminary transformations.** We need the analytic functions $\xi$ and $t$ of $x$ defined by

$$\xi(x_1, x) = \int_{x_1}^{x} p^{1/2}(\tau)\,d\tau, \qquad \xi(x_1, x_1) = 0, \tag{2.1}$$

$$t(x_1, x) = \left[\frac{3}{2}\xi(x_1, x)\right]^{2/3}. \tag{2.2}$$

Whenever no misunderstanding is to be feared, we shall replace $\xi(x_1, x)$, $t(x_1, x)$ by the shorter notation $\xi(x)$, $t(x)$. While $\xi(x)$ has a branch point at $x = x_1$, the right member of (2.2) represents three holomorphic functions in a neighborhood of $x_1$, which are obtained from each other by multiplication with a cube root of unity (see Wasow [9, pp. 161–162]). We need the following facts on the mappings induced by these functions.

At $x = x_1$, there meet three curves on which $\operatorname{Re} \xi(x) = 0$, forming equal angles there. Generally speaking, any curve of the family $\operatorname{Re} \xi(x) = \text{const.}$ that contains a turning point will be called a Stokes curve. Evgrafov and Fedoryuk [2] have shown that the Stokes curves divide the $x$-plane into a set of unbounded simply connected regions that contain no Stokes curves. They may be called *Stokes regions.* In each of the three Stokes regions that abut on $x = x_1$, each holomorphic branch of $\xi(x_1, x)$ has a nonvanishing real part. A branch of $\xi(x_1, x)$ in such a Stokes region will be referred to as *recessive* or *dominant* according as $\operatorname{Re} \xi(x_1, x) < 0$ or $\operatorname{Re} \xi(x_1, x) > 0$ there. The mapping (2.1) takes each Stokes region into a region of the $\xi$-plane that is either a vertical strip or a half-plane bounded by a parallel to the imaginary $\xi$-axis. (All these facts and many related ones are explained in detail in Evgrafov and Fedoryuk [2].) Accordingly, we shall distinguish between Stokes regions of *strip type* and of *half-plane type.* The example and the figures in § 6 may be helpful.

A *canonical region* in the $x$-plane is defined as a union of two or more Stokes regions together with the boundaries between them which has the property that its image under a holomorphic branch of $\xi(x_1, x)$ is the whole schlicht $\xi$-plane except for a finite number of cuts. These cuts are images of Stokes curves and are therefore parallel to the imaginary axis, extending from the image of some turning point to infinity. Let us call a canonical region *consistent* or *inconsistent*, according as these cuts in the $\xi$-plane do or do not all tend to infinity in the same direction.

We shall be primarily concerned with a consistent canonical region $D$ having the turning point $x_1$ on its boundary. Let $\xi(D)$ designate the image of such a $D$ in the $\xi$-plane. The condition $\xi(x_1, x_1) = 0$, together with the stipulation that the boundary cuts of $\xi(D)$ are to approach infinity in the direction of *increasing* imaginary parts, determines a unique branch of the function $\xi(x_1, x)$ in $D$. Since $d\xi/dx = p^{1/2}(x)$ does not vanish in the simply connected region $D$, the function $\xi(x_1, x)$ has a holomorphic inverse in $\xi(D)$.

The function $t = (\frac{3}{2}\xi)^{2/3}$ from $\xi(D)$ into the $t$-plane maps $\xi(D)$ onto a simply connected region consisting of a sector of central angle $4\pi/3$, bounded by two of the three rays $\operatorname{arc} t = \mp\pi/3, \pi$, except that the images of the boundary cuts of $\xi(D)$ other than the one beginning at $\xi = 0$ have to be deleted. We specify the branch of $t = (\frac{3}{2}\xi)^{2/3}$ in $\xi(D)$ by requiring that the corresponding sector in the

$t$-plane be $\pi/3 < \arg t < 5\pi/3$. The function $t = (\frac{3}{2}\xi(x_1, x))^{2/3}$ is now a uniquely defined function, holomorphic in $D$. It can even be continued, as a holomorphic function, into the larger domain $D \cup K$ if $K$ is a sufficiently small disk with center at $x_1$. Let $R_t$ be the image of $D \cup K$ in the $t$-plane under the mapping $t = t(x_1, x) = (\frac{3}{2}\xi(x_1, x))^{2/3}$. For $K$ small enough this mapping is univalent and $R_t$ is simply connected. The inverse of $t(x_1, x)$ is then a holomorphic function in $R_t$.

The importance of the function $t(x_1, x)$ stems from the fact—which can be verified directly—that the transformation (2.2) takes the given differential equation (1.3) into

$$\varepsilon^2 \frac{d^2u}{dt^2} - tu + \varepsilon^2 g(t)\frac{du}{dt} = 0, \tag{2.3}$$

where

$$g(t) = -\frac{d^2x}{dt^2}\left(\frac{dx}{dt}\right)^{-1} = \frac{d^2t}{dx^2}\left(\frac{dt}{dx}\right)^{-2}. \tag{2.4}$$

Thus, $g(t)$ is holomorphic in $R_t$, and (2.3) can be regarded as a perturbation of the simpler equation

$$\varepsilon^2 \frac{d^2v_1}{dt^2} - tv_1 = 0, \tag{2.5}$$

which is equivalent to Airy's equation

$$\frac{d^2\tilde{v}}{dz^2} - z\tilde{v} = 0 \tag{2.6}$$

by virtue of the transformation

$$z = t\varepsilon^{-2/3}. \tag{2.7}$$

The further simplifications of (2.3) become more transparent if (2.3) is replaced by the equivalent system

$$\varepsilon\frac{dy}{dt} = \left[\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} + \varepsilon\begin{pmatrix} 0 & 0 \\ 0 & -g(t) \end{pmatrix}\right] y = [A_0(t) + \varepsilon A_1(t)]y, \tag{2.8}$$

where

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} u \\ \varepsilon du/dt \end{pmatrix}, \quad A_0(t) = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}, \quad A_1(t) = \begin{pmatrix} 0 & 0 \\ 0 & -g(t) \end{pmatrix}. \tag{2.9}$$

**3. Formal reductions to Airy's equation.** Our aim is to reduce the system (2.8) to the system form of (2.5), i.e.,

$$\varepsilon\frac{dv}{dt} = A_0(t)v, \tag{3.1}$$

by means of a change of the dependent variable of the form

$$y = P(t, \varepsilon)v. \tag{3.2}$$

This will be achieved if and only if $P(t, \varepsilon)$ satisfies the matrix differential equation

$$\varepsilon \frac{dP}{dt} = A_0(t)P - PA_0(t) + \varepsilon A_1(t)P. \tag{3.3}$$

The essence of Langer's method and its generalizations in Wasow [8], [9] is to satisfy (3.3) formally by a power series

$$\sum_{r=0}^{\infty} P_r(t)\varepsilon^r \tag{3.4}$$

for $P$ in such a way that all $P_r(t)$ are holomorphic in a fixed neighborhood of the turning point $t = 0$. We shall now describe a variant of the procedure in Wasow [9] that has the advantage of leading to a unified analysis of solutions of Langer's type as well as of those of Fedoryuk's type. Insertion of (3.4) for $P$ into (3.3) and a formal rearrangement produces the recursive sequence of equations

$$A_0(t)P_r - P_rA_0(t) - \frac{dP_{r-1}}{dt} = -A_1(t)P_{r-1}, \qquad P_{-1} = 0, \quad r = 0, 1, 2, \cdots. \tag{3.5r}$$

In scalar form (3.5r) reads

$$\begin{pmatrix} p_{21}^r - tp_{12}^r & p_{22}^r - p_{11}^r \\ t(p_{11}^r - p_{22}^r) & tp_{12}^r - p_{21}^r \end{pmatrix} = \begin{pmatrix} \dot p_{11}^{r-1} & \dot p_{12}^{r-1} \\ \dot p_{21}^{r-1} + gp_{21}^{r-1} & \dot p_{22}^{r-1} + gp_{22}^{r-1} \end{pmatrix}. \tag{3.6r}$$

Here, $P_r = \{p_{jk}^r\}$, the dot means differentiation with respect to $t$, and the argument $t$ has been omitted for brevity.

The four linear algebraic equations for the $p_{jk}^r$, $j, k = 1, 2$, collected in (3.6r) are compatible if and only if the right members satisfy the same linear identities as the left members, i.e., if

$$\begin{aligned} \dot p_{11}^{r-1} + \dot p_{22}^{r-1} + gp_{22}^{r-1} &= 0, \\ t\dot p_{12}^{r-1} + \dot p_{21}^{r-1} + gp_{21}^{r-1} &= 0 \end{aligned} \tag{3.7r}$$

(cf. Wasow [9, p. 164]). If $P_{r-1}(t)$ has already been determined so that the compatibility conditions (3.7r) are fulfilled, then any $P_r(t)$ that satisfies the two equations

$$p_{21}^r - tp_{12}^r = \dot p_{11}^{r-1}, \qquad p_{22}^r - p_{11}^r = \dot p_{12}^{r-1} \tag{3.8r}$$

from (3.6r) will satisfy all four conditions (3.6r). To find the ones among these matrices $P_r(t)$ for which the next set of compatibility conditions, namely (3.7($r+1$)), is also fulfilled we differentiate (3.8r) and eliminate $p_{22}^r, p_{21}^r$. There result then the two uncoupled linear differential equations

$$\dot p_{11}^r + \frac{g}{2}p_{11}^r = -\frac{1}{2}(\ddot p_{12}^{r-1} + g\dot p_{12}^{r-1}), \qquad r = 0, 1, \cdots, \tag{3.9ra}$$

$$t\dot p_{12}^r + \frac{1}{2}(1 + gt)p_{12}^r = -\frac{1}{2}(\ddot p_{11}^{r-1} + g\dot p_{11}^{r-1}), \qquad r = 0, 1, \cdots. \tag{3.9rb}$$

To any solution $p_{11}^r(t)$, $p_{12}^r(t)$ of (3.9r) one finds then corresponding functions $p_{21}^r(t)$, $p_{22}^r(t)$ from (3.8r). Every formal solution (3.4) of (3.3) can be obtained in this manner. This proves the following lemma.

LEMMA 3.1. *The series* $\sum_{r=0}^{\infty} P_r(t)\varepsilon^r$ *is a formal solution of the system* (3.3) *if and only if the entries* $p_{jk}^r(t)$, $j,k = 1,2$, $r = 0,1,2,\cdots$, *of* $P_r(t)$ *satisfy the recursive conditions* (3.8r), (3.9r), *where* $P_{-1} \equiv 0$, *by definition.*

Let us write, for abbreviation,

$$q(t) = \left(\frac{dx}{dt}\right)^{1/2} = \left(\frac{t}{p(x)}\right)^{1/4}, \tag{3.10}$$

with a fixed, but arbitrary holomorphic determination of the square root in $R_t$. Then the general solutions of (3.9r) can be written, by using also formula (2.4) for $g(t)$,

$$\begin{aligned} p_{11}^r(t) &= c_{1r}q(t) - \frac{1}{2}q(t)\int_{\alpha_{1r}}^{t} q(\tau)\frac{d}{d\tau}[\dot{p}_{12}^{r-1}(\tau)q^{-2}(\tau)]\,d\tau, \\ p_{12}^r(t) &= c_{2r}t^{-1/2}q(t) - \frac{1}{2}t^{-1/2}q(t)\int_{\alpha_{2r}}^{t} \tau^{-1/2}q(\tau)\frac{d}{d\tau}[\dot{p}_{11}^{r-1}(\tau)q^{-2}(\tau)]\,d\tau, \end{aligned} \tag{3.11r}$$

where $c_{1r}, c_{2r}$ are arbitrary complex constants and $\alpha_{1r}, \alpha_{2r}$ are arbitrary points in $R_t$.

The choice of the constants in (3.11r) depends on the properties we want the solutions to possess. Two sets of conditions are of interest to us.

CONDITION L. In some disk $|t| \leqq t_0$,

(a) $\det P_0(t) \neq 0$,

(b) all $P_r(t)$ are holomorphic.

CONDITION F. For every $t_1 > 0$ and for $t \in R_t$, $|t| \geqq t_1$,

(a) $\det P_0(t) \neq 0$,

(b) there exist constants $c_r(t_1)$ such that

$$\|P_r(t)\| \leqq c_r(t_1)|t|^{-\beta r}, \qquad r = 0,1,2,\cdots, \tag{3.12}$$

where $\beta$ is a positive constant.

These conditions lead to solutions of Langer's or of Fedoryuk's type, respectively, as is shown below.

LEMMA 3.2. *There exists a formal solution* $\sum_{r=0}^{\infty} P_r^L(t)\varepsilon^r$ *of the system* (3.3) *which satisfies Condition* L. *This solution is not unique.*

*Proof.* In (3.11r) choose $a_{1r} = \alpha_{2r} = 0$ for all $r$; let $c_{10} = 1$, $c_{20} = 0$, and $c_{1r} = c_{2r} = 0$ for $r = 1,2,\cdots$. Then one finds

$$P_0^L(t) = q(t)I. \tag{3.13}$$

The right members of (3.11r) and the solutions of (3.8r) turn out to be holomorphic, as required.

However, the conditions $c_{1r} = 0$, $\alpha_{1r} = 0$, $r > 0$ are not all necessary, as the next lemma shows.

LEMMA 3.3. *The series* $\sum_{r=0}^{\infty}\tilde{P}_r(t)\varepsilon^r$ *is a formal solution of* (3.3) *which satisfies Condition* L *if and only if*

$$\sum_{r=0}^{\infty} \tilde{P}_r(t)\varepsilon^r = \left(\sum_{r=0}^{\infty} \gamma_r\varepsilon^r\right)\left(\sum_{r=0}^{\infty} P_r^L(t)\varepsilon^r\right) \tag{3.14}$$

*as a relation among formal power series. The* $\gamma_r$ *are arbitrary scalar constants with* $\gamma_0 \neq 0$.

*Proof.* The right member of (3.14) when expanded and arranged according to powers of $\varepsilon$ satisfies (3.3) formally, because the formal power series in $\varepsilon$ with holomorphic coefficients form a ring with differentiation. The coefficients $\tilde{P}_r(t)$ of this power series are holomorphic in $|t| \leqq t_0$, and $\tilde{P}_0(t) = \gamma_0 P_0^L(t)$, which is a nonsingular matrix. Conversely, let the left member of (3.14) be a formal solution of (3.3) satisfying Condition L. Then the formal transformation $y = (\sum_{r=0}^{\infty} \tilde{P}_r(t)\varepsilon^r)v$ takes (2.8) into (3.1), as does the transformation $y = (\sum_{r=0}^{\infty} P_r^L(t)\varepsilon^r)v$. Hence, the composite transformation with matrix

$$\left(\sum_{r=0}^{\infty} P_r^L(t)\varepsilon^r\right)^{-1}\left(\sum_{r=0}^{\infty} \tilde{P}_r(t)\varepsilon^r\right)$$

takes (3.1) into itself. The last matrix is to be interpreted as the formal series $\sum_{r=0}^{\infty} Q_r(t)\varepsilon^r$, obtained by expanding and multiplying the product above. This implies that the $Q_r(t)$ satisfy the same recursion formulas (3.5r) as the $P_r(t)$ with the specialization that $q(t) \equiv 1$. Solving the equations (3.11r) successively in this special case and remembering that all $Q_r(t)$ are holomorphic at $t = 0$, one finds, successively, that $Q_r(t) = \gamma_r I$, as was to be proved.

LEMMA 3.4. *There exists a formal solution* $\sum_{r=0}^{\infty} P_r^F(t)\varepsilon^r$ *of the system* (3.3) *which satisfies Condition* F *in the strengthened form that*

$$\begin{bmatrix} |p_{11}^r(t)| & |p_{12}^r(t)| \\ |p_{21}^r(t)| & |p_{22}^r(t)| \end{bmatrix} \leqq c_r(t_1)|t|^{-(m-1)/[2(m+2)]-3r/2} \begin{bmatrix} 1 & |t|^{-1/2} \\ |t|^{1/2} & 1 \end{bmatrix} \tag{3.15}$$

*for* $|t| \geqq t_1 > 0$, $t \in R_t$. (Here $m$ is the degree of $p(x)$. This inequality combines four scalar inequalities in a self-explanatory manner.) *The inequalities may be differentiated.*

*Proof.* Let $P_0^F = P_0^L$. As

$$q(t) = O(t^{-(m-1)/[2(m+2)]}) \quad \text{as } t \to \infty, \tag{3.16}$$

the inequalities (3.15) are then true for $r = 0$. It is clear that

$$q'(t) = O(t^{-(m-1)/[2(m+2)]-1}),$$

i.e., that (3.16) may be formally differentiated. Assume that (3.15) and the corresponding inequalities for the derivatives are true for $r = k - 1, k \geqq 1$. Then choose $c_{jk} = 0, \alpha_{jk} = \infty, j = 1, 2$, and take the paths of integration in (3.11$k$) in the region $R_t$ at a positive distance from the boundary. A short calculation based on (3.11$k$), the inductive hypothesis and (3.16) then yields the required order of magnitude for $p_{11}^k(t)$, $p_{12}^k(t)$. Calculation of $p_{21}^k(t)$, $p_{22}^k(t)$ from (3.8$k$) then completes the proof by induction.

LEMMA 3.5. *Every formal solution of the system* (3.3) *which satisfies Condition* F *is of the form* $\gamma_0 \sum_{r=0}^{\infty} P_\gamma^F(t)\varepsilon^r$, $\gamma_0 \neq 0$.

*Proof.* By analogy with the reasoning in the proof of Lemma 3.3 the argument can be reduced to showing that the equations (3.11) and (3.8), with $P_{-1} \equiv 0$, when specialized by setting $q(t) \equiv 1$, have no solutions other than

$$P_0 = \gamma_0 I, \qquad p^r_{jk} = 0, \qquad j = 1, 2, \quad r > 0,$$

if Condition F is imposed. That this is the case is easily verified by induction.

THEOREM 3.1. *The formal solution of Fedoryuk's type,* $\sum_{r=0}^{\infty} P_r^F(t)\varepsilon^r$, *also satisfies Condition* L *if and only if the infinitely many conditions*

$$\int_0^{\infty} \tau^{-1/2} q(\tau) \frac{d}{d\tau}[\dot{p}^{r-1}_{11,F}(\tau) q^{-2}(\tau)]\, d\tau = 0, \qquad r = 1, 2, \cdots, \tag{3.17r}$$

*are satisfied. The path of integration is to be in* $R_t$.

*Proof.* If (3.17r) is true, then (3.11r) and (3.8r) with the choice of the constants $c_{jr}, \alpha_{jr}, j = 1, 2$, as in the proof of Lemma 3.4, show that all $P_r^F(t)$ are holomorphic at $t = 0$. Conversely, with this choice of the constants, (3.11r) yields a $p^r_{12}(t)$ holomorphic at $t = 0$ only if (3.17r) is satisfied.

The relations (3.17r) are truly restrictions on $p(x)$. Even for $r = 1$, i.e., for $p^{r-1}_{11,F}(t) = q(t)$, it is not difficult to construct polynomials $p(x)$ with the required properties for which the integral is not zero. This can, e.g., be done with the help of the Weierstrass approximation theorem. We omit the details.

**4. On the analytic reduction to Airy's equation.**

THEOREM 4.1. *There exists a transformation* $y = P^L(t, \varepsilon)v$ *of the given differential equation in the form* (2.8) *into Airy's equation in the form* (3.1) *such that the matrix* $P^L(t, \varepsilon)$ *has, uniformly in some disk* $|t| \leqq t_0$, *the asymptotic expansion*

$$P^L(t, \varepsilon) \sim \sum_{r=0}^{\infty} P_r^L(t)\varepsilon^r, \qquad \varepsilon \to 0+. \tag{4.1}$$

This theorem was stated in Wasow [9, p. 181]. The proof there contains an error, but a correct proof has been given in the expanded Russian translation of that book.

The formal solution of Fedoryuk's type described in Lemma 3.4 is the asymptotic expansion of another solution of (3.3). The domain of validity of that expansion can be described as follows: Let $\delta > 0$ be arbitrary. From $\xi(D)$ delete circular neighborhoods of radius $\delta$ about the endpoints of the cuts, as well as sectors of central angle $\delta$ that have their vertices at the endpoints of the cuts and are bisected by the cuts. The resulting domain in the $\xi$-plane may be called $\xi(D, \delta)$. The corresponding domains in the $x$- or $t$-plane will be denoted by $D_\delta$ or $D_{t\delta}$, respectively.

THEOREM 4.2. *There exists a transformation* $y = P^F(t, \varepsilon)v$ *of* (2.8) *into* (3.1) *such that the matrix* $P^F(t, \varepsilon)$ *has in* $R_{t\delta}$ *the doubly asymptotic expansion*

$$P^F(t, \varepsilon) \sim \sum_{r=0}^{\infty} P_r^F(t)\varepsilon^r \quad \text{as } \varepsilon \to 0+ \text{ or } t \to \infty. \tag{4.2}$$

*More precisely, for every integer* $N \geqq 0$ *and every sufficiently small* $\delta > 0$ *there exists a constant* $C(N, \delta)$ *such that*

$$\left| P^F(t, \varepsilon) - \sum_{r=0}^{N} P_r(t)\varepsilon^r \right| \leqq C(N, \delta)|t|^{-(m-1)/[2(m+2)]} \begin{bmatrix} 1 & |t|^{-1/2} \\ |t|^{1/2} & 1 \end{bmatrix} (|t|^{-3/2}\varepsilon)^{N+1} \tag{4.3}$$

*for* $t \in R_{t\delta}$, $0 < \varepsilon \leqq \varepsilon_0$. (Formula (4.3) is to be understood as combining four scalar inequalities. The left member is not a norm, but the matrix formed by the absolute values of the four entries, each of which satisfies the corresponding inequality.)

Theorem 4.2 can be proved in a manner analogous to the method in Wasow [9, § 30.1–§ 30.4]. This leads to independent proofs of the theorems in § 3 and § 4 of Evgrafov and Fedoryuk [2]. For simplicity the opposite route will be taken here. We shall use the following results from the last mentioned paper.

THEOREM 4.3 (Evgrafov and Fedoryuk). *The differential equation* (1.2) *has two particular solutions* $u^{\mp}(x, \varepsilon)$ *with the following asymptotic properties*:

$$u^{\mp}(x, \varepsilon) = p^{-1/4}(x)\hat{u}^{\mp}(x, \varepsilon) \exp\left\{\mp\frac{1}{\varepsilon}\xi(x)\right\}, \tag{4.4}$$

*where*

$$\hat{u}^{\mp}(x, \varepsilon) \sim \sum_{r=0}^{\infty} \hat{u}_r^{\mp}(x)\varepsilon^r, \quad \hat{u}_0^{\mp}(x) \equiv 1 \quad \text{as } \varepsilon \to 0+ \text{ or } x \to \infty \text{ in } D_\delta, \tag{4.5}$$

*in the precise sense that*

$$\left|\hat{u}^{\mp}(x, \varepsilon) - \sum_{r=0}^{N} \hat{u}_r^{\mp}(x)\varepsilon^r\right| \leqq C(N, \delta)(|x|^{-(m+2)/2}\varepsilon)^{N+1} \tag{4.6}$$

*for* $x \in D_\delta$. *Here*, $C(N, \delta)$ *is a constant. The functions* $\hat{u}_r^{\mp}(x)$ *are holomorphic in* $D$, *and*

$$\hat{u}_r^{\mp}(x) = O(|x|^{-(m+2)r/2}) \quad \text{as } x \to \infty. \tag{4.7}$$

*These asymptotic formulas may be formally differentiated.*

Each formula above combines two: take the upper or lower sign, throughout. Each solution is uniquely determined by its asymptotic representation.

The solutions $u^{\mp}(x, \varepsilon)$ are uniquely defined only if the branch of $p^{-1/4}(x)$ is specified. According to (3.10),

$$p^{-1/4}(x) = q(t)t^{-1/4}. \tag{4.8}$$

We shall take

$$\pi/3 < \arg t < 5\pi/3 \quad \text{in } R_{t\delta} \quad \text{and} \quad \arg t^\alpha = \alpha \arg t \quad \text{in (4.8)}. \tag{4.9}$$

*Proof of Theorem* 4.2. If the two scalar solutions $u^+(x, \varepsilon)$, $u^-(x, \varepsilon)$ are transformed to the variable $t$, then the matrix $Y^F(t, \varepsilon)$ whose first row consists of $u^+$, $u^-$, in this order, while the second row is $(\varepsilon\, du^+/dt, \varepsilon\, du^-/dt)$, is a fundamental matrix solution of (2.8). Its asymptotic form in $R_{t\delta}$, as calculated from Theorem 4.3, turns out to be, after some straightforward calculations,

$$Y^F(t, \varepsilon) = (p(x(t)))^{-1/4}\begin{bmatrix} 1 & 0 \\ 0 & t^{1/2} \end{bmatrix}\hat{Y}^F(t, \varepsilon)\begin{bmatrix} \exp\left\{\frac{2}{3\varepsilon}t^{3/2}\right\} & 0 \\ 0 & \exp\left\{-\frac{2}{3\varepsilon}t^{3/2}\right\} \end{bmatrix} \tag{4.10}$$

with the rule (4.9) applying. The matrix $\hat{Y}^F(t, \varepsilon)$ has an asymptotic expansion

$$\hat{Y}^F(t, \varepsilon) \sim \sum_{r=0}^{\infty} \hat{Y}_r(t)\varepsilon^r, \qquad \hat{Y}_0(t) = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{4.11}$$

in the strong sense that, for $0 < \varepsilon \leqq \varepsilon_0$, $t \in R_{t\delta}$,

$$\left\|\hat{Y}^F(t, \varepsilon) - \sum_{r=0}^{N} \hat{Y}_r(t)\varepsilon^r\right\| \leqq C_2(N, \delta)(|t|^{-3/2}\varepsilon)^{N+1}. \tag{4.12}$$

$C_2(N, \delta)$ is a constant whose value depends on the choice of the matrix norm. The $\hat{Y}_r(t)$ are holomorphic in $R_{t\delta}$ and of the order of magnitude

$$\hat{Y}_r(t) = O(x^{-(m+2)r/2}) = O(t^{-3r/2}) \quad \text{as } t \to \infty. \tag{4.13}$$

For $p(x) \equiv x$, in particular, the matrix (4.10) solves the simpler equation (3.1). In that case it will be called $V(t, \varepsilon)$. It has the form

$$V(t,\varepsilon) = t^{-1/4}\begin{bmatrix} 1 & 0 \\ 0 & t^{1/2} \end{bmatrix} \hat{V}(t,\varepsilon) \begin{bmatrix} \exp\left\{\frac{2}{3\varepsilon}t^{3/2}\right\} & 0 \\ 0 & \exp\left\{-\frac{2}{3\varepsilon}t^{3/2}\right\} \end{bmatrix}, \tag{4.14}$$

with an expansion for $\hat{V}(t, \varepsilon)$, which is a special case of (4.11), (4.12). The domain of validity corresponding to $R_{t\delta}$ in this case is defined by

$$\frac{\pi + \delta}{3} < \arg t < \frac{5\pi - \delta}{3}, \qquad |t| > \left(\frac{3}{2}\delta\right)^{2/3}. \tag{4.15}$$

Our choice of branches for the various multivalued functions was made so as to deal with a region symmetric to the real axis in the $t$-plane. This preference implies that $t = xe^{2\pi i/3}$ for (3.1), which is somewhat unnatural. Accordingly, we can take $q(t) \equiv e^{2\pi i/3}$ ($e^{-\pi i/2}$ would have been another possible definition), but not $q(t) \equiv 1$.

Let us define $P(t, \varepsilon)$ by

$$Y^F = P(t,\varepsilon)V, \quad \text{i.e.,} \quad P(t,\varepsilon) = Y^F(t,\varepsilon)V^{-1}(t,\varepsilon), \tag{4.16}$$

and observe that

$$P(t,\varepsilon) = p^{-1/4}(x(t))\begin{bmatrix} 1 & 0 \\ 0 & t^{1/2} \end{bmatrix} \hat{Y}^F(t,\varepsilon)\hat{V}^{-1}(t,\varepsilon)\begin{bmatrix} 1 & 0 \\ 0 & t^{-1/2} \end{bmatrix} t^{1/4}. \tag{4.17}$$

One verifies directly, with the help of formulas (4.11) to (4.13), and the corresponding formulas for $\hat{V}(t, \varepsilon)$, that the expansion of $\hat{Y}^F(t, \varepsilon)\hat{V}^{-1}(t, \varepsilon)$ has the form

$$\hat{Y}^F(t,\varepsilon)\hat{V}^{-1}(t,\varepsilon) \sim \sum_{r=0}^{\infty} Z_r(t)\varepsilon^r, \qquad Z_0(t) \equiv I,$$

where

$$\left\| \hat{Y}^F(t,\varepsilon)\hat{V}^{-1}(t,\varepsilon) - \sum_{r=0}^{N} Z_r(t)\varepsilon^r \right\| \leqq C_3(N,\delta)(|t|^{-3/2}\varepsilon)^{N+1}$$

for $|t| \in R_{t\delta}$, and

$$Z_r(t) = O(t^{-3r/2}) \quad \text{as } t \to \infty.$$

Combining these facts, formulas (3.10), (3.16), (4.16) and (4.17), one is led to

$$\left| P(t,\varepsilon) - \sum_{r=0}^{N} P_r(t)\varepsilon^r \right| \leqq C(N,\delta)|t|^{-(m-1)/[2(m+2)]}\begin{bmatrix} 1 & |t|^{-1/2} \\ |t|^{1/2} & 1 \end{bmatrix}(|t|^{-3/2}\varepsilon)^{N+1}$$

for $t \in R_{t\delta}$ (the meaning of this formula is as in (4.3)), and

$$P_0(t) = q(t)I, \qquad P_r(t) = O(t^{1-(m-1)/[2(m+2)]-3r/2}), \qquad r = 1, 2, \cdots.$$

This last relation shows that the series $\sum_{r=0}^{\infty} P_r(t)\varepsilon^r$ satisfies Condition F. Also, $P(t, \varepsilon)$ must satisfy the system (3.3) because of (4.16). From Lemma 3.5 it follows that $P_r(t) = P_r^F(t)$ for all $r$. This completes the proof of Theorem 4.2.

**5. Central connection formulas.**

LEMMA 5.1. *The particular solution matrix* $V(t, \varepsilon)$ *of* (3.1), *as given in* (4.14), *has the following expression in terms of special functions*:

$$V(t, \varepsilon) = 2\sqrt{\pi}\begin{bmatrix} e^{-\pi i/6}\varepsilon^{-1/6}\mathrm{Ai}\,(e^{-2\pi i/3}\varepsilon^{-2/3}t) & e^{-\pi i/3}\varepsilon^{-1/6}\mathrm{Ai}\,(e^{-4\pi i/3}\varepsilon^{-2/3}t) \\ e^{7\pi i/6}\varepsilon^{1/6}\mathrm{Ai}'\,(e^{-2\pi i/3}\varepsilon^{-2/3}t) & e^{\pi i/3}\varepsilon^{1/6}\mathrm{Ai}'\,(e^{-4\pi i/3}\varepsilon^{-2/3}t) \end{bmatrix}. \tag{5.1}$$

*Here*, $\mathrm{Ai}(z)$ *is Airy's function and* $\mathrm{Ai}'(z) = d\mathrm{Ai}(z)/dz$.

*Proof.* $\mathrm{Ai}(e^{4\pi i/3}\varepsilon^{-2/3}t)$ and $\mathrm{Ai}(e^{2\pi i/3}\varepsilon^{-2/3}t)$ are solutions of (2.5) that tend to zero in $\pi/3 < \arg t < \pi$ and $\pi < \arg t < 5\pi/3$, respectively, as $t \to \infty$. This property characterizes them uniquely except for arbitrary constant factors. Theorem 4.3, applied to (2.5) with $x = t$, also supplies two solutions that tend to zero in the same sectors, respectively. The constant factors of proportionality connecting these two sets of solutions can be determined by comparing the leading terms of their asymptotic expansions for $t \to \infty$ in these sectors, as given by the theory of Airy's function and by formulas (4.14), (4.11). Now, the first row of $V(t, \varepsilon)$ consists precisely of this latter pair of solutions of (2.5), while the second row is then obtained by operating with $\varepsilon\, d/dt$ on the first row (cf. (2.9)). Thus, a straightforward calculation leads to (5.1).

The importance of Lemma 5.1 lies in the fact that the matrix $V(t, \varepsilon)$, introduced in § 4 through its asymptotic expansions in certain regions, is now completely known throughout the $t$-plane.

We now consider the two particular solutions

$$Y^F(t, \varepsilon) = P^F(t, \varepsilon)V(t, \varepsilon) \quad \text{and} \quad Y^L(t, \varepsilon) = P^L(t, \varepsilon)V(t, \varepsilon) \tag{5.2}$$

of the system (2.8) and proceed to calculate the asymptotic form of the matrix $C(\varepsilon)$ in the formula

$$Y^F(t, \varepsilon) = Y^L(t, \varepsilon)C(\varepsilon), \tag{5.3}$$

which must connect them. This can be done by substituting into the right member of the formula

$$C(\varepsilon) = [Y^L(t, \varepsilon)]^{-1}Y^F(t, \varepsilon) = V^{-1}(t, \varepsilon)[P^L(t, \varepsilon)]^{-1}P^F(t, \varepsilon)V(t, \varepsilon) \tag{5.4}$$

the asymptotic expansions of the several matrices at a point $t$ at which they are all valid. Applying (4.1), (4.2) and (4.14) and remembering that $P^L$ and $P^F$ have the same leading term in their expansions, we find, after some manipulation, that

$$C(\varepsilon) = \begin{bmatrix} c_{11}(\varepsilon) & c_{12}(\varepsilon) \\ c_{21}(\varepsilon) & c_{22}(\varepsilon) \end{bmatrix} = \begin{bmatrix} 1 + \varepsilon\gamma_{11}(\varepsilon) & \gamma_{12}(t, \varepsilon)\exp\left\{-\frac{4}{3\varepsilon}t^{3/2}\right\} \\ \gamma_{21}(t, \varepsilon)\exp\left\{\frac{4}{3\varepsilon}t^{3/2}\right\} & 1 + \varepsilon\gamma_{22}(\varepsilon) \end{bmatrix}, \tag{5.5}$$

where the $\gamma_{jk}$ have asymptotic series in powers of $\varepsilon$ as $\varepsilon \to 0$, valid in $R_{t\delta} \cap (|t| \leqq t_0)$. We choose $t = t_0\, e^{4\pi i/3}$ to calculate $c_{12}(\varepsilon)$ and $t = t_0\, e^{2\pi i/3}$ to calculate $c_{21}(\varepsilon)$, and find that

$$C(\varepsilon) = \begin{bmatrix} 1 + \varepsilon\gamma_{11}(\varepsilon) & \tilde{\gamma}_{12}(\varepsilon) \exp\left\{-\frac{4}{3\varepsilon} t_0^{3/2}\right\} \\ \tilde{\gamma}_{21}(\varepsilon) \exp\left\{-\frac{4}{3\varepsilon} t_0^{3/2}\right\} & 1 + \varepsilon\gamma_{22}(\varepsilon) \end{bmatrix}. \tag{5.6}$$

Here $\tilde{\gamma}_{21}(\varepsilon)$ and $\tilde{\gamma}_{12}(\varepsilon)$ have asymptotic series in powers of $\varepsilon$. In obvious scalar notation (5.3) now becomes

$$\begin{aligned} y_{11}^F(t, \varepsilon) &= (1 + \varepsilon\gamma_{11}(\varepsilon) y_{11}^L(t, \varepsilon) + \tilde{\gamma}_{21}(\varepsilon) \exp\left\{-\frac{4}{3\varepsilon} t_0^{3/2}\right\} y_{12}^L(t, \varepsilon), \\ y_{12}^F(t, \varepsilon) &= \tilde{\gamma}_{12}(\varepsilon) \exp\left\{-\frac{4}{3\varepsilon} t_0^{3/2}\right\} y_{11}^L(t, \varepsilon) + (1 + \varepsilon\gamma_{22}(\varepsilon)) y_{12}^L(t, \varepsilon), \\ y_{21}^F(t, \varepsilon) &= (1 + \varepsilon\gamma_{11}(\varepsilon)) y_{21}^L(t, \varepsilon) + \tilde{\gamma}_{21}(\varepsilon) \exp\left\{-\frac{4}{3\varepsilon} t_0^{3/2}\right\} y_{22}^L(t, \varepsilon), \\ y_{22}^F(t, \varepsilon) &= \tilde{\gamma}_{12}(\varepsilon) \exp\left\{-\frac{4}{3\varepsilon} t_0^{3/2}\right\} y_{21}^L(t, \varepsilon) + (1 + \varepsilon\gamma_{22}(\varepsilon)) y_{22}^L(t, \varepsilon). \end{aligned} \tag{5.7}$$

Since the asymptotic form of $C(\varepsilon)$ is known, formulas (5.7) extend our asymptotic knowledge of $Y^F$ into the disk $|t| \leqq t_0$, where $Y^L$ is asymptotically known.

We now return from the system (2.8) to the original scalar equation (1.3) and define solutions $u_L^+(x, \varepsilon)$, $u_L^-(x, \varepsilon)$ of that equation by

$$u_L^+(x, \varepsilon) = y_{11}^L(t(x), \varepsilon), \qquad u_L^-(x, \varepsilon) = y_{12}^L(t(x), \varepsilon).$$

These are two of the solutions studied, in different notation, in Wasow [9, § 30]. Their asymptotic expansions anywhere in a full neighborhood of the turning point $x = x_1$ are known, thanks to Theorem 4.1, (5.2) and Lemma 5.1. Formula (5.7) implies the following theorem.

THEOREM 5.1. *Let $a > 0$ be such that $|t(x)| < t_0$ for $|x - x_1| \leqq a$. Then there is a function $w(\varepsilon)$, with $w(\varepsilon) \sim 0$ as $\varepsilon \to 0+$, such that*

$$\left.\begin{aligned} &|u^+(x, \varepsilon) - (1 + \varepsilon\gamma_{11}(\varepsilon)) u_L^+(x, \varepsilon)| \\ &|u^-(x, \varepsilon) - (1 + \varepsilon\gamma_{22}(\varepsilon)) u_L^-(x, \varepsilon)| \end{aligned}\right\} < w(\varepsilon) \exp\left\{-\frac{2}{3\varepsilon} t_0^{3/2}\right\} \tag{5.8}$$

*for $|x - x_1| \leqq a$.*

*Proof.* The asymptotic properties of Airy's function applied to $V(t, \varepsilon)$ in (5.1), combined with Theorem 4.1 and (5.2), show that

$$\| Y^L(t, \varepsilon)\| \leqq K \exp\left\{\frac{2}{3\varepsilon} t_1^{3/2}\right\}, \qquad |t| \leqq t_1 < t_0, \quad 0 < \varepsilon \leqq \varepsilon_0,$$

for any $t$, with $0 < t_1 < t_0$. The constant $K$ depends on $\varepsilon_0$ and $t_1$. Hence, for $j, k = 1, 2$,

$$\begin{aligned} &\left| y_{jk}^L(t, \varepsilon) \exp\left\{-\frac{4}{3\varepsilon} t_0^{3/2}\right\}\right| \\ &\qquad \leqq K \exp\left\{\frac{2}{3\varepsilon}(t_1^{3/2} - t_0^{3/2})\right\} \exp\left\{-\frac{2}{3\varepsilon} t_0^{3/2}\right\} \end{aligned}$$

$$= w(\varepsilon) \exp \left\{ -\frac{2}{3\varepsilon} t_0^{3/2} \right\},$$

$w(\varepsilon)$ having the required property. Insertion of the last inequality into (5.7) proves the theorem.

The right member of (5.8) is small of higher order of magnitude than $u_L^+(x, \varepsilon)$ or $u_L^-(x, \varepsilon)$ as $\varepsilon \to 0+$ as long as $|t(x)| \leqq t_0$, provided $t$ is not a zero of $u_L^+$ or $u_L^-$. The zeros of Airy's function are all on the negative real axis. From (5.1) it follows then that the zeros of $u_L^{\mp}(x, \varepsilon)$ are, for small $\varepsilon$, in narrow sectors containing the boundary of $D$. With these remarks in mind, (5.8) can be replaced by the simpler, but weaker and less precise statement, that

(5.9) $\quad u^+(x, \varepsilon) \sim (1 + \varepsilon\gamma_{11}(\varepsilon))u_L^+(x, \varepsilon), \qquad u^-(x, \varepsilon) \sim (1 + \varepsilon\gamma_{22}(\varepsilon))u_L^-(x, \varepsilon)$

for $x \in D \cap (|x - x_1| \leqq a)$.

I do not know if $w(\varepsilon)$ is actually different from zero. This is a theoretically interesting question, but it has little bearing on the actual connection formulas.

**6. An example.** Let

(6.1) $$p(x) = x^3 - 1.$$

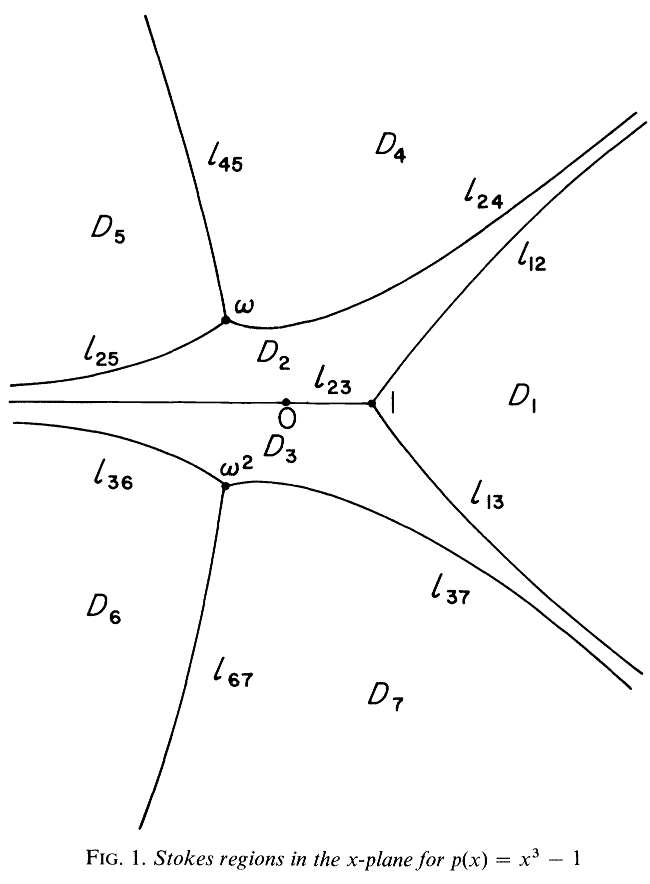


FIG. 1. *Stokes regions in the x-plane for* $p(x) = x^3 - 1$

An elementary, but not entirely trivial analysis, which will not be reproduced here, shows that the patterns of the Stokes regions are as illustrated in Fig. 1. The turning points are at 1, $\omega = e^{2\pi i/3}$ and $\omega^2 = e^{4\pi i/3}$.

To fix the ideas, we choose the turning point $x_1 = 1$ and the consistent canonical region $D = D_1 \cup l_{12} \cup D_2 \cup l_{24} \cup D_4$, which is the unshaded part of Fig. 2. (For comparison, the canonical region $D_1 \cup l_{12} \cup D_2 \cup l_{25} \cup D_5$ is inconsistent.) The image $\xi(D)$ of $D$ in the $\xi$-plane, with the branch of $\xi(x_1, x)$ chosen according to the rule in § 2, can be seen in Fig. 3. Corresponding parts of $D$ and $\xi(D)$ are designated by the same letters, with a tilde over the images in $\xi(D)$. Finally, Fig. 4 shows the region $R_t$ in the $t$-plane for this example.

The ray $x > 1$ is mapped onto the ray $\arg t = 2\pi/3$. By indicating the correct branches to be taken on these two rays for the various analytic functions that enter our calculations, those branches are specified for the whole regions in question.

On $\arg t = 2\pi/3$ we have $\arg(dt/dx) = 2\pi/3$ and we may take $\arg(q(t)) = \arg(\dot{x}^{1/2}) = 2\pi/3$. ($-\pi/3$ would have been the other possible choice.) Observe that $\xi(1, x)$ has to be calculated with $\arg(p(x)^{1/2}) = \pi$, for $x > 1$, to obtain the correct branch in $\xi(D)$. Furthermore, (4.8) gives

$$(x^3 - 1)^{-1/4} = |(x^3 - 1)^{-1/4}|e^{\pi i/2} \quad \text{for } x > 1 \tag{6.2}$$

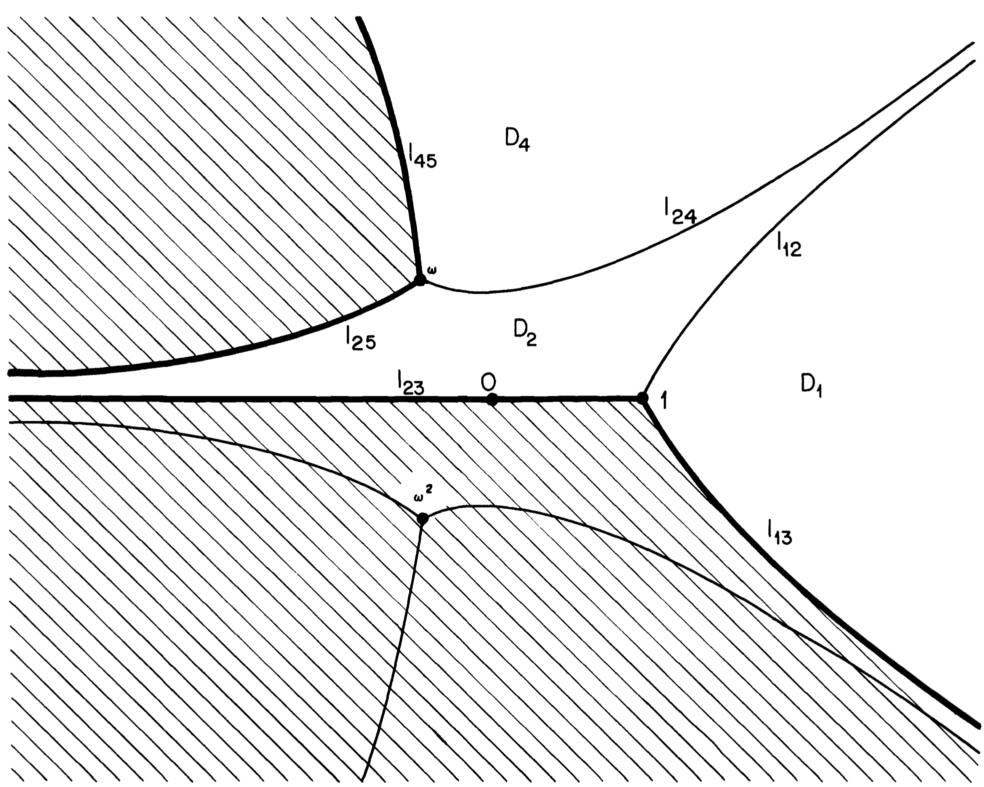


FIG. 2. *A canonical domain for* $p(x) = x^3 - 1$

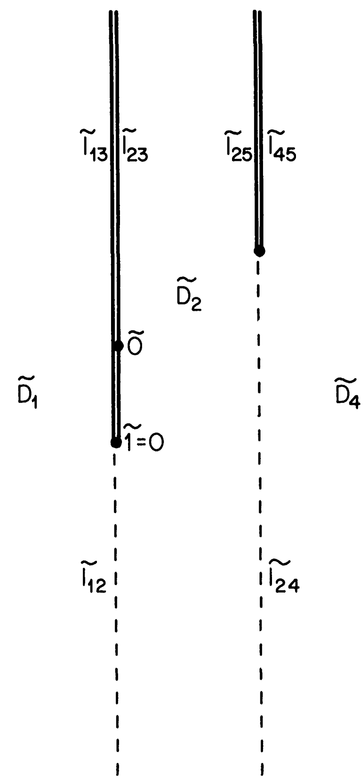


FIG. 3. *Image in the $\xi$-plane of the canonical region of Fig.* 2

as the correct branch to be used in (4.4), by our conventions. Finally, $dq/dt$ is real on $\arg t = 2\pi/3$.

The following facts are easily established. As $x \to 1$ on $x > 1$,

$$p(x) = 3(x-1)[1 + (x-1) + \tfrac{1}{3}(x-1)^2],$$

$$p^{1/2}(x) = e^{\pi i}|3(x-1)|^{1/2}(1 + \tfrac{1}{2}(x-1) + \tfrac{1}{24}(x-1)^2 + O((x-1)^3)),$$

$$\xi(x) = e^{\pi i/2}|3^{-1/2}||(x-1)^{3/2}|(1 + \tfrac{3}{10}(x-1) + \tfrac{1}{48}(x-1)^2 + O((x-1)^3)),$$

$$(6.3)\qquad t(x) = e^{2\pi i/3}|3^{1/3}|(x-1)(1 + \tfrac{1}{5}(x-1) - \tfrac{7}{360}(x-1)^2 + O((x-1)^3)),$$

$$dt/dx = e^{2\pi i/3}|3^{1/3}|(1 + \tfrac{2}{5}(x-1) - \tfrac{7}{120}(x-1)^2 + O((x-1)^3)),$$

$$dx/dt = e^{-2\pi i/3}|3^{-1/3}|(1 - \tfrac{2}{5}(x-1) + \tfrac{131}{600}(x-1)^2 + O((x-1)^3)),$$

$$q = \dot{x}^{1/2} = e^{2\pi i/3}|3^{-1/6}|(1 - \tfrac{1}{5}(x-1) - \tfrac{119}{1200}(x-1)^2 + O((x-1)^3)),$$

$$\dot{q} = \frac{dq}{dx}\cdot\frac{dx}{dt} = e^{\pi i}|3^{-1/2}|(1 + \tfrac{1}{125}(x-1) + O((x-1)^2)).$$

For abbreviation, we set

$$(6.4)\qquad h(t) = \frac{1}{2}\int_0^t \tau^{-1/2} q(\tau)\frac{d}{d\tau}[\dot{q}(\tau) q^{-2}(\tau)]\, d\tau.$$

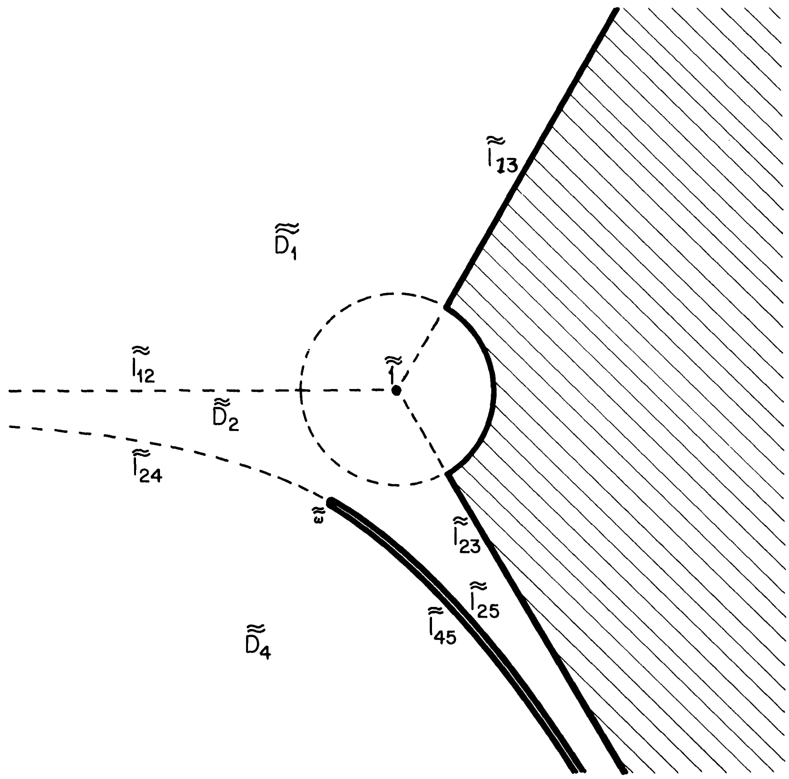


FIG. 4. *The region $R_t$ in the t-plane corresponding to the canonical domain of Fig.* 2

An inspection of the arguments of the factors in the integrand on $\arg \tau = 2\pi/3$ shows that $h(t)$ is real on the ray $\arg t = 2\pi/3$. With the help of (6.3) one can verify that

$$\lim_{t \to 0} [t^{-1/2} h(t)] = e^{-\pi i/3} \tfrac{119}{600} |3^{-5/6}| . \tag{6.5}$$

This result will be needed below.

From (3.11r) and (3.8r) with $r = 1$ and the initial conditions prescribed in Lemmas 3.2 and 3.4, we find that

$$P^L(t, \varepsilon) = q(t)I + \begin{bmatrix} 0 & -t^{-1/2}q(t)h(t) \\ -t^{1/2}q(t)h(t) + \dot{q}(t) & 0 \end{bmatrix} \varepsilon + O(\varepsilon^2), \tag{6.6}$$

$$P^F(t, \varepsilon) = q(t)I + \begin{bmatrix} 0 & t^{-1/2}q(t)(h(\infty) - h(t)) \\ t^{1/2}q(t)(h(\infty) - h(t)) + \dot{q}(t) & 0 \end{bmatrix} \varepsilon + O(\varepsilon^2), \tag{6.7}$$

and, hence,

$$(P^L)^{-1}P^F = I + \begin{bmatrix} 0 & t^{-1/2} \\ t^{1/2} & 0 \end{bmatrix} h(\infty)\varepsilon + O(\varepsilon^2). \tag{6.8}$$

If this is combined with (4.14) and (5.4), there results, after some manipulations, the formula

$$C(\varepsilon) = \begin{bmatrix} 1 + h(\infty)\varepsilon & 0 \\ 0 & 1 - h(\infty)\varepsilon \end{bmatrix} + O(\varepsilon^2). \tag{6.9}$$

Next, we calculate the asymptotic expansion for $V(t, \varepsilon)$ in $R_{t\delta}$, up to the second term. This can be done by expanding the Airy functions in (5.1) into their asymptotic series. We find

$$\begin{aligned} V(t,\varepsilon) = \begin{bmatrix} t^{-1/4} & 0 \\ 0 & t^{1/4} \end{bmatrix} & \left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} - \frac{1}{48}\begin{bmatrix} -5 & 5 \\ 7 & 7 \end{bmatrix} t^{-3/2}\varepsilon + O(t^{-3}\varepsilon^2) \right\} \\ & \cdot \begin{bmatrix} \exp\left\{\dfrac{2}{3\varepsilon}t^{3/2}\right\} & 0 \\ 0 & \exp\left\{-\dfrac{2}{3\varepsilon}t^{-3/2}\right\} \end{bmatrix}. \end{aligned} \tag{6.10}$$

Combining (6.7) with (6.10) one can calculate $Y^F(t, \varepsilon)$ as defined in (5.2):

$$\begin{aligned} Y^F(t,\varepsilon) = q(t) & \begin{bmatrix} t^{-1/4} & 0 \\ 0 & t^{1/4} \end{bmatrix} \left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right. \\ & + \begin{bmatrix} h(\infty) - h(t) + \dfrac{5}{48}t^{-3/2} & -h(\infty) + h(t) + \dfrac{5}{48}t^{-3/2} \\ h(\infty) - h(t) - \dfrac{7}{48}t^{-3/2} + \dfrac{\dot q(t)}{q(t)}t^{-1/2} & h(\infty) - h(t) + \dfrac{7}{48}t^{-3/2} - \dfrac{\dot q(t)}{q(t)}t^{-1/2} \end{bmatrix}\varepsilon \\ & \left. + O(t^{-3}\varepsilon^2) \right\} \begin{bmatrix} \exp\left\{\dfrac{2}{3\varepsilon}t^{3/2}\right\} & 0 \\ 0 & \exp\left\{-\dfrac{2}{3\varepsilon}t^{3/2}\right\} \end{bmatrix}. \end{aligned} \tag{6.11}$$

This implies, in particular, that

$$u^+(x,\varepsilon) \sim p^{-1/4}(x)(1 + (h(\infty) - h(t(x)) + \tfrac{5}{72}\xi^{-1}(x)\varepsilon + \cdots)e^{\xi(x)/\varepsilon}. \tag{6.12}$$

The expansion (6.12) is valid in $D_\delta$. To obtain an expansion for the same solution that is valid near the turning point $x = 1$ we first calculate $Y^L$ from (5.1), (5.2) and (6.6), concentrating on the first entry, $y_{11}^L$. Clearly,

$$\begin{aligned} y_{11}^L(t,\varepsilon) &= q(t)v_{11}(t,\varepsilon) - t^{-1/2}q(t)h(t)v_{21}(t,\varepsilon) \\ &= 2\sqrt{\pi}q(t)\{e^{-\pi i/6}\varepsilon^{-1/6}\text{Ai}\,(e^{-2\pi i/3}\varepsilon^{-2/3}t) \\ &\qquad + t^{-1/2}h(t)e^{\pi i/6}\varepsilon^{1/6}\text{Ai}'\,(e^{-2\pi i/3}\varepsilon^{-2/3}t)\}. \end{aligned}$$

Therefore, the formula

$$u^+(x,\varepsilon) \sim (1 + h(\infty)\varepsilon + \cdots)y_{11}^L(t,\varepsilon)$$

enables us to calculate $u^+(x, \varepsilon)$ asymptotically throughout $|x - 1| \leqq a$. For $x = 1$, i.e., at the turning point itself, the result is

$$u^+(1, \varepsilon) = 2\sqrt{\pi}q(0)(1 + h(\infty)\varepsilon + O(\varepsilon^2)) \cdot \{e^{-\pi i/6}\varepsilon^{-1/6}\mathrm{Ai}\,(0) + \lim_{t \to 0}(t^{-1/2}h(t))e^{\pi i/6}\varepsilon^{1/6}\mathrm{Ai}'(0)\}$$

or

$$u^+(1, \varepsilon) = 2\sqrt{\pi}i(1 + h(\infty)\varepsilon)\left(\frac{3^{-5/6}}{\Gamma(2/3)}\varepsilon^{-1/6} - \frac{119 \cdot 3^{-4/3}}{600\Gamma(1/3)}\varepsilon^{1/6}\right) + O(\varepsilon^{11/6}).$$

The factor $i$ in this formula is a consequence of our rules concerning the branches of the analytic function to be taken, as explained in (6.2). As $p(x) = x^3 - 1$ is real for real $x$, the solution $-iu^+(x, \varepsilon)$, which is real on the real axis, might be preferable.

## REFERENCES

[1] T. M. CHERRY, *Uniform asymptotic formulae for functions with transition points*, Trans. Amer. Math. Soc., 68 (1950), pp. 224–257.

[2] M. A. EVGRAFOV AND M. V. FEDORYUK, *Asymptotic behavior of the equation* $w''(z) - p(z, \lambda)w(z) = 0$ *as* $\lambda \to \infty$ *in the complex z-plane*, Uspekhi Mat. Nauk, 21 (1966), no. 1 (127), pp. 3–50; English transl., Russian Math. Surveys, 21 (1966), pp. 1–48.

[3] M. V. FEDORYUK, *Topology of Stokes curves for the equation of the second order*, Izv. Akad. Nauk SSSR Ser. Mat., 29 (1965), pp. 645–656.

[4] R. E. LANGER, *The asymptotic solutions of ordinary linear differential equations with special reference to the Stokes phenomenon*, Bull. Amer. Math. Soc., 40 (1934), pp. 545–582.

[5] F. W. J. OLVER, *The asymptotic solution of linear differential equations of the second order for the large values of a parameter*, Philos. Trans. Roy. Soc. London Ser. A, 247 (1954), pp. 307–327.

[6] ———, *Uniform asymptotic expansions of linear second-order differential equations for large values of a parameter*, Ibid., 250 (1958), pp. 479–517.

[7] ———, *Error analysis of phase-integral methods*, J. Res. Nat. Bur. Standards Sect. B, 69B (1965), pp. 271–290.

[8] W. WASOW, *Simplification of turning point problems for systems of linear differential equations*, Trans. Amer. Math. Soc., 106 (1963), pp. 100–114.

[9] ———, *Asymptotic Expansions for Ordinary Differential Equations*, Interscience, New York, 1965.

SIAM J. MATH. ANAL.
Vol. 1, No. 2, May 1970

# LIE THEORY AND $q$-DIFFERENCE EQUATIONS*

WILLARD MILLER, JR.†

**Abstract.** A factorization method is established for systems of second order linear $q$-difference equations. The factorization types are shown to correspond to irreducible representations of infinite-dimensional Lie algebras. If the $q$-difference equations degenerate to differential equations (as $q$ approaches 1) a Lie theory of hypergeometric and related functions is obtained in the limit. If the $q$-difference equations degenerate to ordinary difference equations a Lie theory of special functions of a discrete variable is obtained in the limit.

**Introduction.** In 1951 Infeld and Hull discussed a technique, the factorization method, which could be used to construct solutions of families of second order linear differential equations arising in mathematical physics [1]. The hypergeometric functions and their various limits were obtained from this analysis. In all there were shown to be eight factorization types: A, B, C′, C″, D′, D″, E, F, each one associated with a class of special functions. Later it was realized that the factorization types corresponded to irreducible representations of the four-dimensional Lie algebras $\mathscr{G}(a, b)$ and the six-dimension Lie algebra $\mathscr{T}_6$ (see § 5) [2], [3]. The special functions corresponded to basis vectors and matrix elements of these representations. This second approach yielded many more properties of special functions than did the original factorization method since it allowed application of the machinery of group theory, particularly the Lie theory of local transformation groups.

Recently the author constructed a factorization method for families of second order linear difference equations [4]. Six factorization types were constructed $(\alpha, \alpha', \beta, \beta', \gamma, \gamma')$, and among the special functions obtained were the polynomials of Chebyshev, Hahn, Charlier, Meixner and Krawtchouk. Again the factorization types corresponded to irreducible representations of the Lie algebras $\mathscr{G}(a, b)$. In addition it was shown that the Hahn and Chebyshev polynomials also transformed according to representations of the six-dimensional Lie algebra $\mathcal{O}_4$.

In the present paper, which is essentially a commentary on some aspects of the work of W. Hahn [5], [6], we construct a factorization method for families of second order $q$-difference equations. It turns out that there is an infinite variety of factorization types. This is not unexpected since there is an infinite variety of $q$-analogies of well-known special functions. Here, we choose two families of factorizations which are as simple as possible, yet rich enough to obtain all of the fourteen factorization types for differential and difference equations in the limit as $q \to 1$. It is well known that many $q$-difference equations can be parametrized so that as $q \to 1$ they converge to either differential or difference equations [5]. Similarly, we show that we can parametrize our factorizations so that as $q \to 1$ we get all fourteen factorization types mentioned above. The special functions associated with the factorization types are $q$-analogies of familiar special functions.

---

* Received in revised form by the editors September 22, 1969. Presented by invitation at the Symposium on Special Functions, sponsored by the Air Force Office of Scientific Research, at the 1969 National Meeting of Society for Industrial and Applied Mathematics, held in Washington, D.C., June 10–12, 1969.

† Mathematics Department, University of Minnesota, Minneapolis, Minnesota 55455.

As usual, the factorizations for $q$-difference equations have a Lie algebraic interpretation. In this case the Lie algebras are infinite-dimensional. However, they have the special property that the centers of their universal enveloping algebras contain elements of the second order ($q$-analogies of Casimir operators). With the parametrizations mentioned above we find that as $q \to 1$ the infinite-dimensional Lie algebras degenerate to one of the Lie algebras $\mathcal{G}(a, b)$, $\mathcal{T}_6$, $\mathcal{O}_4$.

Since there is no convenient analogy of the local Lie theory of transformation groups for $q$-difference operators, our relation of factorizations to Lie algebras is not as useful as it was in the case of differential operators. Nevertheless, we show that all of the $q$-analogies of special functions discussed by Hahn have factorizations and transform according to irreducible representations of certain Lie algebras. This provides us with useful algebraic tools for uncovering the properties of these functions. In particular many of the techniques for differential operators due to Weisner [7], [8], [9] can be carried over to $q$-difference operators, though the computations are much more difficult. We work out a few examples in § 6.

It should be noted that there are many factorizations for $q$-difference operators in addition to those constructed in this paper. The factorizations selected here are introduced primarily on the basis of their simplicity. The point is that the $q$-analogies of familiar special functions can be studied by the factorization and related Lie algebraic methods.

**1. Resumé of the factorization method.** The basic idea underlying the factorization method has been presented many times before [1], [3], [10], but will be reviewed here for the convenience of the reader. Let $\{X_m\}$, $m \in S = \{m_0, m_0 \pm 1, m_0 \pm 2, \cdots\}$ where $m_0$ is a complex number, be a sequence of linear operators defined on the complex vector space $\mathcal{V}$. The operators $\{X_m\}$ admit a *factorization* if there exist sequences of linear operators $\{L_m^+\}$, $\{L_m^-\}$ on $\mathcal{V}$ and constants $\{a_m\}$ such that

$$X_m \equiv L_m^+ L_m^- + a_m \equiv L_{m+1}^- L_{m+1}^+ + a_{m+1} \qquad \text{for all } m \in S. \tag{1.1}$$

If the $\{X_m\}$ admit a factorization then the eigenvalue equation

$$X_m Y_\lambda(m) = \lambda Y_\lambda(m), \qquad m \in S, \tag{1.2}$$

is equivalent to the two equations

$$L_m^+ L_m^- Y_\lambda(m) = (\lambda - a_m) Y_\lambda(m), \qquad L_{m+1}^- L_{m+1}^+ Y_\lambda(m) = (\lambda - a_{m+1}) Y_\lambda(m). \tag{1.3}$$

Furthermore, the following lemma is easily verified.

LEMMA. *Let $Y_\lambda(l)$ be a solution of* (1.2) *for $m = l$. Then $L_{l+1}^+ Y_\lambda(l)$ is a solution of* (1.2) *for $m = l + 1$ and $L_l^- Y_\lambda(l)$ is a solution for $m = l - 1$.*

Thus, the existence of a factorization implies the existence of recurrence relations for the eigenvectors $Y_\lambda(m)$. There is an important class of factorizations with the property: There exists an $m_0 \in S$ such that $a_{m_0} = \lambda$. In this case any nonzero solution of $L_{m_0}^- Y_\lambda(m_0) = 0$ is automatically a solution of $X_m Y_\lambda(m_0) = \lambda Y_\lambda(m_0)$, and from the solution $Y_\lambda(m_0)$ we can construct a family of eigenvectors $Y_\lambda(m_0 + n)$, $n = 0, 1, 2, \ldots$, by applying the "raising operators" $L_{m+1}^+$ recursively. Such a factorization is said to be *bounded below*. Similarly, if there is an $m_1 \in S$ such that $a_{m_1+1} = \lambda$ the factorization is *bounded above*.

**2. Solutions of $q$-difference equations.** Let $a, b, c, d, x, q$ be complex numbers, $0 < |q| < 1$. The Heine series [5], [6], [11],

$$
{}_3\varphi_2(a, b, x; c, d; q) = \sum_{n=0}^{\infty} h_n q^n,
$$

$$
h_n = \frac{(a-1)(aq-1)\cdots(aq^{n-1}-1)(b-1)(bq-1)\cdots(bq^{n-1}-1)(x-1)}{(c-1)(cq-1)\cdots(cq^{n-1}-1)(d-1)(dq-1)\cdots(dq^{n-1}-1)} \cdot \frac{(xq-1)\cdots(xq^{n-1}-1)}{(q-1)(q^2-1)\cdots(q^n-1)}, \tag{2.1}
$$

$$
h_0 = 1,
$$

converges absolutely and satisfies the $q$-difference equation

$$
(x-1)(qabx-cd)F(qx) - [x^2(a+b)q - x(abq+cd+(c+d)q) + cd(1+q)]F(x) + q(x-c)(x-d)F(q^{-1}x) = 0. \tag{2.2}
$$

A second solution of (2.2) is

$$
F(x) = \prod_{j=0}^{\infty} \frac{1-q^j x}{1-q^{j+1}x/c} \cdot {}_3\varphi_2\left(\frac{aq}{c}, \frac{bq}{c}, \frac{xq}{c}; \frac{dq}{c}, \frac{q^2}{c}; q\right).
$$

The basic hypergeometric function

$$
{}_2\varphi_1(a, x; c; qb) = \sum_{n=0}^{\infty} h_n (qb)^n,
$$

$$
h_n = \frac{(a-1)(aq-1)\cdots(aq^{n-1}-1)(x-1)(xq-1)\cdots(xq^{n-1}-1)}{(c-1)(cq-1)\cdots(cq^{n-1}-1)(q-1)(q^2-1)\cdots(q^n-1)}, \tag{2.3}
$$

$$
h_0 = 1,
$$

converges absolutely for $|b| < 1$, $|q| < 1$ and satisfies the $q$-difference equation

$$
(x-1)(abx-c)F(qx) - [x^2 b - x(ab+c+q) + c(1+q)]F(x) - q(x-c)F(q^{-1}x) = 0. \tag{2.4}
$$

A second solution is

$$
F(x) = \prod_{j=0}^{\infty} \frac{1-q^j x}{1-q^{j+1}x/c} \cdot {}_2\varphi_1\left(\frac{qa}{c}, \frac{qx}{c}; \frac{q^2}{c}; b\right).
$$

Also, the function ${}_2\varphi_1(a, b; c; x)$ is a solution of

$$
(c-abx)F(qx) - [c+q-(a+b)x]F(x) + (q-x)F(q^{-1}x) = 0. \tag{2.5}
$$

Another solution is

$$
F(x) = x^{1-\gamma}{}_2\varphi_1\left(\frac{aq}{c}, \frac{bq}{c}; \frac{q^2}{c}; x\right)
$$

where $q^\gamma = c$.

The basic confluent hypergeometric function is defined by

$$
(2.6)\qquad \begin{aligned}
{}_1\varphi_1(a;c;x) &= \lim_{|b|\to\infty} {}_2\varphi_1\left(a,b;c;\frac{x}{b}\right) = \sum_{n=0}^{\infty} h_n x^n,\\
h_n &= \frac{(a-1)(aq-1)\cdots(aq^{n-1}-1)q^{n(n-1)/2}}{(c-1)(cq-1)\cdots(cq^{n-1}-1)(q-1)(q^2-1)\cdots(q^n-1)},\\
h_0 &= 1,
\end{aligned}
$$

and converges absolutely for all $a$, $c$, $x$ if $|q| < 1$. This function is a solution of the equation

$$
(2.7)\qquad (c - ax)F(qx) - (c + q - x)F(x) + qF(q^{-1}x) = 0.
$$

A linearly independent solution is

$$
F(x) = x^{1-\gamma}{}_1\varphi_1\left(\frac{aq}{c};\frac{q^2}{c};x\right)
$$

where $c = q^\gamma$.

We note for future use that in the case where $a = q^\alpha$, $b = q^\beta$, $c = q^\gamma$, $x = q^\xi$, $d = q^\delta$ and $\alpha$, $\beta$, $\gamma$, $\xi$, $\delta$ are constants, then

$$
(2.8)\qquad \begin{aligned}
&\lim_{q\to 1} {}_3\varphi_2(a,b,x;c,d;q) = {}_3F_2(\alpha,\beta,\xi;\gamma,\delta;1),\\
&\lim_{q\to 1} {}_2\varphi_1(a,b;c;z) = {}_2F_1(\alpha,\beta;\gamma;z),\\
&\lim_{q\to 1} {}_1\varphi_1(a;c;(q-1)z) = {}_1F_1(\alpha;\gamma;z),
\end{aligned}
$$

where the ${}_3F_2$, ${}_2F_1$, ${}_1F_1$ are hypergeometric functions. Furthermore, if $f$ is a differentiable function of $x$ then the relation

$$
(2.9)\qquad \lim_{q\to 1}\frac{f(qx) - f(x)}{(q-1)x} = \frac{df}{dx}(x)
$$

is valid [5], [11]. It follows from these last remarks that in the limit as $q \to 1$ the $q$-difference equations (2.5), (2.7) become the usual differential equations satisfied by the hypergeometric and confluent hypergeometric functions, respectively.

**3. Factorizations by $q$-difference operators.** To begin, we seek factorizations of the form

$$
(3.1)\qquad \begin{aligned}
L_m^+ &= r(x)T^+ + g(x)q^{-m} + h(x),\\
L_m^- &= R(x)q^m T^- + G(x)q^m + H(x),
\end{aligned}
$$

where

$$
T^{\pm}f(x) = f(q^{\pm 1}x)
$$

and $r$, $g$, $h$, $R$, $G$, $H$ are functions to be determined. It follows immediately from the form of the $q$-difference operators $L_m^{\pm}$ that the constants $a_m$ can be written as

$$
(3.2)\qquad a_m = Aq^m + Bq^{-m},
$$

where $A$ and $B$ are to be determined. Substitution of (3.1), (3.2) into (1.1) and a straightforward though tedious computation yield the result:

$$
\begin{aligned}
L_{m+1}^{+} &= (k_2 x + c_5)T^{+} + c_3 q^{-m-1} + c_2 x, \\
L_m^{-} &= (k_1 x + c_4)q^m T^{-} + c_1 x q^m + c_6,
\end{aligned} \tag{3.3}
$$

where

$$
k_1 k_2 = c_1 c_2, \qquad A = -c_4 c_5, \qquad B = -c_3 c_6. \tag{3.4}
$$

Except for the requirements (3.4) the $k$'s and $c$'s are arbitrary constants. The second order $q$-difference operators $X_m$ defined by (1.1) become

$$
\begin{aligned}
X_m = {} & (k_2 x + c_5)(c_6 + c_1 x q^{m+1})T^{+} + \{x(c_1 c_3 + c_2 c_6) \\
& + q^m[(k_1 k_2 q + c_1 c_2)x^2 + (k_1 c_5 q + k_2 c_4)x]\} \\
& + (k_1 x + c_4)(c_3 + q^m x c_2)T^{-}.
\end{aligned} \tag{3.5}
$$

*Note.* Expressions (3.3) are not the most general solution of the factorization equations. The general solution is of the form $L_{m+1}^{+\prime} = [\rho(x)]^{-1} L_{m+1}^{+} \rho(x)$, $L_m^{-\prime} = [\rho(x)]^{-1} L_m^{-} \rho(x)$ where $\rho$ is an arbitrary function which is never zero. We have normalized our solutions so that $r(x)$ and $R(x)$ are first order polynomials in $x$.

We shall investigate a number of special cases of the above factorizations and show that they yield $q$-analogies of familiar special functions.

*Example* 1. Set $k_1 = -q^{-1}, k_2 = c, c_1 = q^{-1}, c_2 = -c, c_3 = qab/d, c_4 = q^{-1}$, $c_5 = -1, c_6 = -d^{-1}$. Then,

$$
\begin{aligned}
L_{m+1}^{+} &= (cx - 1)T^{+} + \frac{ab}{d} q^{-m} - cx, \\
L_m^{-} &= (-x + 1)q^{m-1} T^{-} + x q^{m-1} - d^{-1}, \\
a_m &= q^{m-1} + \frac{ab}{d^2} q^{-m+1}.
\end{aligned} \tag{3.6}
$$

The basic vectors $f_m(x)$ associated with this factorization are of the form ${}_3\varphi_2$. In fact, the functions

$$
f_m(x) = {}_3\varphi_2(a, b, x^{-1}; c, dq^m; q) \tag{3.7}
$$

satisfy the recursion relations

$$
\begin{aligned}
L_{m+1}^{+} f_m &= \frac{(aq^{-m} - d)(dq^m - b)}{d(dq^m - 1)} f_{m+1}, \\
L_m^{-} f_m &= \frac{(dq^{m-1} - 1)}{d} f_{m-1}
\end{aligned} \tag{3.8}
$$

and the $q$-difference equations

$$
X_m f_m = \lambda f_m, \qquad \lambda = \frac{a+b}{d}.
$$

If we set $x = q^z, a = q^\alpha, b = q^\beta, c = q^\gamma, d = q^\delta$ and let $q \to 1$ the above factorization becomes

$$L^+_{m+1} f_m = \frac{(\alpha - \delta - m)(\delta - \beta + m)}{(\delta + m)} f_{m+1}, \qquad L^-_m f_m = (\delta + m - 1) f_{m-1},$$

$$\begin{aligned} &L^+_{m+1} = (\gamma + z)E + (\alpha + b - \gamma - \delta - z - m), \\ (3.9)\qquad &L^-_m = -zL + (\delta + z + m - 1), \\ &f_m(z) = {}_3F_2(\alpha, \beta, -z; \gamma, \delta + m; 1), \end{aligned}$$

where

$$Ef(z) = f(z + 1), \qquad Lf(z) = f(z - 1).$$

(Note: In this and some of the following examples it is necessary to multiply one or both of the operators $L^+_{m+1}, L^-_m$ by the factor $(q - 1)^{-1}$ before going to the limit.) If $\alpha$ is a negative integer, the eigenvectors $f_m(z)$ are the Hahn polynomials [12]. In any case the factorization (3.9) is the type $\alpha''$ factorization by first order difference operators classified in [4].

*Example* 2. Set $k_1 = 1$, $k_2 = 0$, $c_1 = -a$, $c_2 = 0$, $c_3 = dq$, $c_4 = -d^{-1}$, $c_5 = -bd$, $c_6 = c/bd$. Then the above factorization becomes

$$\begin{aligned} &L^+_{m+1} = -bdT^+ + dq^{-m}, \\ (3.10)\qquad &L^-_m = (x - d^{-1})q^m T^- - axq^m + c/bd, \\ &a_m = -bq^m - \frac{c}{b} q^{-m+1}. \end{aligned}$$

The second order $q$-difference equations obtained from this factorization are closely related to (2.5). Indeed the functions

$$(3.11)\qquad f_m(x) = {}_2\varphi_1(a, bq^m; c; q\,dx)$$

satisfy the recursion relations

$$(3.12)\qquad L^+_{m+1} f_m = d(q^{-m} - b) f_{m+1}, \qquad L^-_m f_m = \frac{(c - bq^m)}{bd} f_{m-1}$$

and the $q$-difference equations

$$X_m f_m = \lambda f_m, \qquad \lambda = -c - q.$$

Setting $a = q^\alpha, b = q^\beta, c = q^\gamma, d = 1$ and letting $q \to 1$, we obtain

$$\begin{aligned} &L^+_{m+1} f_m = -(\beta + m) f_{m+1}, \qquad L^-_m f_m = (\gamma - \beta - m) f_{m-1}, \\ (3.13)\qquad &L^+_{m+1} = -x\frac{d}{dx} - \beta - m, \qquad L^-_m = -x(x - 1)\frac{d}{dx} - \alpha x + \gamma - \beta - m, \\ &f_m(x) = {}_2F_1(\alpha, \beta + m; \gamma; x). \end{aligned}$$

This is the type A factorization of the hypergeometric equation by first order differential operators [1], [2], [3].

*Example* 3. Set $k_1 = 0, k_2 = -qd, c_1 = 0, c_2 = 0, c_3 = -q, c_4 = 1, c_5 = c/a$, $c_6 = -a$. Then we obtain

(3.14)
$$\begin{aligned} L^+_{m+1} &= (-qdx + c/a)T^+ - q^{-m}, \\ L^-_m &= q^m T^- - a, \\ a_m &= -\frac{c}{a}q^m - aq^{-m+1}. \end{aligned}$$

The functions

(3.15)
$$f_m(x) = {}_1\varphi_1(aq^{-m}; c; dq^m x)$$

satisfy the relations

$$\begin{aligned} L^+_{m+1} f_m &= a^{-1}(c - aq^{-m}) f_{m+1}, \\ L^-_m f_m &= (q^m - a) f_{m-1}, \\ X_m f_m &= -(c + q) f_m. \end{aligned}$$

If we set $a = q^\alpha$, $c = q^\gamma$, $d = 1$, $z = (q-1)x$, and let $q \to 1$ it follows that

(3.16)
$$\begin{aligned} &L^+_{m+1} f_m = (m + \gamma - \alpha) f_{m+1}, \qquad L^-_m f_m = (m - \alpha) f_{m-1}, \\ &L^+_{m+1} = z\frac{d}{dz} - z + \gamma - \alpha + m, \qquad L^-_m = -z\frac{d}{dz} + m - \alpha, \\ &f_m(x) = {}_1F_1(\alpha - m; \gamma; z). \end{aligned}$$

This is the type B factorization of the confluent hypergeometric equation [1], [2], [3].

*Example* 4. Set $k_1 = wb, k_2 = 1, c_1 = -bw, c_2 = -1, c_3 = a^{-1}, c_4 = -w$, $c_5 = 0, c_6 = qa^{-1}$. The factorization reads

(3.17)
$$\begin{aligned} L^+_{m+1} &= xT^+ + a^{-1}q^{-m-1} - x, \\ L^-_m &= w(bx - 1)q^m T^- - bwxq^m + qa^{-1}, \\ a_m &= -a^{-2}q^{-m+1}. \end{aligned}$$

The functions

(3.18)
$$f_m(x) = \prod_{j=0}^{\infty} (1 - x^{-1}q^j) \cdot {}_2\varphi_1(aq^m, b; x^{-1}; w)$$

satisfy

(3.19)
$$\begin{aligned} &L^+_{m+1} f_m = -q^{-m-1}a^{-1}(aq^m - 1) f_{m+1}, \qquad L^-_m f_m = aq^{-1} f_{m-1}, \\ &X_m f_m = -a^{-1} f_m. \end{aligned}$$

If we set $a = q^{\alpha}$, $b = q^{\beta}$, $x = q^{z}$ and let $q \to 1$ we obtain in the limit

$$\begin{aligned} &L^{+}_{m+1} f_m = (\alpha + m) f_{m+1}, \qquad L^{-}_{m} f_m = f_{m-1}, \\ (3.20) \qquad &L^{+}_{m+1} = E + \alpha + z + m + 1, \qquad L^{-}_{m} = -w(\beta + z)L + 1 - w, \\ &f_m(z) = \frac{1}{\Gamma(-z)} {}_2F_1(\alpha + m, \beta; -z; w). \end{aligned}$$

This is a type $\beta''$ factorization by difference operators [4].

*Example* 5. Set $k_1 = -1$, $k_2 = 1$, $c_1 = 1$, $c_2 = -1$, $c_3 = qbc^{-1}$, $c_4 = 1$, $c_5 = 0$, $c_6 = -qc^{-1}$. The factorization becomes

$$\begin{aligned} &L^{+}_{m+1} = xT^{+} + bc^{-1}q^{-m} - x, \\ (3.21) \qquad &L^{-}_{m} = (1 - x)q^{m}T^{-} + xq^{m} - qc^{-1}, \\ &a_m = bq^{2-m}/c^{2}. \end{aligned}$$

The functions

$$(3.22) \qquad f_m(x) = {}_1\varphi_1(x^{-1}; cq^{m}; qb)$$

satisfy

$$\begin{aligned} &L^{+}_{m+1} f_m = \frac{-bq^{-m}}{c(cq^{m} - 1)} f_{m+1}, \qquad L^{-}_{m} f_m = qc^{-1}(cq^{m-1} - 1) f_{m-1}, \\ (3.23) \qquad &X_m f_m = 0. \end{aligned}$$

Setting $x = q^{z}$, $c = q^{\gamma}$, $b = (q - 1)\beta$ we obtain in the limit as $q \to 1$:

$$\begin{aligned} &L^{+}_{m+1} f_m = \frac{-\beta}{(\gamma + m)} f_{m+1}, \qquad L^{-}_{m} f_m = (\gamma + m - 1) f_{m-1}, \\ (3.24) \qquad &L^{+}_{m+1} = E - 1, \qquad L^{-}_{m} = -zL + z + m + \gamma - 1, \\ &f_m(z) = {}_1F_1(-z; \gamma + m; \beta). \end{aligned}$$

This is a type $\gamma''$ factorization by difference operators [4].

*Example* 6. Set $k_1 = 0$, $k_2 = -1$, $c_1 = 0$, $c_2 = 1$, $c_3 = -ac^{-1}$, $c_4 = c$, $c_5 = 0$, $c_6 = -q$. The resulting factorization is

$$\begin{aligned} (3.25) \qquad &L^{+}_{m+1} = -xT^{+} - ac^{-1}q^{-m-1} + x, \qquad L^{-}_{m} = cq^{m}T^{-} - q, \\ &a_m = -ac^{-1}q^{-m+1}. \end{aligned}$$

The functions

$$(3.26) \qquad f_m = {}_1\varphi_1(a; cq^{m}; x^{-1})$$

satisfy

$$\begin{aligned} &L^{+}_{m+1} f_m = (ac^{-1}q^{-m} - 1)q^{-1}(cq^{m} - 1)^{-1} f_{m+1}, \\ (3.27) \qquad &L^{-}_{m} f_m = q(cq^{m-1} - 1) f_{m-1}, \\ &X_m f_m = -f_m. \end{aligned}$$

If we set $a = q^\alpha$, $c = q^\gamma$, $x = z^{-1}(q-1)^{-1}$ and let $q \to 1$ we find

$$L^+_{m+1} f_m = \frac{\alpha - \gamma - m}{\gamma + m} f_{m+1}, \qquad L^-_m f_m = (\gamma + m - 1) f_{m-1},$$

$$(3.28) \qquad L^+_{m+1} = \frac{d}{dz} - 1, \qquad L^-_m = z\frac{d}{dz} + \gamma + m - 1,$$

$$f_m(z) = {}_1F_1(\alpha; \gamma + m; z).$$

This is a type C′ factorization by first order differential operators [1], [2], [3].

**4. A second class of factorizations.** To complete the derivation of $q$-analogies for the factorization approach to differential and difference equations we need a new class of factorizations for $q$-difference equations. Here, we investigate factorizations of the form

$$(4.1) \qquad \begin{aligned} L^+_m &= r(x)T^+ + k(x)q^m + h(x)q^{-m} + j(x), \\ L^-_m &= R(x)T^+ + K(x)q^{-m} + J(x), \\ a_m &= Aq^{-2m} + Bq^m + Cq^{-m} + D. \end{aligned}$$

We require that $r(x)R(x) \neq 0$. Substituting these expressions into (1.1) we see that (4.1) defines a factorization if and only if the following requirements are satisfied:

$$(4.2) \qquad \begin{aligned} &\text{(i) } r(x) = c_1 R(x), \qquad \text{(ii) } T^+[c_1 K(x) - q^{-1}h(x)] = c_1 q^{-1} K(x) - h(x), \\ &\text{(iii) } k(x) = c_2 x^{-1}, \qquad \text{(iv) } j(x) = c_1 J(x) + c_3, \\ &\text{(v) } h(x)K(x) = -A, \quad \text{(vi) } h(x)J(x) + j(x)K(x) = -C, \\ &\text{(vii) } k(x)J(x) = -B. \end{aligned}$$

It is a tedious exercise to write down all solutions of (4.2). Here we list only the most important factorization types which arise from a study of (4.1).

*Example* 7. It is easy to verify that the expressions

$$(4.3) \qquad \begin{aligned} L^+_{m+1} &= (1 - cx)T^+ + bq^m(x^{-1} - aq^{-m}) + c(x - q^{-1}) + (aq^{-m-1} - 1), \\ L^-_m &= (1 - cx)T^+ + cx - aq^{-m}, \\ a_m &= -q^{-1}(aq^{-m+1} - c)(aq^{-m} - 1)(bq^m - 1) \end{aligned}$$

define a factorization of the form (4.1). Indeed the functions

$$(4.4) \qquad f_m(x) = {}_2\varphi_1(aq^{-m}, x^{-1}; c; bq^{m+1})$$

satisfy the relations

$$(4.5) \qquad \begin{aligned} &L^+_{m+1} f_m = q^{-1}(aq^{-m} - c) f_{m+1}, \qquad L^-_m f_m = (aq^{-m} - 1)(bq^m - 1) f_{m-1}, \\ &X_m f_m = 0. \end{aligned}$$

If we set $a = q^\alpha, c = q^\gamma, x = q^z$ and let $q \to 1$ we obtain

$$(4.6)\qquad \begin{aligned} &L^+_{m+1} f_m = (\alpha - \gamma - m) f_{m+1}, \qquad L^-_m f_m = (\alpha - m)(b - 1) f_{m-1},\\ &L^+_{m+1} = -(\gamma + z)E - b(z + \alpha - m) + z + \alpha - m,\\ &L^-_m = -(\gamma + z)E + z + \gamma - \alpha + m,\\ &f_m(z) = {}_2F_1(\alpha, -z; \gamma; b). \end{aligned}$$

This is a type $\alpha'$ factorization by first order difference operators [4]. When the factorization is bounded below, the solutions $f_m(z)$ are Meixner and Krawtchouk polynomials.

*Example* 8. The expressions

$$(4.7)\qquad \begin{aligned} L^+_{m+1} &= x^{-1}T^+ - x^{-1} + bq(1 - axq^{-m-1}) + a,\\ L^-_m &= x^{-1}T^+ - x^{-1},\\ a_m &= abq(1 - q^{-m}) \end{aligned}$$

satisfy (4.1), (4.2) and define a factorization. The functions

$$(4.8)\qquad f_m(x) = \prod_{j=0}^{\infty} \frac{(1 - axq^{-m+j})}{(1 - axq^j)} \cdot {}_1\varphi_1\left(q^{-m}; aq^{-m}x; \frac{a}{b}\right)$$

satisfy the relations

$$(4.9)\qquad \begin{aligned} &L^+_{m+1} f_m = bq f_{m+1}, \qquad L^-_m f_m = -a(1 - q^{-m}) f_{m-1},\\ &X_m f_m = 0. \end{aligned}$$

Setting $a = q^\alpha$, $b = \beta^{-1}(q - 1)^{-1}$, $x = q^z$ and letting $q \to 1$ we obtain

$$(4.10)\qquad \begin{aligned} &L^+_{m+1} f_m = \beta^{-1} f_{m+1}, \qquad L^-_m f_m = -m f_{m-1},\\ &L^+_{m+1} = E + \beta^{-1}(m + 1 - \alpha - z), \qquad L^-_m = E - 1,\\ &f_m(z) = \frac{\Gamma(m - \alpha - z + 1)}{\Gamma(1 - \alpha - z)} {}_1F_1(-m; z + \alpha - m; \beta). \end{aligned}$$

This is a type $\beta'$ factorization by first order difference operators [4]. When the factorization is bounded below, the solutions $f_m(z)$ are proportional to the Charlier polynomials.

*Example* 9.

$$(4.11)\qquad \begin{aligned} L^+_{m+1} &= x^{-1}(q - 1)^{-1}[T^+ - 1] + q^{-m-1}cx,\\ L^-_m &= x^{-1}(q - 1)^{-1}[T^+ - 1],\\ a_m &= c(q - 1)^{-1}(q^{-m} - 1). \end{aligned}$$

The functions associated with this factorization are $q$-analogies of the parabolic cylinder functions and have a fairly complicated representation. Denote the basic hypergeometric function by ${}_2\varphi_1(a, b; c; q; x)$ where now the notation explicitly exhibits the $q$ dependence of the function. Then the functions

$$
\begin{aligned}
f_m(x) &= b_{m\,2}\varphi_1(q_1^{-m/2}, 0; q_1^{1/2}; q_1; -c(q-1)x^2) \\
&\quad + b_{m+1}x\,{}_2\varphi_1(q_1^{(1-m)/2}, 0; q_1^{3/2}; q_1; -c(q-1)x^2), \qquad (4.12)\\
b_m &= c^{m/2}(q-1)^{-m/2}\prod_{j=0}^{\infty}(1 - q_1^{(-m+1)/2+j}),
\end{aligned}
$$

where $q_1 = q^2$, satisfy the relations

$$
\begin{aligned}
&L_{m+1}^{+} f_m = f_{m+1}, \qquad L_m^{-} f_m = c\frac{(1-q^{-m})}{q-1} f_{m-1}, \qquad (4.13)\\
&X_m f_m = 0.
\end{aligned}
$$

If we let $q \to 1$ the $f_m(x)$ become parabolic cylinder functions and the factorization becomes

$$
\begin{aligned}
&L_{m+1}^{+} = \frac{d}{dx} + cx, \qquad L_m^{-} = \frac{d}{dx}, \qquad (4.14)\\
&a_m = -mc.
\end{aligned}
$$

In this case the eigenfunctions corresponding to a factorization bounded below ($m$ a nonnegative integer) are just the Hermite polynomials. Expressions (4.14) define a type D′ factorization by linear differential operators [1], [2], [3].

*Example* 10. The expressions

$$
\begin{aligned}
&L_{m+1}^{+} = x^{-1}(q-1)^{-1}[T^{+} - q^{m}], \qquad L_m^{-} = x^{-1}(q-1)^{-1}[T^{+} - q^{-m}], \qquad (4.15)\\
&a_m = 0
\end{aligned}
$$

satisfy (4.1), (4.2) and thus define a factorization. We define the functions

$$
\begin{aligned}
f_m(x) &= (1-q)^m q^{m(m-1)/2} x^m \qquad (4.16)\\
&\quad \cdot \prod_{j=0}^{\infty}(1 - q_1^{m+j}) \cdot {}_0\varphi_1(; q_1^{m+1}; q_1; (q-1)^2 q x^2) = I_m^q(x),
\end{aligned}
$$

where $q_1 = q^2$ and

$$
{}_0\varphi_1(; a; x) \equiv {}_0\varphi_1(; a; q; x). \qquad (4.17)
$$

The $q$-dependence of ${}_0\varphi_1$ is exhibited explicitly on the right-hand side of (4.17). The $f_m(x)$ are $q$-analogies of Bessel functions. They satisfy the relations

$$
\begin{aligned}
&L_{m+1}^{+} f_m = f_{m+1}, \qquad L_m^{-} f_m = f_{m-1}, \qquad (4.18)\\
&X_m f_m = f_m.
\end{aligned}
$$

If we let $q \to 1$ the factorization retains the form (4.18) where now

$$
L_{m+1}^{+} = \frac{d}{dx} - \frac{m}{x}, \qquad L_m^{-} = \frac{d}{dx} + \frac{m}{x} \qquad (4.19)
$$

and the eigenfunctions $f_m(x)$ become modified Bessel functions $I_m(x)$, [12]. This is the type C″ factorization of Bessel's equation [1], [2], [3].

*Example* 11. The expressions

$$(4.20)\qquad \begin{aligned} &L_{m+1}^{+} = T^{+} + w^{-1}(q-1)^{-1}(xq^{-m} - x^{-1}q^{m}), \qquad L_m^{-} = T^{+},\\ &a_m = 0 \end{aligned}$$

define a factorization of the form (4.1). If we set $x = q^z$, it is easy to show that the functions

$$(4.21)\qquad f_m(z) = I_{m-z}^{q}(w)$$

satisfy the relations

$$(4.22)\qquad \begin{aligned} &L_{m+1}^{+} f_m = f_{m+1}, \qquad L_m^{-} f_m = f_{m-1},\\ &X_m f_m = f_m. \end{aligned}$$

The functions $I_z^q(w)$ are defined by (4.16). In the limit as $q \to 1$ the factorization maintains the form (4.22) where

$$(4.23)\qquad \begin{aligned} &L_{m+1}^{+} = E + \frac{2(z-m)}{w}, \qquad L_m^{-} = E,\\ &f_m(z) = I_{m-z}(w). \end{aligned}$$

The basic functions $f_m(z)$ are now modified Bessel functions and the factorization is type $\gamma'$, [4].

*Example* 12. Consider the factorization

$$(4.24)\qquad \begin{aligned} &L_{m+1}^{+} = x^{-1}(q-1)^{-1}[T^{+} - 1], \qquad L_m^{-} = x^{-1}(q-1)^{-1}[T^{+} - 1],\\ &a_m = 0. \end{aligned}$$

The eigenfunctions

$$(4.25)\qquad f_m(x) = {}_2\varphi_1(0,0;0;(1-q)x), \quad \text{independent of } m,$$

satisfy the relations

$$(4.26)\qquad \begin{aligned} &L_{m+1}^{+} f_m = f_{m+1}, \qquad L_m^{-} f_m = f_{m-1},\\ &X_m f_m = f_m. \end{aligned}$$

In the limit as $q \to 1$ the factorization retains the form (4.26) where now

$$(4.27)\qquad \begin{aligned} &L_{m+1}^{+} = L_m^{-} = \frac{d}{dx},\\ &f_m(x) = \exp x. \end{aligned}$$

This is the rather trivial, type D″ factorization [1], [3].

**5. Factorizations and Lie algebras.** To illustrate the connection between our factorizations and infinite-dimensional Lie algebras we consider only the factorizations of § 3. The results for the factorizations of § 4 are analogous and are left to the reader.

Given a factorization defined by expressions (3.2)–(3.4) we define the operators $J^+$, $J^-$, $J^3$ as follows:

$$J^+ = t\left[(k_2x + c_5)T_x^+ + \frac{c_3}{q}T_t^- + c_2x\right],$$

$$J^- = t^{-1}[(k_1x + c_4)T_t^+T_x^- + c_1xT_t^+ + c_6], \tag{5.1}$$

$$J^3 = (q-1)^{-1}[T_t^+ - 1].$$

It will be shown that the eigenfunctions of the operators (3.5) are related to a representation of the Lie algebra generated by the $J$-operators. The $J$-operators act on a space $V$ of functions of the two complex variables $x$, $t$. The subscripts on the $T$-operators denote the variables on which the operators are acting. It is a straightforward computation to verify the commutation relations

$$[J^3, J^+] = J^+T_t^+, \qquad [J^3, J^-] = -q^{-1}J^-T_t^+,$$

$$[J^+, J^-] = (A(q-1)^2 + q^{-1}B(q-1)^2T_t^-)J^3 + (A(q-1) - q^{-1}B(q-1))I, \tag{5.2}$$

where the constants $A$, $B$ are defined by (3.4) and $I$ is the identity operator. It follows from relations (5.2) that the operators $J^\pm$, $J^3$, $I$ generate an infinite-dimensional Lie algebra $\mathscr{G}$ acting on $\mathscr{V}$. This in itself is not very interesting. However, since our operators were defined from a factorization it is easy to show from (5.2) that the operator

$$C_{A,B} = J^+J^- + A(q-1)J^3 - B(q-1)T_t^-J^3 + (A+B)I \tag{5.3}$$

commutes with the elements of $\mathscr{G}$. In fact,

$$[C_{A,B}, J^\pm] = [C_{A,B}, J^3] = 0.$$

The invariant operator $C_{A,B}$ can be considered as a $q$-analogy of the Casimir operator.

Let $X_m$ be the operator (3.5) and set $f_m(x,t) = Y_\lambda(m,x)t^m$ where $X_mY_\lambda(m,x) = \lambda Y_\lambda(m,x)$. Then we find

$$J^+f_m(x,t) = L_{m+1}^+Y_\lambda(m,x)t^{m+1},$$

$$J^-f_m(x,t) = L_m^-Y_\lambda(m,x)t^{m-1},$$

$$J^3f_m(x,t) = \frac{q^{m-1}}{q-1}Y_\lambda(m,x)t^m, \tag{5.4}$$

$$C_{A,B}f_m(x,t) = X_mY_\lambda(m,x)t^m = \lambda Y_\lambda(m,x)t^m.$$

It follows from these expressions that the ladders of eigenfunctions obtained from the factorization method correspond to models of irreducible representations of the infinite-dimensional Lie algebra $\mathscr{G}$.

Given any pair of complex numbers $(a,b)$ we define a four-dimensional Lie algebra $\mathscr{G}(a,b)$ with basis $\mathscr{J}^+$, $\mathscr{J}^-$, $\mathscr{J}^3$, $\mathscr{J}$ by the commutation relations

$$[\mathscr{J}^+, \mathscr{J}^-] = 2a^2\mathscr{J}^3 - b\mathscr{J}, \qquad [\mathscr{J}^3, \mathscr{J}^\pm] = \pm\mathscr{J}^\pm,$$

$$[\mathscr{J}^\pm, \mathscr{J}] = [\mathscr{J}^3, \mathscr{J}] = \theta. \tag{5.5}$$

Here, $\theta$ is the additive identity element of the Lie algebra. The following isomorphisms are valid [3]:

$$
\begin{aligned}
&\mathscr{G}(a,b) \cong \mathscr{G}(1,0) \cong sl(2) \oplus (\mathscr{I}) \quad \text{if } a \neq 0,\\
(5.6) \qquad &\mathscr{G}(a,b) \cong \mathscr{G}(0,1) \quad \text{if } a = 0, \quad b \neq 0,\\
&\mathscr{G}(a,b) \cong \mathscr{G}(0,0) \cong \mathscr{T}_3 \oplus (\mathscr{I}) \quad \text{if } a = b = 0,
\end{aligned}
$$

where $sl(2)$ is the Lie algebra of the $2 \times 2$ complex unimodular group and $\mathscr{T}_3$ is the Lie algebra of the complex Euclidean group in the plane.

In references [2]–[4] it is shown that the factorization types A, B, $\alpha'$, $\alpha''$ are associated with irreducible representations of $\mathscr{G}(1,0)$, the factorization types C′, D′, $\beta'$, $\beta''$ are associated with irreducible representations of $\mathscr{G}(0,1)$, and the factorization types C″, D″, $\gamma'$, $\gamma''$ are associated with irreducible representations of $\mathscr{G}(0,0)$. In the first twelve examples considered in this paper we have shown that we can obtain these twelve factorization types from $q$-factorizations in the limit as $q \to 1$. We can shed light on these results by the following construction: Form a one-parameter family (parameter $q$) of factorizations such that $L_m^{\pm}$, $A(q-1)^2$, $B(q-1)^2$, and $A(q-1) - q^{-1} \cdot B(q-1)$ approach finite limits as $q \to 1$. Then in the limit as $q \to 1$ the commutation relations (5.2) become

$$
(5.7) \qquad
\begin{aligned}
&[J^3, J^{\pm}] = \pm J^{\pm},\\
&[J^+, J^-] = 2a^2 J^3 - bI,
\end{aligned}
$$

where

$$
\begin{aligned}
2a^2 &= \lim_{q\to 1} \{(A + q^{-1}B)(q-1)^2\},\\
b &= \lim_{q\to 1} \{(q^{-1}B - A)(q-1)\}.
\end{aligned}
$$

(Note that $\lim_{q\to 1} J^3 = t(\partial/\partial t)$.) Since expressions (5.7) are just the commutation relations of the Lie algebra $\mathscr{G}(a,b)$ it is not surprising that we obtain the factorizations associated with this Lie algebra in the limit. Of course, the one-parameter families chosen in the twelve examples are highly nonunique. We have chosen the families for simplicity and to agree with certain well-known $q$-analogies of the hypergeometric functions.

The infinite-dimensional Lie algebra $\mathscr{G}(q)$, $q \neq 1$, which we have associated with each factorization is also not unique. For example, instead of the operators (5.1) we could choose operators $J'^{\pm}$, $J'^3$, $I'$ such that $J'^+ = J^+$, $J'^- = J^-$, $I' = I$, $J'^3 = t(\partial/\partial t)$. The primed operators generate a Lie algebra $\mathscr{G}'(q)$ which is not isomorphic to $\mathscr{G}(q)$. However, in the limit as $q \to 1$ this nonuniqueness disappears and we obtain again the known representation theory of the Lie algebras $\mathscr{G}(a,b)$.

At this point we briefly indicate the relationship between $q$-factorizations and certain six-dimensional Lie algebras. Consider the infinite-dimensional Lie algebra generated by the operators $J^{\pm}$, $J^3$ (given by (5.1)) and $P^3 = ax^{-1} + b$, $a, b \in \mathscr{C}$. In analogy with the discussion earlier in this section we parametrize the $J$- and $P$-operators (parameter $q$) and investigate the conditions such that as $q \to 1$ the operators generate a finite-dimensional Lie algebra. In the following cases we obtain a six-dimensional Lie algebra in the limit. (Here, only the limits will be listed.

The construction of corresponding one-parameter families, though simple, will be omitted.)

*Example* 13. A basis for the Lie algebra $\mathcal{O}_4$ generated by $J^{\pm}$, $J^3$, $P^3$ is given by

$$J^+ = t\left(-(\gamma + z)E + (z + \gamma + \alpha) + t\frac{\partial}{\partial t}\right), \quad J^- = t^{-1}\left(zL - z + \alpha - t\frac{\partial}{\partial t}\right),$$

$$(5.8) \qquad J^3 = t\frac{\partial}{\partial t}, \qquad P^3 = z + \frac{\gamma}{2},$$

$$P^+ = t(z + \gamma)E, \qquad P^- = -t^{-1}zL.$$

Here, $\mathcal{O}_4$ is the Lie algebra of the complex orthogonal group in four-space. This case is closely related to Example 1 and leads to recursion relations for the Hahn and Chebyshev polynomials [4].

*Example* 14. The operators

$$J^+ = t\left(x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \lambda\right), \quad J^3 = t\frac{\partial}{\partial t},$$

$$(5.9) \qquad J^- = t^{-1}\left(x(1 - x)\frac{\partial}{\partial x} - t\frac{\partial}{\partial t} + x(\alpha + \lambda) - \lambda\right),$$

$$P^- = 2\omega(1 - x^{-1})t^{-1}, \quad P^+ = 2\omega x^{-1}t, \quad P^3 = \omega(2x^{-1} - 1)$$

form a basis for $\mathcal{T}_6$, the Lie algebra of the complex Euclidean group in three-space. This case is closely related to Example 2. The operators (5.9) define a type E factorization studied in [1]–[3], [13]. This factorization yields recursion relations for the hypergeometric functions.

*Example* 15. The operators

$$(5.10) \qquad \begin{aligned} J^{\pm} &= t^{\pm 1}\left(x\frac{\partial}{\partial x} \pm t\frac{\partial}{\partial t} \mp \frac{x}{2}\right), \qquad J^3 = t\frac{\partial}{\partial t}, \\ P^{\pm} &= \pm 2t^{\pm 1}x^{-1}, \qquad P^3 = 2\alpha x^{-1} \end{aligned}$$

again form a basis for $\mathcal{T}_6$. These operators, closely related to Example 3, define a type F factorization. This factorization yields recursion relations for confluent hypergeometric functions and is studied in [1]–[3], [13].

**6. Some identities.** It is well known that factorizations expressed in terms of linear differential operators lead, via the Lie theory of local transformation groups, to addition theorems and generating functions for special functions [3]. There are analogous identities obeyed by special functions corresponding to factorizations by $q$-difference and difference operators. However, in the latter cases there is no local Lie theory and the formulas are much more difficult to derive. Rather than attempt to sketch a complete theory we shall merely present a few simple examples of identities whose proofs are motivated by Lie algebraic ideas.

Suppose we have a model of the Lie algebra $\mathcal{G}(a, b)$ in terms of differential operators $J^{\pm}$, $J^3$, $E$ and suppose the functions $f_m$ form a basis for an irreducible representation of $\mathcal{G}(a, b)$ using this model. Then, choosing an element of the Lie algebra, say $J^+$, we find that the expressions

$$\exp(\alpha J^+)f_m = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!}(J^+)^k f_m \tag{6.1}$$

lead to identities for the "special functions" $f_m$. In fact, the left-hand side of (6.1) can be computed using local Lie theory while the right-hand side can be evaluated from the structure of the irreducible representation. A large number of known identities for special functions have this basic structure.

Similar ideas lead to identities for basis functions $f_m$ corresponding to a model of an irreducible representation of an infinite-dimensional Lie algebra $\mathcal{G}$ in which the elements of the Lie algebra are $q$-difference operators. However, in this case the left-hand side of (6.1) cannot be obtained from local Lie theory and must be computed directly. Furthermore, rather than make use of the exponential function it is usually simpler to use a $q$-analogy such as

$$\exp_q(\alpha) = {}_2\varphi_1(0,0;0;(1-q)\alpha),$$

i.e., Example 12.

As a simple illustration of an identity obtained in such a fashion we consider Example 2. There, the operator $J^+$ and basis vectors $f_m(x,t)$ are given by $(d = b = 1)$

$$J^+ = \frac{t}{(q-1)}(-T_x^+ + T_t^-),$$

$$f_m(x,t) = {}_2\varphi_1(a, q^m; c; qx)t^m.$$

We now evaluate both sides of

$$\exp_q(\alpha J^+)f_m(x,t) = \sum_{k=0}^{\infty} \frac{\alpha^k(q-1)^k(J^+)^k}{(q-1)(q^2-1)\cdots(q^k-1)} f_m(x,t).$$

The right-hand side follows from (3.12) and the left-hand side can be computed directly by operating term by term on the series for $f_m$, with the result

$$\begin{aligned} &{}_2\varphi_1(a, q^m; c; q[x, \alpha t]_m)t^m \\ &\quad = \sum_{k=0}^{\infty} \frac{(q^{-m}-1)(q^{-m-1}-1)\cdots(q^{-m-k+1}-1)}{(q-1)(q^2-1)\cdots(q^k-1)}(\alpha t)^k \\ &\quad \cdot {}_2\varphi_1(a, q^{m+k}; c; qx)t^m. \end{aligned} \tag{6.2}$$

Here, the symbol on the left side of (6.2) is defined by replacing $x^n$ in the power series expansion of ${}_2\varphi_1$ by

$$x^n {}_1\varphi_1(q^{-n-m}; 0; q^{-1}; -t\alpha q^{n-1}).$$

This notation is due to Hahn [6]. If we set $a = q^\alpha, c = q^\gamma$ and let $q \to 1$, (6.2) becomes

$$\begin{aligned} &{}_2F_1\left(\alpha, m; \gamma; \frac{x}{1-\alpha t}\right)(1-\alpha t)^{-m} \\ &\quad = \sum_{k=0}^{\infty} (\alpha t)^k \binom{m+k-1}{k} {}_2F_1(\alpha, m+k; \gamma; x), \end{aligned} \tag{6.3}$$

an identity whose group-theoretic interpretation is well known [3], [7].

As another illustration, consider the operator $J^- = (xt(q-1))^{-1}(T_x^+ - 1)$ of Example 9. Here, the basis vectors $f_m(x,t) = f_m(x)t^m$, equation (4.12), are $q$-analogies of the parabolic cylinder functions. Computing $\exp_q(\alpha J^-)f_m(x,t)$ in two different ways we obtain ($c = 1$):

$$(6.4)\quad f_m([x+\alpha]) = \sum_{k=0}^{\infty} \frac{(1-q^{-m})(1-q^{-m+1})\cdots(1-q^{-m+k-1})}{(q-1)(q^2-1)\cdots(q^k-1)} \cdot f_{m-k}(x)\alpha^k.$$

Here, the left side of (6.4) is defined by replacing $x^n$ in the power series expansion of $f_m(x)$ by $x^n{}_1\varphi_1(q^n; 0; q^{-1}; \alpha/qx)$. If $m$ is a nonnegative integer and $q \to 1$ then (6.4) becomes

$$H_m(x+\alpha) = \sum_{k=0}^{m} \binom{m}{m-k}(-\alpha)^k H_{m-k}(x),$$

an identity for the Hermite polynomials.

Another method for deriving identities obeyed by functions related to a model of the Lie algebra $\mathcal{G}$ is based on elementary facts about commuting operators. In this method, essentially due to Weisner [8]–[10], one looks for simultaneous eigenvectors of the commuting operators $C_{A,B}$ and $\alpha J^3 + \beta J^+ + \gamma J^-$, equations (5.2), (5.3). As an especially simple example consider the operators

$$(6.5)\quad \begin{aligned} &J^+ = tE_z, \qquad J^- = t^{-1}\left(-q^z T_t^+ E_z - \frac{2}{w(q-1)}(q^z T_t^+ - 1)\right), \\ &J^3 = (T_t^+ - 1)/(q-1), \end{aligned}$$

where $E_z f(z,t) = f(z+1,t)$. The operators (6.5) satisfy exactly the same commutation relations as the operators obtained from Example 11, but they correspond to a class of factorizations not encountered in §§ 3 and 4. The functions

$$j_m^q(w) = \left[\prod_{k=0}^{\infty} \frac{(1-q^{m+k+1})}{(1-q^{k+1})}\right](w(q-1))^m {}_2\varphi_1(0,0;q^{m+1}; -(w(q-1)/2)^2)$$

satisfy the relation

$$J^+ f_m = f_{m+1}, \qquad J^- f_m = f_{m-1}, \qquad J^3 f_m = m f_m, \qquad J^+J^- f_m = f_m,$$

where

$$f_m(x,t) = j_{m+z}^q(w)t^m.$$

Note that $J^+$, $J^-$, $J^+J^-$ are pairwise commuting operators. We search for a simultaneous eigenvector of $J^+$ and $J^-$; in particular, a function $f(z,t)$ such that $J^+f = J^-f = f$. An elementary computation shows $f(z,t) = t^{-z}g(t)$ where $g(t)$ satisfies the $q$-difference equation

$$g(qt) = \frac{(1 - wt(q-1)/2)}{(1 + w(q-1)/2qt)} g(t).$$

It follows easily from Example 12 that

$$g(t) = \exp_q\left(-\frac{wt}{2}\right)\exp_q\left(\frac{w}{2t}\right)$$

to within a constant factor. Writing

$$f(z, t) = \sum_{n=-\infty}^{+\infty} g_n(w) t^{-z+n},$$

we see that the functions $g_n(w)$ are solutions of the difference equations

$$q^{n+1} g_{n+1}(w) + \frac{2}{w(q-1)}(q^n - 1) g_n(w) + g_{n-1}(w) = 0.$$

The only solutions of these equations which are bounded in a neighborhood of $w = 0$ are $g_n(w) = c j_n^q(-w)$, $n = 0, \pm 1, \pm 2, \cdots$. Thus,

$$\exp_q\left(-\frac{wt}{2}\right) \exp_q\left(\frac{w}{2t}\right) = \sum_{n=-\infty}^{\infty} c j_n^q(-w) t^n.$$

By comparing coefficients of $t^n$ for $n = 0$ we can easily show that $c = 1$. Hence,

$$\exp_q\left(\frac{wt}{2}\right) \exp_q\left(-\frac{w}{2t}\right) = \sum_{n=-\infty}^{\infty} j_n^q(w) t^n.$$

As $q \to 1$ this identity reduces to a well-known generating function for Bessel functions [12]. The above example was treated by Hahn [6], using a different method. The same technique would work on Examples 10 and 11 but the generating functions turn out to be more complicated.

## REFERENCES

[1] L. INFELD AND T. HULL, *The factorization method*, Rev. Modern Phys., 23 (1951), pp. 21–68.
[2] W. MILLER, *On Lie algebras and some special functions of mathematical physics*, Mem. Amer. Math. Soc., no. 50, Providence, 1964.
[3] ———, *Lie theory and special functions*, Academic Press, New York, 1968.
[4] ———, *Lie theory and difference equations. I*, J. Math. Anal. Appl., 28 (1969), pp. 383–399.
[5] W. HAHN, *Über Orthogonalpolynome, die q-Differenzengleichen genügen*, Math. Nachr., 2 (1949), pp. 4–34.
[6] ———, *Beiträge zur Theorie der Heineschen Reihen*, Ibid., 2 (1949), pp. 340–379.
[7] L. WEISNER, *Group-theoretic origin of certain generating functions*, Pacific J. Math., 5 (1955), pp. 1033–1039.
[8] ———, *Generating functions for Hermite functions*, Canad. J. Math., 11 (1959), pp. 141–147.
[9] ———, *Generating functions for Bessel functions*, Ibid., 11 (1959), pp. 148–155.
[10] P. MORSE AND H. FESHBACK, *Methods of Mathematical Physics*, Part 1, McGraw-Hill, New York, 1953.
[11] L. J. SLATER, *Generalized Hypergeometric Junctions*, Cambridge University Press, Cambridge, 1966.
[12] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions*, vol. II, McGraw-Hill, New York, 1953.
[13] W. MILLER, *Special functions and the complex Euclidean group in 3-space. I, II, III*, J. Math. Phys., 9 (1968), pp. 1162–1175, 1175–1187, 1434–1444.

SIAM J. MATH. ANAL.
Vol. 1, No. 2, May 1970

# FURTHER RESULTS ON THE BOUNDEDNESS AND THE STABILITY OF SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS OF THE FOURTH ORDER*

MARTIN HARROW†

**1.** The equation considered here is of the form

$$x^{(4)} + a\dddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t). \tag{1.1}$$

It is assumed that the functions $f(\ddot{x})$, $g'(\dot{x})$, $h'(x)$ and $p(t)$ depend only on the arguments displayed, are continuous, and are of such a nature that the existence and uniqueness of the solutions, as well as their continuous dependence on the initial values, are assured. Let $x(t)$ be a solution. Then we may write

$$\frac{dx}{dt} = \dot{x}, \quad \frac{d^2x}{dt^2} = \ddot{x}, \quad \frac{d^3x}{dt^3} = \dddot{x}, \quad \frac{d^4x}{dt^4} = x^{(4)}.$$

In what follows, we shall also use the following system, which is equivalent to (1.1):

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = w, \quad \dot{w} = -aw - f(z) - g(y) - h(x) + p(t). \tag{1.2}$$

THEOREM 1. *Given are* (1.2) *with* $p(t) = 0$; *and*

(i) *positive constants* $a, b, c, d$ *such that* $\Delta = abc - c^2 - a^2d > 0$;

(ii) $f(0) = g(0) = h(0) = 0$, $g(y)/y \geqq c$ *for all* $y$ $(y \neq 0)$,

$$d - \frac{2a\Delta_0}{c} < h'(x) \leqq d \text{ for all } x,$$

$$0 < \beta \leqq \frac{h(x)}{x} \text{ for all } x \qquad (x \neq 0),$$

*where* $\beta$ *is a constant and* $\Delta_0$ *is a positive constant such that*:

(iii) $\{ab - g'(y)\}c - a^2d \geqq \Delta_0$ *for all* $y$;

(iv) $0 \leqq g'(y) - g(y)/y \leqq \alpha$ *for all* $y$ $(y \neq 0)$, *where the constant* $\alpha$ *is such that* $\alpha < \Delta d/(2ac^2)$;

(v) $0 \leqq f(z)/z - b \leqq \varepsilon_0 c^3/d^2$ *for all* $z$ $(z \neq 0)$, *where* $\varepsilon_0$ *is a positive constant defined as*

$$\varepsilon_0 < \varepsilon = \min\{d/c, \Delta/(6a^2c), \Delta_0/(4a^2c)\}.$$

*Then every solution of the differential equation* (1.1) *satisfies*

$$x \to 0, \quad \dot{x} \to 0, \quad \ddot{x} \to 0, \quad \dddot{x} \to 0 \quad \text{as } t \to \infty. \tag{1.3}$$

The following special cases should be noted:

(a) $f(z) = bz$, $g(y) = cy$, $h(x) = dx$,

(b) $f(z) = bz$, $g(y) = cy$,

(c) $f(z) = bz$.

---

* Received by the editors July 22, 1969.

† Department of Mathematics, Sir George Williams University, Montreal, Canada, and McGill University, Montreal, Canada.

In case (a), all conditions in hypotheses (i) to (v) are trivially fulfilled and reduce to the Routh–Hurwitz criteria for the asymptotic stability in the large of the trivial solution of the equation

$$x^{(4)} + a\dddot{x} + b\ddot{x} + c\dot{x} + dx = 0.$$

In the case (b), the theorem reduces to a special case of an earlier stability result obtained by Harrow [1].

For the case $p(t) \not\equiv 0$, we shall prove the following theorem.

THEOREM 2. *If the conditions in hypotheses* (i) *to* (v) *of Theorem* 1 *hold, and if further*

$$\int_0^t |p(s)|\, ds \leqq A < \infty$$

*for all* $t \geqq 0$, *where* $A$ *is some positive number, then given any finite numbers* $x_0$, $y_0$, $z_0$, $w_0$ *there is a finite constant* $D \equiv D(x_0, y_0, z_0, w_0)$ *such that there exists a (unique) solution* $x(t)$ *of* (1.1) *which is determined by the initial conditions*

$$x(0) = x_0, \quad \dot{x}(0) = y_0, \quad \ddot{x}(0) = z_0, \quad \dddot{x}(0) = w_0,$$

*and which satisfies*

$$|x(t)| \leqq D, \quad |\dot{x}(t)| \leqq D, \quad |\ddot{x}(t)| \leqq D, \quad |\dddot{x}(t)| \leqq D \tag{1.4}$$

*for all* $t \geqq 0$.

**2. The function** $V(x, y, z, w)$. The proofs of (1.3) and (1.4) depend on the function $V(x, y, z, w)$ defined by

$$\begin{aligned} 2V(x, y, z, w) = {} & \left(b\delta - \frac{d}{a}\right)y^2 + 2\int_0^y g(s)\, ds + 2\delta \int_0^x h(s)\, ds + a^{-1}w^2 \\ & + (a - \delta)z^2 + 2wz + 2\delta wy + 2\delta ayz + 2h(x)y \\ & + 2a^{-1}h(x)z + 2a^{-1}g(y)z + 2a^{-1}\int_0^z f(s)\, ds, \end{aligned}$$

where $\delta = d/(c) + \varepsilon$. We shall prove that $V$ is a Lyapunov function for the system (1.2) with $p(t) \equiv 0$. The proof consists of two lemmas.

LEMMA 1. *Under hypotheses* (i) *to* (v) *of Theorem* 1, *it follows that* $V(0, 0, 0, 0) = 0$ *and there are positive constants* $D_i$ $(i = 1, 2, 3, 4)$ *depending on* $a, b, c, d, \varepsilon, \Delta_0, \alpha, \beta$, *such that for all* $x$, $y$, $z$, $w$,

$$V(x, y, z, w) \geqq D_1x^2 + D_2y^2 + D_3z^2 + D_4w^2. \tag{2.1}$$

*Proof*. Define $g(y)/y = g'(0)$ for $y = 0$. For $x \neq 0$, $2V(x, y, z, w)$ can be written as

$$\begin{aligned} 2V(x, y, z, w) = {} & V_1 + V_2 + V_3 + a^{-1}(w + az + \delta ay)^2 \\ & + \frac{y}{g(y)}\left[\frac{h(x)}{x}x + y\frac{g(y)}{y} + a^{-1}\frac{g(y)}{y}z\right]^2, \end{aligned}$$

where

$$V_1 = \left(b\delta - \frac{d}{a} - a\delta^2\right)y^2 + 2\int_0^y g(s)\,ds - yg(y)$$

$$\geqq \frac{d\Delta}{ac^2}y^2 - a\varepsilon\left(\frac{d}{c} + \varepsilon\right)y^2 + 2\int_0^y g(s)\,ds - yg(y)$$

$$\geqq \frac{d\Delta}{2ac}y^2 + 2\int_0^y g(s)\,ds - yg(y).$$

But we note that

$$yg(y) = \int_0^y g(s)\,ds + \int_0^y sg'(s)\,ds.$$

Therefore

$$V_1 \geqq \int_0^y \left\{\frac{d\Delta}{ac^2} - \left[g'(s) - \frac{g(s)}{s}\right]\right\} s\,ds,$$

and, by hypotheses (iv),

$$V_1 \geqq \int_0^y \left(\frac{d\Delta}{ac^2} - \alpha\right) s\,ds \geqq \int_0^y \frac{d\Delta}{2ac^2} s\,ds.$$

Hence

$$V_1 \geqq \frac{d\Delta y^2}{4ac^2}.$$

Also,

$$V_2 = 2a^{-1}\int_0^z f(s)\,ds - a^{-1}bz^2 + \left[a^{-1}b - \delta - a^{-2}\frac{g(y)}{y}\right]z^2$$

$$= 2a^{-1}\int_0^z \left[\frac{f(s)}{s} - b\right] s\,ds + \left[a^{-1}b - \delta - a^{-2}\frac{g(y)}{y}\right]z^2$$

$$\geqq \left(\frac{\Delta_0}{a^2c} - \varepsilon\right)z^2 \geqq \frac{\Delta_0 z^2}{2a^2c},$$

and

$$V_3 = 2\delta\int_0^x h(s)\,ds - \frac{y}{g(y)}\left[\frac{h(x)}{x}\right]^2 x^2$$

$$= 2\int_0^x \frac{h(s)}{s}\left[\frac{d}{c} - \frac{y}{g(y)}h'(s)\right] s\,ds + 2\varepsilon\int_0^x h(s)\,ds.$$

By hypotheses (ii) $dg(y)/y \geqq ch'(x)$; therefore

$$V_3 \geqq \varepsilon\beta x^2.$$

Hence

$$2V(x, y, z, w) \geqq \varepsilon\beta x^2 + \frac{d\Delta y^2}{4ac^2} + \frac{\Delta_0 z^2}{2a^2 c} + a^{-1}(w + az + a\delta y)^2.$$

The case $x = 0$ is trivial, and the verification of the lemma is complete.

LEMMA 2. *Under hypotheses* (i) *to* (v) *of Theorem* 1 *there exist positive constants* $D_5$ *and* $D_6$ *depending on* $a, b, c, d, \varepsilon, \varepsilon_0, \Delta_0$, *such that if* $(x, y, z, w)$ *is any solution of* (1.2) *with* $p(t) \equiv 0$, *then*

$$\dot{V} \equiv \frac{d}{dt} V(x, y, z, w) \leqq -(D_5 y^2 + D_6 z^2).$$

*Proof.* A straightforward calculation gives that, for the case $z \neq 0$,

$$-\dot{V} = V_4 + V_5 + [d - h'(x)]a^{-1}yz + V_6,$$

where

$$V_4 = \left[\delta \frac{g(y)}{y} - h'(x)\right] y^2 \geqq \varepsilon \frac{g(y)}{y} y^2 + [d - h'(x)]y^2,$$

$$V_5 = [b - \delta a - g'(y)a^{-1}]z^2 \geqq \frac{\Delta_0 z^2}{ac} - \varepsilon a z^2$$

and

$$V_6 = \left[\frac{f(z)}{z} - b\right](z^2 + \delta yz)$$

$$= \left[\frac{f(z)}{z} - b\right]\left(z + \frac{1}{2}\delta y\right)^2 - \frac{1}{4}\delta^2\left[\frac{f(z)}{z} - b\right]y^2.$$

Now

$$V_4 + [d - h'(x)]a^{-1}yz \geqq \varepsilon \frac{g(y)}{y} y^2 + [d - h'(x)]\left(y + \frac{z}{2a}\right)^2 - \frac{[d - h'(x)]}{4a^2} z^2.$$

Therefore

$$\dot{V} \leqq -\varepsilon \frac{g(y)}{y} y^2 + \frac{[d - h'(x)]}{4a^2} z^2 - \left(\frac{\Delta_0}{ac} - \varepsilon a\right) z^2 + \frac{\delta^2}{4}\left[\frac{f(z)}{z} - b\right] y^2.$$

By hypotheses (ii) and (v),

$$V_7 \equiv \frac{\Delta_0}{ac} - \varepsilon a - \frac{[d - h'(x)]}{4a^2} \geqq \frac{\Delta_0}{ac} - \varepsilon a - \frac{2a\Delta_0}{4a^2 c} > \frac{\Delta_0}{4ac}.$$

Again, by hypotheses (v),

$$\frac{1}{4}\delta^2\left[\frac{f(z)}{z} - b\right] < \varepsilon_0 c.$$

Therefore

$$\varepsilon \frac{g(y)}{y} - \frac{1}{4}\delta^2\left[\frac{f(z)}{z} - b\right] > (\varepsilon - \varepsilon_0)c$$

and

$$\dot{V} \leqq -\frac{\Delta_0 z^2}{4ac} - (\varepsilon - \varepsilon_0)cy^2. \tag{2.2}$$

The case $z = 0$ is trivial, and the verification of the lemma is complete.

**3. Proof of Theorem 1.** To prove (1.3) we proceed by a method originated by Barbashin [2] and used by Cartwright, Ezeilo and others. It is obvious that

$$\begin{aligned} V(x, y, z, w) &= 0 \quad \text{if and only if} \quad x^2 + y^2 + z^2 + w^2 = 0, \\ V(x, y, z, w) &> 0 \quad \text{if and only if} \quad x^2 + y^2 + z^2 + w^2 > 0, \\ V(x, y, z, w) &\to \infty \quad \text{if and only if} \quad x^2 + y^2 + z^2 + w^2 \to \infty. \end{aligned} \tag{3.1}$$

Let $\gamma$ denote a trajectory $[x(t), y(t), z(t), w(t)]$ of (1.2) with $p(t) \equiv 0$, such that at $t = 0, x = x_0, y = y_0, z = z_0, w = w_0$, where $(x_0, y_0, z_0, w_0)$ is an arbitrary point in the $x, y, z, w$-space from which motions may originate. Then by Lemma 2, for $t \geqq 0$,

$$V(x, y, z, w) = V[x(t), y(t), z(t), w(t)] = V(t) \leqq V(0).$$

Moreover, $V(t)$ is nonnegative and nonincreasing and therefore tends to a nonnegative limit, $V(\infty)$ say, as $t \to \infty$. Suppose that $V(\infty) > 0$. Consider the set $S = \{(x, y, z, w) | V(x, y, z, w) \leqq V(x_0, y_0, z_0, w_0)\}$. By (3.1) we know that $S$ is bounded, and therefore the set $\gamma \subset S$ is also bounded. Moreover, the nonempty set of all limit points of $\gamma$ consists of whole trajectories of

$$\dot{X} = y, \quad \dot{y} = z, \quad \dot{z} = w, \quad \dot{w} = -aw - f(z) - g(y) - h(x) \tag{3.2}$$

lying on the surface $V(x, y, z, w) = V(\infty)$. Thus if $P$ is a limit point of $\gamma$, then there exists a half-trajectory, say $\gamma_p$ of (3.2), issuing from $P$ and lying on the surface $V(x, y, z, w) = V(\infty)$. Since for every point $(x, y, z, w)$ on $\gamma_p$ we have $V(x, y, z, w) \geqq V(\infty)$, this implies that $\dot{V} = 0$ on $\gamma_p$. Also, by (2.2) $\dot{V} = 0$ implies that $y = z = 0$; and by (3.2) and hypotheses (ii) this means that $x = 0$. Thus the point $(0, 0, 0, 0)$ lies on the surface $V(x, y, z, w) = V(\infty)$ and hence $V(\infty) = 0$. This completes the proof of Theorem 1.

**4. Proof of Theorem 2.** The proof is based on a method devised by Antosiewicz [3]. Using the same function $V(x, y, z, w)$ as in the proof of Theorem 1, we have that, for the system (1.2),

$$\dot{V} \leqq -(D_5 y^2 + D_6 z^2) + (a^{-1}w + z + \delta y)p(t).$$

If $D_7 = \max(a^{-1}, 1, \delta)$, then

$$\dot{V} \leqq D_7[|w| + |z| + |y|]|p(t)|;$$

and since $|w| \leqq 1 + w^2$ and so on, it follows that

$$\dot{V} \leqq D_7[3 + w^2 + z^2 + y^2]|p(t)|.$$

Putting $D_8 = 3D_7$, $D_9 = \min(D_1, D_2, D_3, D_4)$, $D_{10} = D_7/D_9$ we obtain

$$\dot{V} - D_{10}V|p(t)| \leqq D_8|p(t)|,$$

and

$$V(t) \leqq \left\{ V(0) + D_8 \int_0^t |p(s)| K(s)\, ds \right\} K^{-1}(t),$$

where

$$K(t) \equiv \exp\left[ -D_{10} \int_0^t |p(s)|\, ds \right].$$

But $K(t) \leqq 1$; therefore

$$V(t) \leqq \{V(0) + AD_8\} \exp AD_{10}$$

and

$$V(0) = V(x_0, y_0, z_0, w_0).$$

This proves Theorem 2.

## REFERENCES

[1] MARTIN HARROW, *On the stability of solutions of some nonhomogeneous differential equations of the fourth order*, this Journal, 1 (1970), pp. 98–103.

[2] E. A. BARBASHIN, *On the theory of general dynamical systems*, Uchenye Zapiski Mosgov. Gos. Univ. 135, Matematika, vol. II, 1948, pp. 110–133.

[3] H. A. ANTOSIEWICZ, *On non-linear differential equations of the second order with integrable forcing term*, J. London Math. Soc., 30 (1955), pp. 64–69.

SIAM J. MATH. ANAL.
Vol. 1, No. 2, May 1970

# MONOTONEITY PROPERTIES OF SOLUTIONS OF HERMITIAN RICCATI MATRIX DIFFERENTIAL EQUATIONS*

WILLIAM T. REID†

**1. Introduction.** The results of this paper center around certain monotoneity properties possessed by the solutions of matrix differential equations involving an $n$-dimensional Hermitian Riccati matrix differential operator

$$K[W] = W' + WA(t) + A^*(t)W + WB(t)W - C(t),$$

with $B(t)$ and $C(t)$ Hermitian $n \times n$ matrix functions. In particular, the results obtained include theorems on the existence and range of extensibility of solutions which are Hermitian nonnegative, together with results on the behavior of solutions and allied functions when the linear Hamiltonian system associated with $K[W] = 0$ is subjected to a particular type of linear transformation. A special application of such properties is made to the study of a class of first order matrix differential equations involving with $K[W]$ an additional function, in general nonlinear, and which is monotone on the class of nonnegative Hermitian matrices. Also, earlier work of the author [7], [8], [9], [11] on the existence and nature of distinguished solutions of $K[W] = 0$ is elaborated in two important cases involving assumptions of normality that are of intermediate strength. Finally, the concluding section is devoted to brief comments on relationships between the results of this paper and those of recent papers by Bucy [2], [3] and Wonham [14].

Matrix notation is used throughout; in particular, matrices of one column are termed vectors, and for a vector $y = (y_\alpha)$, $\alpha = 1, \cdots, n$, the norm $|y|$ is given by $(|y_1|^2 + \cdots + |y_n|^2)^{1/2}$; the linear vector space of ordered $n$-tuples of complex numbers, with complex scalars, is denoted by $\mathfrak{C}_n$. The $n \times n$ identity matrix is signified by $E_n$, or by merely $E$ when there is no ambiguity, while 0 is used indiscriminately for the zero matrix of any dimension; the conjugate transpose of a matrix $M$ is denoted by $M^*$. If $M$ is an $n \times n$ matrix the symbol $\nu[M]$ is used for the maximum of $|My|$ on the unit ball $\{y \mid |y| \leqq 1\}$ in $\mathfrak{C}_n$. The notation $M \geqq N$ $(M > N)$ is used to signify that $M$ and $N$ are Hermitian matrices of the same dimensions and $M - N$ is a nonnegative (positive) definite Hermitian matrix. If an Hermitian matrix function $M(t)$, $t \in I$, is such that $M(s) - M(t) \geqq 0$ $(\leqq 0)$ for $(s, t) \in I \times I$, $s < t$, then $M(t)$ is termed nonincreasing (nondecreasing) Hermitian on $I$. If the elements of a matrix $M(t)$ are a.c. (absolutely continuous) on an interval $[a, b]$, then $M'(t)$ signifies the matrix of derivatives at values for which these derivatives exist and the zero matrix elsewhere; correspondingly, if the elements of $M(t)$ are (Lebesgue) integrable on $[a, b]$ then $\int_a^b M(t)\,dt$ denotes the matrix of integrals of respective elements of $M(t)$. If $M(t)$ and $N(t)$ are equal a.e. (almost everywhere) on their domain of definition we write simply $M(t) = N(t)$. A matrix function $M(t)$ is

* Received by the editors August 14, 1969.

† Department of Mathematics, The University of Oklahoma, Norman, Oklahoma 73069. This research was supported by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force, under Grant AF-AFOSR-68-1398.

called continuous, integrable, etc., when each element of the matrix possesses the specified property. Also, $M(t)$ is said to be locally a.c. on an interval $I$ if it is a.c. on arbitrary compact subintervals $[a, b]$ of $I$. If $M = [M_{\alpha j}]$, $N = [N_{\alpha j}]$, $\alpha = 1, \cdots, n$, $j = 1, \cdots, r$, are $n \times r$ matrices, for typographical simplicity the symbol $(M; N)$ is used to denote the $2n \times r$ matrix whose $j$th column has elements $M_{1j}, \cdots, M_{nj}, N_{1j}, \cdots, N_{nj}$.

For a given compact interval $[a, b]$ on the real line the symbols $\mathscr{L}_{nr}[a, b]$, $\mathscr{L}_{nr}^{\infty}[a, b]$ are used to denote the classes of $n \times r$ matrix functions $M(t) = [M_{\alpha\beta}(t)]$, $\alpha = 1, \cdots, n$, $\beta = 1, \cdots, r$, which on $[a, b]$ are respectively (Lebesgue) integrable, (Lebesgue) measurable and essentially bounded. Also, for brevity the symbols $\mathscr{L}_n[a, b]$, $\mathscr{L}_n^{\infty}[a, b]$ are written for the respective classes designated by indices $n, r = 1$.

**2. Related Riccati equations and linear differential systems.** In the following we shall be concerned with a Riccati matrix differential equation

$$K[W] \equiv W' + WA(t) + A^*(t)W + WB(t)W - C(t) = 0, \qquad t \in I, \tag{2.1}$$

where on a given interval $I$ on the real line the $n \times n$ coefficient matrix functions satisfy the following hypothesis:

$\mathfrak{H}$ $A(t), B(t), C(t)$ are of class $\mathscr{L}_{nn}[a, b]$ on arbitrary compact subintervals $[a, b]$ of $I$, while $B(t)$ and $C(t)$ are Hermitian for $t \in I$.

Intimately related to (2.1) is the linear Hamiltonian vector differential system

$$\begin{aligned} L_1[u, v](t) &\equiv -v'(t) + C(t)u(t) - A^*(t)v(t) = 0, \qquad t \in I, \\ L_2[u, v](t) &\equiv u'(t) - A(t)u(t) - B(t)v(t) = 0, \qquad t \in I, \end{aligned} \tag{2.2}$$

and the corresponding matrix differential system

$$\begin{aligned} L_1[U, V](t) &\equiv -V'(t) + C(t)U(t) - A^*(t)V(t) = 0, \qquad t \in I, \\ L_2[U, V](t) &\equiv U'(t) - A(t)U(t) - B(t)V(t) = 0, \qquad t \in I, \end{aligned} \tag{2.2$_M$}$$

in general $n \times r$ dimensional matrix functions $U(t)$, $V(t)$.

If $y = (y_\sigma)$, $\sigma = 1, \cdots, 2n$, with $y_\alpha = u_\alpha$, $y_{n+\alpha} = v_\alpha$, $\alpha = 1, \cdots, n$, then (2.2) may be written as the $2n$-dimensional vector differential equation

$$\mathscr{L}[y](t) \equiv \mathscr{J} y'(t) + \mathscr{A}(t)y(t) = 0, \qquad t \in I, \tag{2.2$'$}$$

where $\mathscr{J}$ and $\mathscr{A}(t)$ are the $2n \times 2n$ matrices

$$\mathscr{J} = \begin{bmatrix} 0 & -E_n \\ E_n & 0 \end{bmatrix}, \qquad \mathscr{A}(t) = \begin{bmatrix} C(t) & -A^*(t) \\ -A(t) & -B(t) \end{bmatrix}.$$

As $\mathscr{A}(t)$ is Hermitian, and $\mathscr{J}$ is skew-Hermitian, the vector differential operator $\mathscr{L}[y](t)$ is identical with its formal Lagrange adjoint $\mathscr{L}^*[y](t) = -\mathscr{J}^*y'(t) + \mathscr{A}^*(t)y(t)$. Correspondingly, if $Y(t) = (Y_{\sigma j}(t))$, $\sigma = 1, \cdots, 2n$, $j = 1, \cdots, r$, with $Y_{\alpha j}(t) = U_{\alpha j}(t)$, $Y_{n+\alpha, j}(t) = V_{\alpha j}(t)$, then (2.2$_M$) may be written as

$$\mathscr{L}[Y](t) \equiv \mathscr{J} Y'(t) + \mathscr{A}(t)Y(t) = 0, \qquad t \in I. \tag{2.2$'_M$}$$

The following interrelations between (2.1) and (2.2$_M$) are well known (see, for example, Reid [8, § 2] and [9, § II]).

LEMMA 2.1. *A solution $W = W(t)$ of* (2.1) *exists on a nondegenerate subinterval $I_\circ$ of $I$ if and only if there is a solution $Y(t) = (U(t); V(t))$ of* ($2.2_M$) *such that $U(t)$ is nonsingular on $I_\circ$ and $W(t) = V(t)U^{-1}(t)$.*

If $y_\alpha(t) = (u_\alpha(t); v_\alpha(t))$, $\alpha = 1, 2$, are solutions of (2.2) it follows readily that the function $\{u_1; v_1|u_2; v_2\}(t) = v_2^*(t)u_1(t) - u_2^*(t)v_1(t) = y_2^*\mathscr{J}y_1$ is constant on $I$; in particular, if the constant value of $\{u_1; v_1|u_2; v_2\}$ is zero then $(u_1(t); v_1(t))$ and $(u_2(t); v_2(t))$ are said to be *(mutually) conjoined solutions* of (2.2). If $Y(t) = (U(t); V(t))$ is a solution of ($2.2_M$) whose column vectors are $n$ linearly independent solutions of (2.2) which are mutually conjoined, then for brevity we say that $Y(t)$ is a *conjoined basis* for (2.2). If $Y(t) = (U(t); V(t))$ is a solution of ($2.2_M$) with $U(t)$ nonsingular, and $W(t) = V(t)U^{-1}(t)$, it follows readily that $\{U; V|U; V\}(t) = U^*(t)[W^*(t) - W(t)]U(t)$, and consequently we have the following result.

LEMMA 2.2. *If $W = W(t)$ and $Y(t) = (U(t); V(t))$ are solutions of* (2.1) *and* ($2.2_M$), *respectively, which are related as in Lemma* 2.1, *then $Y(t)$ is a conjoined basis for* (2.2) *if and only if $W(t)$ is Hermitian for $t \in I_\circ$.*

Now if $W = W_\circ(t)$ is a solution of (2.1) on a subinterval $I_\circ$ of $I$, and for $s \in I_\circ$ the matrix functions $G(t) = G(t, s|W_\circ)$, $H(t) = H(t, s|W_\circ)$ are determined as the solutions of the differential systems

$$G' + (A^* + W_\circ B)G = 0, \qquad G(s) = E, \tag{2.3a}$$

$$H' + H(A + BW_\circ) = 0, \qquad H(s) = E, \tag{2.3b}$$

and

$$\Theta(t, s|W_\circ) = \int_s^t H(r, s|W_\circ)B(r)G(r, s|W_\circ)\, dr, \tag{2.4}$$

then from Lemma 2.1 of Reid [8] it follows that a matrix function $W(t)$ is a solution of (2.1) on $I_\circ$ if and only if the constant matrix $\Gamma = W(s) - W_\circ(s)$ is such that $E + \Theta(t, s|W_\circ)\Gamma$ is nonsingular on $I_\circ$, and

$$W(t) = W_\circ(t) + G(t, s|W_\circ)\Gamma[E + \Theta(t, s|W_\circ)\Gamma]^{-1}H(t, s|W_\circ). \tag{2.5}$$

In particular, if $U_\circ(t)$ is the solution on $I_\circ$ of the matrix differential system

$$U_\circ'(t) = [A(t) + B(t)W_\circ(t)]U_\circ(t), \qquad U_\circ(s) = E,$$

then $Y_\circ(t) = (U_\circ(t); V_\circ(t))$, with $V_\circ(t) = W_\circ(t)U_\circ(t)$, is the solution of ($2.2_M$) satisfying the initial condition $Y_\circ(s) = (E; W_\circ(s))$; moreover, the corresponding solution of (2.3b) is given by $H(t, s|W_\circ) = U_\circ(s)U_\circ^{-1}(t)$. Also, if $W_\circ(t)$ is an Hermitian solution of (2.1) so that the above defined $Y_\circ(t) = (U_\circ(t); V_\circ(t))$ is a conjoined basis for (2.2), then $G(t, s|W_\circ) = H^*(t, s|W_\circ)$, and

$$\Theta(t, s|W_\circ) = U_\circ(s)S(t, s; U_\circ)U_\circ^*(s), \tag{2.6}$$

where $S(t, s; U_\circ)$ is the Hermitian matrix function

$$S(t, s; U_\circ) = \int_s^t U_\circ^{-1}(r)B(r)U_\circ^{*-1}(r)\, dr. \tag{2.7}$$

It is to be remarked that if $B(t) \geqq 0$ for $t$ a.e. on a subinterval $[s, s_1]$ of $I_o$ then $S(t, s; U_o) \geqq 0$ for $t \in [s, s_1]$. These relations, together with elementary algebraic transformations, yield the following result.

LEMMA 2.3. *Suppose that* $W = W_o(t)$ *is an Hermitian solution of* (2.1) *on a subinterval* $I_o$ *of* $I$, *and that* $Y_o(t) = (U_o(t); V_o(t))$ *is the corresponding conjoined basis for* (2.2) *determined by the initial condition* $Y_o(s) = (E; W_o(s))$, *where* $s \in I_o$. *Then* $W(t)$ *is a solution of* (2.1) *on* $I_o$ *if and only if the matrix* $\Gamma_1 = U_o^*(s)[W(s) - W_o(s)]U_o(s)$ *is such that* $E + S(t, s; U_o)\Gamma_1$ *is nonsingular on* $I_o$, *and*

$$W(t) = W_o(t) + U_o^{*-1}(t)\Gamma_1[E + S(t, s; U_o)\Gamma_1]^{-1}U_o^{-1}(t), \qquad t \in I_o. \tag{2.8}$$

COROLLARY. *Suppose that* $B(t) \geqq 0$ *for* $t$ *a.e. on a subinterval* $[a, b]$ *of* $I$, $W = W_o(t)$ *is an Hermitian solution of* (2.1) *on* $[a, b]$, *and* $W(t)$ *is a solution of* (2.1) *such that* $W(a) \geqq W_o(a)(W(a) > W_o(a))$. *Then the interval of existence of* $W(t)$ *includes* $[a, b]$ *and* $W(t) \geqq W_o(t)(W(t) > W_o(t))$ *for* $t \in [a, b]$.

If $W(a) > W_o(a)$, then $\Gamma_1 = U_o^*(a)[W(a) - W_o(a)]U_o(a) > 0$, and the result is a ready consequence of (2.8) for $s = a$ and $I_o = [a, b]$, since $\Gamma_1^{-1} + S(t, a; U_o) \geqq \Gamma_1^{-1} > 0$ for $t \in [a, b]$, and $[\Gamma_1^{-1} + S(t, a; U_o)]^{-1} = \Gamma_1[E + S(t, a; U_o)\Gamma_1]^{-1}$. Now if we have merely $W(a) - W_o(a) \geqq 0$, for $\varepsilon > 0$ let $W(t; \varepsilon)$ be the solution of (2.1) satisfying the initial condition $W(a; \varepsilon) = W(a) + \varepsilon E$. Then by the result just established the interval of existence of $W(t; \varepsilon)$ includes $[a, b]$, and $W(t; \varepsilon) > W_o(t)$ for $\varepsilon > 0$, $t \in [a, b]$. Indeed, if $0 < \varepsilon_1 < \varepsilon_2$, then application of this result with $W_o(t)$ replaced by $W(t; \varepsilon_1)$ and $W(t; \varepsilon)$ replaced by $W(t; \varepsilon_2)$ yields the result that $W_o(t) < W(t; \varepsilon_1) < W(t; \varepsilon_2)$ for $0 < \varepsilon_1 < \varepsilon_2$ and $t \in [a, b]$. From this boundedness condition it then follows that for $\varepsilon_o > 0$ the family of solutions $W(t; \varepsilon), 0 < \varepsilon \leqq \varepsilon_o$, $t \in [a, b]$, of (2.1) is uniformly bounded and equi-continuous. Consequently, by the Ascoli theorem there is a monotone decreasing sequence $\{\varepsilon_n\}$ converging to zero, and such that $\{W(t; \varepsilon_n)\}$ converges uniformly on $[a, b]$ to a limit matrix function $\hat{W}(t)$, which is such that $\hat{W}(t) \geqq W_o(t)$ for $t \in [a, b]$, and by a classical argument $W = \hat{W}(t)$ is a solution of (2.1) on $[a, b]$. Since $\hat{W}(a) = W(a)$ it then follows that $W(t) = \hat{W}(t)$ for $t \in [a, b]$, so that the interval of existence of $W(t)$ includes $[a, b]$ and $W(t) \geqq W_o(t)$ for $t \in [a, b]$.

Now (2.1) may also be written as

$$W' = (E; W)^*\mathscr{A}(t)(E; W) + (W^* - W)[A(t) + B(t)W].$$

In particular, if $W(t)$ is an Hermitian solution of this equation on a subinterval $I_o$, then $W' = (E; W)^*\mathscr{A}(t)(E; W)$ and

$$W(t) = W(s) + \int_s^t (E; W(r))^*\mathscr{A}(r)(E; W(r))\,dr \quad \text{for } (s, t) \in I_o \times I_o. \tag{2.9}$$

Also, if $W(t)$ is an Hermitian solution on $I_o$ and $Y(t) = (U(t); V(t))$ is a conjoined basis for (2.2) such that $W(t) = V(t)U^{-1}(t)$, then $U^*(t)W'(t)U(t) = Y^*(t)\mathscr{A}(t)Y(t)$ for $t \in I_o$. Moreover, with the aid of (2.2′) it is immediate that $Y^*\mathscr{A}Y' + Y^{*\prime}\mathscr{A}Y = Y^*\mathscr{A}[\mathscr{J}\mathscr{A}Y] + [-Y^*\mathscr{A}\mathscr{J}]\mathscr{A}Y = 0$. Therefore, if $\mathscr{A}(t)$ is locally a.c. on $I_o$ then $U^*(t)W'(t)U(t)$ is also locally a.c. on this interval and $[U^*(t)W'(t)U(t)]' = Y^*(t)\mathscr{A}'(t)Y(t)$ for $t$ a.e. on $I_o$, so that

$$W'(t) = U^{*-1}(t)[U^*(s)W'(s)U(s) + \int_s^t Y^*(r)\mathscr{A}'(r)Y(r)\,dr]U^{-1}(t) \tag{2.10}$$

for $(s, t) \in I_o \times I_o$.

In particular, (2.9) and (2.10) imply the following results.

THEOREM 2.1. *If $W(t)$ is an Hermitian solution of* (2.1) *on a subinterval $I_o$ of $I$, and $Y(t) = (U(t); V(t))$ is a conjoined basis for* (2.2) *such that $W(t) = V(t)U^{-1}(t)$, then*:

(a) *if $\mathscr{A}(t) \geqq 0$ ($\mathscr{A}(t) \leqq 0$) for $t$ a.e. on $I_o$, then $W(t)$ is a nondecreasing (nonincreasing) Hermitian matrix function on $I_o$*;

(b) *if $\mathscr{A}(t)$ is locally a.c. on $I_o$ with $\mathscr{A}'(t) \geqq 0$ ($\mathscr{A}'(t) \leqq 0$) for $t$ a.e. on $I_0$, then $U^*(t)W'(t)U(t)$ is a nondecreasing (nonincreasing) Hermitian matrix function on $I_o$; in particular, if $s \in I_o$ and*

$$W'(s) = -W(s)A(s) - A^*(s)W(s) - W(s)B(s)W(s) + C(s)$$

*satisfies $W'(s) \geqq 0$ ($W'(s) \leqq 0$), then $W(t)$ is a nondecreasing (nonincreasing) Hermitian matrix function on $I_o^+(s) = \{t | t \in I_o, t \geqq s\}$.*

For the special important case in which $A(t) \equiv 0$ and $B(t)$ is positive definite the system (2.2) is equivalent to the second order differential equation $[R(t)u'(t)]' - C(t)u(t) = 0$, where $R(t) = B^{-1}(t) > 0$. In this instance $\mathscr{A}(t) = \operatorname{diag}\{C(t); -R^{-1}(t)\}$ and $\mathscr{A}(t) \leqq 0$ if and only if $C(t) \leqq 0$; moreover, if $B(t)$ and $C(t)$ are locally a.c. then $\mathscr{A}'(t) \geqq 0$ ($\mathscr{A}'(t) \leqq 0$) if and only if $C'(t) \geqq 0$ and $R'(t) \geqq 0$ ($C'(t) \leqq 0$ and $R'(t) \leqq 0$).

For a nondegenerate subinterval $I_o$ of $I$, let $\Lambda(I_o)$ denote the linear space of $n$-dimensional vector functions $v(t)$ which are solutions of $v'(t) + A^*(t)v(t) = 0$, and $B(t)v(t) = 0$ for $t \in I_o$; clearly $v \in \Lambda(I_o)$ if and only if $u(t) \equiv 0$, $v(t)$ is a solution of (2.2) on $I_o$. If $\Lambda(I_o)$ is zero-dimensional then (2.2) is said to be *normal on* $I_o$, or to have *abnormality of order zero* on $I_o$, whereas if $\Lambda(I_o)$ has dimension $d = d(I_o) > 0$ the system (2.2) is said to be *abnormal*, with *order of abnormality* $d$ on $I_o$. A system (2.2) is said to be *identically normal* if it is normal on every nondegenerate subinterval $I_o$ of $I$. If $I_o = [r, s]$, for brevity we write $d[r, s]$ instead of the more precise $d([r, s])$, with similar contractions in case $I_o$ is of the form $[r, s), (r, s]$ or $(r, s)$. For $I_o$ a subinterval of $I$, clearly, $0 \leqq d(I_o) \leqq n$. Moreover, if $s \in I$ then $d[s, t]$ is an integral-valued monotone nonincreasing function on $\{t | t \in I, t > s\}$ with at most $n$ points of discontinuity, at each of which $d[s,t]$ is left-hand continuous. In particular, if $[r, s] \subset I$, $L_2[\eta, \zeta](t) = 0$ for $t \in [r, s]$ with $\zeta(t) \in \mathscr{L}_n^\infty[r, s]$ and $v \in \Lambda[r, s]$, then $[v^*(t)\eta(t)]' = 0$ on $[r, s]$, and hence $v^*(t)\eta(t)$ is constant on this interval.

The preceding discussion clearly gives preferential treatment to one of the component vector functions $u(t)$, $v(t)$, and this is to be expected in view of the individual roles assumed by these vector functions in such applications as the canonical accessory differential equations for a variational problem (see, for example, Bliss [1, Chaps. III, IV]). From the formal point of view, however, one may interchange the roles of $u(t)$ and $v(t)$, leading to the differential system

$$\mathscr{J}\tilde{y}'(t) + \tilde{\mathscr{A}}(t)\tilde{y}(t) = 0, \qquad t \in I, \tag{2.11}$$

in $\tilde{y}(t) = (\tilde{u}(t); \tilde{v}(t))$, where

$$\tilde{\mathscr{A}}(t) = \begin{bmatrix} \tilde{C}(t) & -\tilde{A}^*(t) \\ -\tilde{A}(t) & -\tilde{B}(t) \end{bmatrix},$$

with

$$\tilde{A}(t) = -A^*(t), \qquad \tilde{B}(t) = C(t), \qquad \tilde{C}(t) = B(t). \tag{2.12}$$

For brevity, (2.11) will be referred to as the *system obverse to* (2.2). Clearly $(\tilde{u}(t); \tilde{v}(t))$ is a solution of (2.11) if and only if $(u(t); v(t)) = (\tilde{v}(t); \tilde{u}(t))$ is a solution of (2.2). The Riccati matrix differential equation related to (2.11) in the manner that (2.1) is related to (2.2) is given by

$$\tilde{K}[\tilde{W}] \equiv \tilde{W}' + \tilde{W}\tilde{A}(t) + \tilde{A}^*(t)\tilde{W} + \tilde{W}\tilde{B}(t)\tilde{W} - \tilde{C}(t). \tag{2.13}$$

If $W(t)$ is a nonsingular solution of (2.1) on a subinterval $[a, b]$ of $I$, then $\tilde{W}(t) = W^{-1}(t)$ is a nonsingular solution of (2.13) on this subinterval; in particular, $\tilde{W}(t)$ is positive definite if $W(t)$ is positive definite.

**3. Transformations for (2.2) and (2.1).** If $T(t)$ is an $n \times n$ matrix function which is nonsingular and locally a.c. on $I$, then under the transformation

$$u(t) = T(t)u^\circ(t), \qquad v(t) = T^{*-1}(t)v^\circ(t), \tag{3.1}$$

the linear differential system (2.2) is equivalent to the system

$$\begin{aligned} L_1^\circ[u^\circ, v^\circ](t) &\equiv -v^{\circ\prime}(t) + C^\circ(t)u^\circ(t) - A^{\circ*}(t)v^\circ(t) = 0, \qquad t \in I, \\ L_2^\circ[u^\circ, v^\circ](t) &\equiv u^{\circ\prime}(t) - A^\circ(t)u^\circ(t) - B^\circ(t)v^\circ(t) = 0, \qquad t \in I, \end{aligned} \tag{2.2°}$$

where the matrix functions $A^\circ(t)$, $B^\circ(t)$, $C^\circ(t)$ are defined as

$$A^\circ = T^{-1}[AT - T'], \qquad B^\circ = T^{-1}BT^{*-1}, \qquad C^\circ = T^*CT. \tag{3.2}$$

If $(u_\alpha(t); v_\alpha(t))$, $\alpha = 1, 2$, are solutions of (2.2), and $(u_\alpha^\circ(t); v_\alpha^\circ(t))$ are the associated solutions of (2.2°) given by the corresponding equations (3.1), then it follows readily that $\{u_1; v_1|u_2; v_2\}(t) \equiv \{u_1^\circ; v_1^\circ|u_2^\circ; v_2^\circ\}(t)$; in particular, $(u_1; v_1)$ and $(u_2; v_2)$ are conjoined solutions of (2.2) if and only if the corresponding $(u_1^\circ; v_1^\circ)$ and $(u_2^\circ; v_2^\circ)$ are conjoined solutions of (2.2°).

Corresponding to (2.2°) we have the matrix differential system

$$\begin{aligned} L_1^\circ[U^\circ, V^\circ](t) &\equiv -V^{\circ\prime}(t) + C^\circ(t)U^\circ(t) - A^{\circ*}(t)V^\circ(t) = 0, \qquad t \in I, \\ L_2^\circ[U^\circ, V^\circ](t) &\equiv U^{\circ\prime}(t) - A^\circ(t)U^\circ(t) - B^\circ(t)V^\circ(t) = 0, \qquad t \in I. \end{aligned} \tag{$2.2^\circ_M$}$$

Now $Y(t) = (U(t); V(t))$ is a conjoined basis for (2.2) if and only if the corresponding $Y^\circ(t) = (U^\circ(t); V^\circ(t)) = (T^{-1}(t)U(t); T^*(t)V(t))$ is a conjoined basis for (2.2°). Moreover, if $W(t)$ is a solution of (2.1) and $Y(t) = (U(t); V(t))$ is an associated solution of $(2.2_M)$ such that $W(t) = V(t)U^{-1}(t)$, then for $Y^\circ(t) = (U^\circ(t); V^\circ(t)) = (T^{-1}(t)U(t), T^*(t)V(t))$, the associated solution of $(2.2^\circ_M)$, we have that $W^\circ(t) = V^\circ(t)U^{\circ-1}(t) = T^*(t)W(t)T(t)$ is a solution of the Riccati matrix differential equation

$$K^\circ[W^\circ] \equiv W^{\circ\prime} + W^\circ A^\circ(t) + A^{\circ*}(t)W^\circ + W^\circ B^\circ(t)W^\circ - C^\circ(t) = 0. \tag{2.1°}$$

Also, $W(t)$ is an Hermitian solution of (2.1) if and only if the associated $W^\circ(t) = T^*(t)W(t)T(t)$ is an Hermitian solution of (2.1°).

Corresponding to (2.3), for a solution $W^\circ = W_\circ^\circ(t)$ of (2.1°) we now have the system

$$\begin{aligned} G^{\circ\prime} + (A^{\circ*} + W_\circ^\circ B^\circ)G^\circ &= 0, \qquad G^\circ(s) = E, \\ H^{\circ\prime} + H^\circ(A^\circ + B^\circ W_\circ^\circ) &= 0, \qquad H^\circ(s) = E, \end{aligned} \tag{2.3°}$$

in $G^\circ(t) = G^\circ(t, s| W^\circ_\circ)$, $H^\circ(t) = H^\circ(t, s| W^\circ_\circ)$, and the associated

$$(2.4^\circ) \qquad \Theta^\circ(t, s| W^\circ_\circ) = \int_s^t H^\circ(r, s| W^\circ_\circ) B^\circ(r) G^\circ(r, s| W^\circ_\circ)\, dr.$$

With $W_\circ(t)$ the solution of (2.1) such that $W^\circ_\circ(t) = T^*(t) W_\circ(t) T(t)$, it may be verified readily that

$$(3.3a) \qquad H^\circ(t, s| W^\circ_\circ) = T^{-1}(s) H(t, s| W_\circ) T(t),$$

$$(3.3b) \qquad G^\circ(t, s| W^\circ_\circ) = T^*(t) G(t, s| W_\circ) T^{*-1}(s),$$

$$(3.3c) \qquad \Theta^\circ(t, s| W^\circ_\circ) = T^{-1}(s) \Theta(t, s| W_\circ) T^{*-1}(s).$$

If $Z(t)$ is a fundamental matrix solution of $Z' + A^*(t)Z = 0$, then $T(t) = Z^{*-1}(t)$ is a fundamental matrix solution of $T' - A(t)T = 0$, and with this choice of $T(t)$ the matrices of (3.2) are given by

$$(3.4) \qquad A^\circ = 0, \qquad B^\circ = Z^* B Z, \qquad C^\circ = Z^{-1} C Z^{*-1}.$$

For brevity, such a $T(t)$ will be referred to as a *reducing transformation* for (2.2), and the resulting system (2.2°) as a *reduced system.*

In particular, in terms of the matrix functions $B^\circ(t)$, $C^\circ(t)$ defined by (3.4) one has the following result.

LEMMA 3.1. *If $B(t) \geqq 0$ for $t$ a.e. on $I$, and $[a, b]$ is a compact subinterval of $I$, then* (2.2) *is normal on $[a, b]$ if and only if*

$$(3.5) \qquad \int_a^b B^\circ(t)\, dt = \int_a^b Z^*(t) B(t) Z(t)\, dt > 0.$$

*Correspondingly, if $C(t) \geqq 0$ for $t$ a.e. on $I$, then the obverse system* (2.12) *is normal on $[a, b]$ if and only if*

$$(3.6) \qquad \int_a^b C^\circ(t)\, dt = \int_a^b Z^{-1}(t) C(t) Z^{*-1}(t)\, dt > 0.$$

If (2.1) has order of abnormality equal to $d$ on a subinterval $I_\circ$ of $I$, let the fundamental matrix solution $Z(t)$ of $Z' + A^*(t)Z = 0$ be chosen such that the last $d$ column vectors of $Z(t)$ form a basis for $\Lambda(I_\circ)$. Then $T(t) = Z^{*-1}(t)$ is such that the resulting matrix function $B^\circ(t)$ of (3.4) is of the form $B^\circ(t) = \operatorname{diag}\{\hat{B}(t); 0\}$, where $\hat{B}(t)$ is an $(n - d) \times (n - d)$ Hermitian matrix function. For brevity, such a choice of $T(t)$ will be referred to as a *preferred reducing transformation* for (2.2).

In particular, if $C^\circ(t) = T^*(t) C(t) T(t)$ is written as

$$(3.7) \qquad C^\circ(t) = \begin{bmatrix} \hat{C}_{11}(t) & \hat{C}_{12}(t) \\ \hat{C}_{21}(t) & \hat{C}_{22}(t) \end{bmatrix}$$

where $\hat{C}_{11}(t) = \hat{C}^*_{11}(t)$ is $(n - d) \times (n - d)$, $\hat{C}_{12}(t) = \hat{C}^*_{21}(t)$ is $(n - d) \times d$ and $\hat{C}_{22}(t) = \hat{C}^*_{22}(t)$ is $d \times d$, then in terms of the vector functions $\eta(t) = (u_\alpha(t))$, $\zeta(t) = (v_\alpha(t))$, $\alpha = 1, \cdots, n - d$, and $\rho(t) = (u_{n-d+\beta}(t))$, $\sigma(t) = (v_{n-d+\beta}(t))$, $\beta = 1$,

$\cdots$, $d$, the vector differential system (2.2) becomes

$$\begin{aligned} \eta'(t) &= \hat{B}(t)\zeta(t), \\ \rho'(t) &= 0, \\ \zeta'(t) &= \hat{C}_{11}(t)\eta(t) + \hat{C}_{12}(t)\rho(t), \\ \sigma'(t) &= \hat{C}_{21}(t)\eta(t) + \hat{C}_{22}(t)\rho(t). \end{aligned} \qquad t \in I, \tag{3.8}$$

Moreover, $t_1$ and $t_2$ are conjugate points with respect to (2.2) if and only if these points are conjugate with respect to the *truncated preferred reduced system*

$$\begin{aligned} \eta'(t) &= \hat{B}(t)\zeta(t), \qquad t \in I, \\ \zeta'(t) &= \hat{C}_{11}(t)\eta(t), \qquad t \in I. \end{aligned} \tag{3.9}$$

Corresponding to (3.9) one has the truncated preferred reduced matrix system

$$\begin{aligned} H'(t) &= \hat{B}(t)Z(t), \qquad t \in I, \\ Z'(t) &= \hat{C}_{11}(t)H(t), \qquad t \in I, \end{aligned} \tag{3.9$_M$}$$

and the truncated preferred reduced Riccati matrix differential equation

$$\Omega' + \Omega\hat{B}(t)\Omega - \hat{C}_{11}(t) = 0, \qquad t \in I. \tag{3.10}$$

Indeed, for $A^\circ(t) = 0$, $B^\circ(t) = \operatorname{diag}\{\hat{B}(t); 0\}$, and $C^\circ(t)$ given by (3.7), upon writing $W^\circ(t)$ as the corresponding partitioned matrix

$$W^\circ(t) = \begin{bmatrix} W^\circ_{11}(t) & W^\circ_{12}(t) \\ W^\circ_{21}(t) & W^\circ_{22}(t) \end{bmatrix}$$

the Riccati matrix differential equation (2.1°) may be written as the system

$$W^{\circ\prime}_{\alpha\beta} + W^\circ_{\alpha 1}\hat{B}(t)W^\circ_{1\beta} - \hat{C}_{\alpha\beta}(t), \qquad t \in I, \quad \alpha, \beta = 1, 2. \tag{3.11}$$

Clearly the interval of existence of $W^\circ(t)$ is that determined by the equation in $W^\circ_{11}(t)$ given by $\alpha = 1$, $\beta = 1$ in (3.11), which is the truncated preferred reduced Riccati matrix differential equation (3.10). With the value of $W^\circ_{11}(t)$ determined the matrix functions $W^\circ_{12}(t)$ and $W^\circ_{21}(t)$ are solutions of related linear matrix differential equations, and $W^\circ_{22}(t)$ is obtained by integration.

Now since $d$ is the order of abnormality of (2.2) on $I_\circ$, it follows that if $\xi$ is a nonnull $(n - d)$-dimensional vector then $\hat{B}(t)\xi$ is not the null vector throughout $I_\circ$. Moreover, when $B(t) \geqq 0$ for $t$ a.e. on $I$ we have that correspondingly $\hat{B}(t) \geqq 0$ for $t$ a.e. on $I_\circ$; in particular, if $I_\circ$ is the compact subinterval $[a, b]$ then $\int_a^b \hat{B}(t)\,dt > 0$. In the special important case wherein (2.2) has the same order of abnormality on all nondegenerate subintervals of $I$, the corresponding truncated preferred reduced system (3.9) is identically normal on $I$.

**4. Comparison theorems.** Two distinct points $t_1$ and $t_2$ on $I$ are said to be (*mutually*) *conjugate* with respect to (2.2) if there exists a solution $y(t) = (u(t); v(t))$ of this differential system with $u(t) \not\equiv 0$ on the subinterval with endpoints $t_1$ and

$t_2$, while $u(t_1) = 0 = u(t_2)$. The system is called *disconjugate* on a subinterval $I_0$ of $I$ provided no two distinct points of this subinterval are conjugate. Moreover, (2.2) is said to be *disconjugate for* large $t$ if there exists a subinterval $(c, \infty)$ of $I$ on which this system is disconjugate.

For $[a, b] \subset I$, the symbol $\mathscr{D}[a, b]$ will denote the linear space of $n$-dimensional vector functions $\eta(t)$ which are a.c. on $[a, b]$, and for which there exists a corresponding $\zeta(t) \in \mathscr{L}_n^\infty[a, b]$ such that $\eta'(t) - A(t)\eta(t) = B(t)\zeta(t)$ on $[a, b]$. The subspace of $\mathscr{D}[a, b]$ on which $\eta(a) = 0 = \eta(b)$ will be designated by $\mathscr{D}_0[a, b]$. The fact that $\eta(t)$ belongs to $\mathscr{D}[a, b]$ or $\mathscr{D}_0[a, b]$ with an associated $\zeta(t)$ will be indicated by the respective symbol $\eta \in \mathscr{D}[a, b] : \zeta$ or $\eta \in \mathscr{D}_0[a, b] : \zeta$. For $[a, b] \subset I$ and $\eta \in \mathscr{D}[a, b] : \zeta$ we shall denote by $J[\eta ; a, b]$ the functional

$$J[\eta ; a, b] = \int_a^b \{\zeta^*(t)B(t)\zeta(t) + \eta^*(t)C(t)\eta(t)\}\, dt. \tag{4.1}$$

It is to be noted that if $\eta \in \mathscr{D}[a, b] : \zeta_1$ and $\eta \in \mathscr{D}[a, b] : \zeta_2$ then $B(t)\zeta_1(t) = B(t)\zeta_2(t)$ on $[a, b]$, and the value of the integral in (4.1) is independent of the choice of the corresponding $\zeta(t)$.

The basic results concerning disconjugacy on a compact subinterval $[a, b]$ of $I$, positive definiteness of the functional (4.1) on $\mathscr{D}_0[a, b]$, and the existence of Hermitian solutions of the Riccati matrix differential equation (2.1), are given in the following theorem.

THEOREM 4.1. *For $[a, b] \subset I$, the functional $J[\eta ; a, b]$ is positive definite on $\mathscr{D}_0[a, b]$ if and only if $B(t) \geqq 0$ for $t$ a.e. on $[a, b]$, and one of the following conditions holds*:

(a) (2.2) *is disconjugate on $[a, b]$*;

(b) *there exists no point $s \in (a, b]$ which is conjugate to $t = a$*;

(c) *there exists a conjoined basis $Y(t) = (U(t); V(t))$ for* (2.2) *with $U(t)$ nonsingular on $[a, b]$*;

(d) *there exists on $[a, b]$ an Hermitian solution $W(t)$ of* (2.1).

Moreover, in view of the comparison results that are immediate consequences of the above theorem, one has the following additional results involving differential inequalities.

COROLLARY. *If $[a, b] \subset I$, and $B(t) \geqq 0$ for $t$ a.e. on $[a, b]$, then* (2.2) *is disconjugate on $[a, b]$ if and only if one of the following conditions holds*:

(i) *there exists on $[a, b]$ a nonsingular $n \times n$ matrix function $U(t)$ such that $U \in \mathscr{D}[a, b] : V$ with an a.c. matrix function $V(t)$, while $\{U ; V | U ; V\}(t) \equiv 0$ and $U^*(t)L_1[U, V](t) \geqq 0$ for $t$ a.e. on $[a, b]$*;

(ii) *there exists an $n \times n$ Hermitian matrix function $W(t)$ which is a.c. and satisfies $K[W](t) \leqq 0$ for $t$ a.e. on $[a, b]$.*

If (2.2) is identically normal on $I$, then the proof of the results of Theorem 4.1 and its corollary are particularly simple, and can be established by methods which are essentially classical for the second order matrix differential equation to which (2.2) is equivalent when $A(t)$, $B(t)$, $C(t)$ are continuous on $I$ and $B(t)$ is nonsingular. For a discussion of this case the reader is referred to Reid [6, § 2]; for the relation of such differential systems to problems of the calculus of variations, see also Bliss [1, §§ 89–91]. For the proof of the above results when no assumptions of normality are made, reference is made to Reid [9, Theorem 5.1] and [11, Theorem 5.1].

For a given subinterval $[a, b]$ of $I$, let $\mathscr{D}_{*0}[a, b]$ denote the subspace of $\mathscr{D}[a, b]$ on which $\eta(b) = 0$. The fundamental relation between the existence of an Hermitian solution of (2.1) on $[a, b]$ and the extremum of an associated quadratic functional is presented in the following theorem (see Reid [11, Theorem 5.5]).

THEOREM 4.2. *If* $[a, b] \subset I$ *and* $Q$ *is a given* $n \times n$ *Hermitian matrix then the functional*

$$J_Q[\eta; a, b] = \eta^*(a)Q\eta(a) + \int_a^b \{\zeta^*(t)B(t)\zeta(t) + \eta^*(t)C(t)\eta(t)\}\, dt \tag{4.2}$$

*is positive definite on* $\mathscr{D}_{*0}[a, b]$ *if and only if* $B(t) \geqq 0$ *for* $t$ *a.e. on* $[a, b]$, *and one of the following conditions holds*:

(a) *if* $Y(t) = (U(t); V(t))$ *is the solution of* $(2.2_M)$ *satisfying the initial condition* $Y(a) = (E; Q)$, *then* $U(t)$ *is nonsingular on* $[a, b]$;

(b) *the Hermitian solution* $W(t)$ *of* (2.1) *determined by the initial condition* $W(a) = Q$ *exists on* $[a, b]$;

(c) *there exists an* $n \times n$ *Hermitian matrix function* $W(t)$ *which is a.c. on* $[a, b]$ *and satisfies the conditions*

$$Q \geqq W(a), \qquad K[W](t) \leqq 0 \quad \textit{for } t \in [a, b]. \tag{4.3}$$

COROLLARY. *If* $B(t) \geqq 0$ *and* $C(t) \geqq 0$ *for* $t$ *a.e. on a subinterval* $[a, b]$ *of* $I$ *and* $Q > 0$ $(Q \geqq 0)$ *then the Hermitian solution* $W(t)$ *of* (2.1) *determined by the initial condition* $W(a) = Q$ *exists on* $[a, b]$, *and* $W(t) > 0$ $(W(t) \geqq 0)$ *for* $t \in [a, b]$.

Let $Y(t) = (U(t); V(t))$ be the solution of $(2.2_M)$ satisfying the initial condition $Y(a) = (E; Q)$, and consider first the case of $Q > 0$. If $c$ is a value on $(a, b]$, and $\pi$ is an $n$-dimensional vector such that either $U(c)\pi = 0$ or $V(c)\pi = 0$, let $(u(t); v(t)) = (U(t)\pi; V(t)\pi)$. Then $(u(t); v(t))$ is a solution of (2.2), with either $u(c) = 0$ or $v(c) = 0$, $v(a) - Qu(a) = 0$, and

$$\begin{aligned} J_Q[u; a, c] &= u^*(a)Qu(a) + \int_a^c \{v^*B(t)v + u^*C(t)u\}\, dt \\ &= u^*(a)Qu(a) + u^*(t)v(t)\Big|_a^c \\ &= u^*(a)[Qu(a) - v(a)] = 0. \end{aligned}$$

Since $Q > 0$ and $B(t) \geqq 0$, $C(t) \geqq 0$ for $t$ a.e. on $[a, c]$, it follows that $u(a) = 0$, $B(t)v(t) = 0$ and $C(t)u(t) = 0$ on $[a, c]$. In particular $0 = u(a) = \pi$, thus showing that $U(t)$ and $V(t)$ are both nonsingular for $t \in (a, b]$. Consequently, $W(t) = V(t)U^{-1}(t)$ is a nonsingular Hermitian matrix on $[a, b]$ so that for $t \in [a, b]$ all proper values of $W(t)$ are different from zero. As $W(a) = Q > 0$, all proper values of $W(a)$ are positive, and hence by continuity throughout $[a, b]$ all proper values of $W(t)$ are positive, and $W(t)$ is positive definite.

If we have merely $Q \geqq 0$, for $\varepsilon > 0$ let $Q_\varepsilon = Q + \varepsilon E$ and $W_\varepsilon(t)$ be the solution of (2.1) satisfying $W_\varepsilon(a) = Q_\varepsilon$. By the result just established the solution $W_\varepsilon(t)$ exists on $[a, b]$ and $W_\varepsilon(t) > 0$ for $t \in [a, b]$. Combining this result with that of the corollary to Lemma 2.3 we have that if $0 < \varepsilon_1 < \varepsilon_2$ then $0 < W_{\varepsilon_1}(t) < W_{\varepsilon_2}(t)$ for $t \in [a, b]$, and by argument similar to that suggested for the corollary to Lemma 2.3 it follows that if $W(t)$ is the solution of (2.1) satisfying the initial condition $W(a) = Q$ then $W(t)$ exists on $[a, b]$ and $W_\varepsilon(t) \to W(t)$ as $\varepsilon \to 0$, so that also $W(t) \geqq 0$ for $t \in [a, b]$.

With the aid of the result of the above corollary, one has a ready proof of the following comparison theorem.

THEOREM 4.3. *Suppose that $B(t) \geqq 0$ for $t$ a.e. on a subinterval $[a, b]$ of $I$, and that $W = W_\circ(t)$ is an Hermitian solution of* (2.1) *on this subinterval. If $C_1(t) \in \mathscr{L}_{nn}[a, b]$ with $C_1(t) \geqq C(t)$ for $t$ a.e. on $[a, b]$, and $W = W_1(t)$ is a solution of the Riccati matrix differential equation*

$$(4.4) \qquad K_1[W_1] \equiv W_1' + W_1A(t) + A^*(t)W_1 + W_1B(t)W_1 - C_1(t) = 0$$

*with $W_1(a) > W_\circ(a)$ ($W_1(a) \geqq W_\circ(a)$) then $W_1(t)$ exists on $[a, b]$ and $W_1(t) > W_\circ(t)$ ($W_1(t) \geqq W_\circ(t)$) for $t \in [a, b]$.*

If we set $W_1(t) = W_\circ(t) + W_2(t)$, then $K_1[W_1] - K[W_\circ] = K_2[W_2]$, where $K_2[W_2] = W_2' + W_2A_2(t) + A_2^*(t)W_2 + W_2B_2(t)W_2 - C_2(t)$, with $A_2(t) = A(t) + B(t)W_\circ(t)$, $B_2(t) = B(t) \geqq 0$ and $C_2(t) = C_1(t) - C_\circ(t) \geqq 0$ for $t \in [a, b]$, while $W_2(a) = W_1(a) - W_\circ(a) \geqq 0$, and the conclusion of the theorem is a ready consequence of the result of the above corollary to Theorem 4.2.

Now if $W(t)$ is a nonsingular solution of (2.1) on a subinterval $[a, b]$ of $I$, then $\tilde{W}(t) = W^{-1}(t)$ is a nonsingular solution on this subinterval of the related Riccati matrix differential equation (2.13) for the system obverse to (2.2); moreover, $\tilde{W}(t)$ is positive definite if $W(t)$ is positive definite. Consequently, if the conditions of $\mathfrak{H}$ hold for (2.1) and $C(t) \geqq 0$ for $t$ a.e. on $I_\circ$, one has for (2.13) a comparison theorem corresponding to that of Theorem 4.3 for (2.1). In particular, this result interpreted in terms of the original equation (2.1) yields the following result.

COROLLARY. *Suppose that $C(t) \geqq 0$ for $t$ a.e. on a subinterval $[a, b]$ of $I$, and that $W = W_\circ(t)$ is a positive definite Hermitian solution of* (2.1) *on this subinterval. If $B_2(t) \in \mathscr{L}_{nn}[a, b]$ with $B_2(t) \geqq B(t)$ for $t$ a.e. on $[a, b]$, and $W = W_2(t)$ is a solution of the Riccati matrix differential equation*

$$K_2[W_2] \equiv W_2' + W_2A(t) + A^*(t)W_2 + W_2B_2(t)W_2 - C(t) = 0$$

*with $0 < W_2(a) < W_\circ(a)$ ($0 < W_2(a) \leqq W_\circ(a)$), then $W_2(t)$ exists on $[a, b]$ and $0 < W_2(t) < W_\circ(t)$ ($0 < W_2(t) \leqq W_\circ(t)$) for $t \in [a, b]$.*

For $I_\circ$ a subinterval of $I$ we shall denote by $\mathfrak{H}_2(I_\circ)$ the following condition:

$\mathfrak{H}_2(I_\circ)$ The matrix functions $A(t)$, $B(t)$, $C(t)$ satisfy $\mathfrak{H}$, and $B(t) \geqq 0$, $C(t) \geqq 0$ for $t$ a.e. on $I_\circ$.

The following comparison theorem is a consequence of the corollary to Theorem 4.2, and the combined results of Theorem 4.3 and its corollary, together with a limit argument similar to that occurring in the proof of the corollary to Theorem 4.2, to treat the particular case in which we have $W_\circ(a) \geqq 0$, but do not have $W_\circ(a) > 0$.

THEOREM 4.4. *Suppose that hypothesis $\mathfrak{H}_2[a, b]$ holds for a subinterval $[a, b]$ of $I$, and that $B_3(t) \in \mathscr{L}_{nn}[a, b]$, $C_3(t) \in \mathscr{L}_{nn}[a, b]$ with $B_3(t) \geqq B(t)$ and $C_3(t) \geqq C(t)$ for $t$ a.e. on $[a, b]$. If $W_\circ(t)$ is a solution of* (2.1) *on $[a, b]$ with $W_\circ(a) \geqq 0$, and $W_3(t)$ is a solution of*

$$(4.5) \qquad K_3[W_3] \equiv W_3' + W_3A(t) + A^*(t)W_3 + W_3B_3(t)W_3 - C_3(t) = 0$$

*satisfying* $W_3(a) > W_o(a)$ $(W_3(a) \geqq W_o(a))$, *then* $W_3(t)$ *exists on* $[a, b]$ *and* $W_3(t) > W_o(t)(W_3(t) \geqq W_o(t))$ *for* $t \in [a, b]$.

**5. A class of monotone matrix differential equations.** For brevity, let $\mathfrak{M}_n$ denote the class of $n \times n$ complex-valued matrices, and $\mathfrak{M}_n^+$ the subclass of $\mathfrak{M}_n$ consisting of the nonnegative definite Hermitian matrices. In the following we shall be concerned with a matrix differential equation

$$(5.1) \qquad W' + WA(t) + A^*(t)W + WB(t)W - C(t) - F(t, W) = 0,$$

where $F(t, W)$ is a function on $I \times \mathfrak{M}_n$ to $\mathfrak{M}_n$ which possesses the following properties:

$\mathfrak{H}_1$ (a) $F$ is continuous in $W$ for fixed $t \in I$;
(b) $F$ is Lebesgue integrable on compact subintervals $[a, b]$ of $I$ for fixed $W \in \mathfrak{M}_n$;
(c) If $W \in \mathfrak{M}_n^+$ then $F(t, W) \in \mathfrak{M}_n^+$ for $t \in I$.

In particular, if $W(t)$ is a continuous Hermitian matrix function with $W(t) \in \mathfrak{M}_n^+$ for each $t$ on a compact subinterval $[a, b]$ of $I$, then $F(t, W(t)) \in \mathscr{L}_{nn}[a, b]$ and $F(t, W(t)) \geqq 0$ for $t \in [a, b]$. Moreover, throughout this section it will be supposed that hypothesis $\mathfrak{H}_2(I)$ holds so that in addition to the conditions of $\mathfrak{H}$ we have that $B(t) \geqq 0$ and $C(t) \geqq 0$ for $t$ a.e. on $I$.

LEMMA 5.1. *Suppose that* $a \in I$, *hypotheses* $\mathfrak{H}_1$-a, b, c *and* $\mathfrak{H}_2(I)$ *hold, and that* $W = W_o(t)$ *is a solution of* (2.1) *with* $W_o(a) \geqq 0$. *If* $W = W(t)$ *is a solution of* (5.1) *on an interval* $[a, c)$ *and* $W(a) > W_o(a)$, *then* $W(t) > W_o(t) \geqq 0$ *for* $t \in [a, c)$.

In view of the corollary to Theorem 4.2 we have that the solution $W = W_o(t)$ of (2.1) exists and satisfies $W_o(t) \geqq 0$ for $t \in I^+(a) = \{t | t \in I, t \geqq a\}$. If $W = W(t)$ is a solution of (5.1) on an interval $[a, c)$ with $W(a) > W_o(a)$ and $W(t) \geqq W_o(t)$ for $t$ on a subinterval $[a, b_1]$ of $[a, c)$, then $C_1(t) = C(t) + F(t, W(t)) \geqq C(t)$ for $t \in [a, b_1]$, and from Theorem 4.3 it follows that $W(t) > W_o(t)$ for $t \in [a, b_1]$. Therefore, $W(t) > W_o(t)$ for all $t \in [a, c)$.

For further considerations it will be supposed that the function $F(t, W)$ satisfies some of the following additional conditions:

$\mathfrak{H}_1$ (d) the solution of (5.1) satisfying given intial data is unique; that is, if $(t^\circ, W^\circ) \in I \times \mathfrak{M}_n$ there is a unique solution $W = W(t; t^\circ, W^\circ)$ of (5.1) such that $W(t^\circ) = W^\circ$;
(e) if $0 \leqq W_1 \leqq W_2$, then $0 \leqq F(t, W_1) \leqq F(t, W_2)$;
(f) there exist nonnegative real-valued functions $\mu_1(t)$, $\mu_2(t)$ which are Lebesgue integrable on arbitrary compact subintervals of $I$, and

$$(5.2) \qquad \nu[F(t, W)] \leqq \mu_1(t) + \mu_2(t)\nu[W] \quad \text{for } (t, W) \in I \times \mathfrak{M}_n.$$

Condition $\mathfrak{H}_1$-e is a rather restrictive condition, which is not satisfied by such a simple function as $F(t, W) = W^2$. On the other hand, all the conditions (a)–(f) of $\mathfrak{H}_1$ are satisfied by such a function as $F(t, W) = F_o(t) + G^*(t)WG(t)$, where $F_o(t) \geqq 0$ for $t$ a.e. on $I$ and $F_o(t) \in \mathscr{L}_{nn}[a, b]$, $G(t) \in \mathscr{L}_{nn}^2[a, b]$ for arbitrary compact subintervals $[a, b]$ of $I$. These conditions also hold for a nonlinear functional such as $F_k(W) = k|W|(E + k|W|)^{-1} = E - (E + k|W|)^{-1}$ for $k$ a positive integer and $|W|$ the nonnegative Hermitian square root matrix of $W^*W$. Indeed, $0 \leqq F_k(W) \leqq E$, and consequently conditions (a)–(f) of $\mathfrak{H}_1$ also hold for $F(t, W)$

$= \sum_{k=1}^{\infty} \phi_k(t)F_k(W)$, where $\phi_k(t)$ are nonnegative Lebesgue measurable functions on $I$ such that $\phi(t) = \sum_{k=1}^{\infty} \phi_k(t)$ is Lebesgue integrable on arbitrary compact subintervals $[a, b]$ of $I$.

THEOREM 5.1. *Suppose that $a \in I$, hypotheses $\mathfrak{H}_1$-a, b, c, d and $\mathfrak{H}_2(I)$ hold, and that $W = W_o(t)$ is a solution of* (2.1) *with $W_o(a) \geqq 0$. If $W = W(t)$ is a solution of* (5.1) *on an interval $[a, c)$ and $W(a) \geqq W_o(a)$, then $W(t) \geqq W_o(t) \geqq 0$ for $t \in [a, c)$.*

For $\varepsilon > 0$, let $W = W(t; \varepsilon)$ be the solution of (5.1) satisfying the initial condition $W(a; \varepsilon) = W(a) + \varepsilon E$. If $[a, c(\varepsilon))$ is the maximal right-hand interval of existence of $W(t; \varepsilon)$, then from Lemma 5.1 we have that $W(t; \varepsilon) > W_o(t)$ for $t \in [a, c(\varepsilon))$. Moreover, from well-known continuity properties of solutions of ordinary differential equations (see, for example, Coddington and Levinson [4, Chap. 2]), we have that if $b_1 \in [a, c)$ then $[a, b_1] \subset [a, c(\varepsilon))$ for $\varepsilon$ sufficiently small, so that $W(t; \varepsilon) \to W(t)$ on $[a, b_1]$ as $\varepsilon \to 0$, and hence $W(t) \geqq W_o(t) \geqq 0$ for $t \in [a, c)$.

THEOREM 5.2. *Suppose that $a \in I$, hypotheses $\mathfrak{H}_1$-a, b, c, d, e and $\mathfrak{H}_2(I)$ hold, and that $W = W_o(t)$ is a solution of* (2.1) *with $W_o(a) \geqq 0$. If $0 \leqq Q_1 \leqq Q_2$ and $W = W_\alpha(t)$, $\alpha = 1, 2$, is the solution of* (5.1) *satisfying the initial condition $W_\alpha(a) = W_o(a) + Q_\alpha$, and with maximal right-hand interval of existence $[a, c_\alpha)$, then $c_2 \leqq c_1$ and*

$$W_2(t) \geqq W_1(t) \geqq W_o(t) \geqq 0 \quad \text{for } t \in [a, c_2). \tag{5.3}$$

*Moreover, if $0 < Q_1$ or $Q_1 < Q_2$, then the respective relation $W_1(t) > W_o(t)$ or $W_2(t) > W_1(t)$ holds for $t \in [a, c_2)$.*

From the corollary to Theorem 4.2 it follows that the solution $W = W_o(t)$ of (2.1) exists and satisfies the condition $W_o(t) \geqq 0$ for $t \in I^+(a) = \{t | t \in I,\ t \geqq a\}$. From the results of Lemma 5.1 and Theorem 5.1 it then follows that if $W = W(t)$ is the solution of (5.1) satisfying $W(a) = W_o(a) + Q$, with $Q \geqq 0$, then the relation $W(t) \geqq W_o(t) \geqq 0$ holds for $t \in [a, c)$, where $[a, c)$ is the right maximal interval of existence of $W(t)$, and $W(t) > W_o(t)$ for $t \in [a, c)$ whenever $Q > 0$. Now suppose that $0 \leqq Q_1 \leqq Q_2$, and $W = W_\alpha(t), \alpha = 1, 2$, is the solution of (5.1) with $W_\alpha(a) = W_o(a) + Q_\alpha$, and denote by $[a, c_\alpha)$ the right maximal interval of existence of $W_\alpha(t)$. Then $\hat{W}(t) = W_2(t) - W_1(t)$ is a solution of the matrix differential system

$$\hat{W}' + \hat{W}\hat{A}(t) + \hat{A}^*(t)\hat{W} + \hat{W}\hat{B}(t)\hat{W} - F_2(t, \hat{W}) = 0, \qquad \hat{W}(a) = Q_2 - Q_1, \tag{5.4}$$

where $\hat{A} = A + BW_1$, $F_2(t, \hat{W}) = F(t, W_1(t) + \hat{W}) - F(t, W_1(t))$. Since the matrix function $F_2(t, \hat{W})$ satisfies conditions $\mathfrak{H}_1$-a, b, c, application of the result of Lemma 5.1 to $W = \hat{W}(t)$ and the solution $\hat{W}_o(t) = 0$ of the corresponding Riccati system

$$\hat{W}_o' + \hat{W}_o\hat{A}(t) + \hat{A}^*(t)\hat{W}_o + \hat{W}_o B(t)\hat{W}_o = 0, \qquad \hat{W}_o(a) = 0, \tag{5.4$_o$}$$

yields the conclusion that $\hat{W}(t) \geqq 0$ for $t \in [a, c_1) \cap [a, c_2)$, and indeed $\hat{W}(t) > 0$ for $t$ on this interval if $\hat{W}(a) = Q_2 - Q_1 > 0$. That is, the conclusions of the theorem have been established for $t \in [a, c_1) \cap [a, c_2)$. Now if $[a, c_1)$ were a proper subinterval of $[a, c_2)$ then $c_1$ would be an interior point of $I$, the relation $0 \leqq W_1(t) \leqq W_2(t)$ would hold on $[a, c_1)$, with $W_2(t)$ continuous and satisfying $0 \leqq W_2(t)$ on $[a, c_2)$, so that $W_1(t)$ would remain bounded as $t \to c_1^-$, a condition which contradicts the assumption that $[a, c_1)$ is the right maximal interval of existence of $W_1(t)$ (see, for example, Coddington and Levinson [4, Chap. 2]). Consequently, we have that $c_1 \geqq c_2$, thus completing the proof of the theorem.

THEOREM 5.3. *Suppose that hypotheses* $\mathfrak{H}_1$-a, b, c, d, e *and* $\mathfrak{H}_2(I)$ *hold, and that* $W = W_o(t)$ *is a solution of* (2.1) *on* $I^+(a)$ *with* $W_o(a) \geqq 0$; *moreover, that* $Q \geqq 0$ *and the solution* $W = W(t)$ *of* (5.1) *satisfying the initial condition* $W(a) = W_o(a) + Q$ *has right maximal interval of existence* $[a, c)$. *If* $\{Q_n\}$ *is a sequence of nonnegative Hermitian matrices such that*

$$0 \leqq Q_1 \leqq Q_2 \leqq \cdots \leqq Q, \quad \text{with } \{Q_m\} \to Q \quad \text{as } m \to \infty, \tag{5.5}$$

*and* $W = W_m(t)$ *is the solution of the differential system*

$$\begin{aligned} W' + WA(t) + A^*(t)W + WB(t)W - C(t) - F(t, W_{m-1}) &= 0, \\ W(a) = W_o(a) + Q_m&, \end{aligned} \tag{5.6.m}$$

for $m = 1, 2, \cdots$, *then*

$$W_o(t) \leqq W_1(t) \leqq \cdots \leqq W_{m-1}(t) \leqq W_m(t) \leqq \cdots \quad \text{for } t \in I^+(a), \tag{5.7a}$$

$$\{W_m(t)\} \to W(t) \quad \text{for } t \in [a, c). \tag{5.7b}$$

From Theorem 5.1 it follows that $0 \leqq W_o(t) \leqq W(t)$ for $t \in [a, c)$. Moreover, since $W_o(t)$ has right maximal interval of existence $I^+(a)$ and $W_o(t) \geqq 0$ on this interval, we have $F(t, W_o(t)) \geqq 0$ for $t \in I^+(a)$, and from Theorem 4.3 it follows that the solution $W = W_1(t)$ of the system (5.6.1) exists on $I^+(a)$ and satisfies $0 \leqq W_o(t) \leqq W_1(t)$. Moreover, since Theorem 5.1 implies that $0 \leqq W_o(t) \leqq W(t)$ for $t \in [a, c)$ we have that $0 \leqq F(t, W_o(t)) \leqq F(t, W(t))$ for $t \in [a, c)$, and as $W(a) = W_o(a) + Q \geqq W_o(a) + Q_1 \geqq 0$ it follows with the aid of Theorem 4.3 that $W_1(t) \leqq W(t)$ for $t \in [a, c)$. By induction it follows that $0 \leqq W_{m-1}(t) \leqq W_m(t)$ for $t \in I^+(a)$ and $W_m(t) \leqq W(t)$ for $t \in [a, c)$. Consequently, $\{W_m(t)\}$ is a monotone nondecreasing sequence of Hermitian matrix functions satisfying $W_m(t) \leqq W(t)$ for $t \in [a, c)$, and hence (see, for example, [12, p. 263]), there exists an Hermitian matrix function $\hat{W}(t)$ such that $\hat{W}(t) \leqq W(t)$ and $\{W_m(t)\} \to \hat{W}(t)$ for $t \in [a, c)$. Now from the differential equation satisfied by $W_m(t)$, and the fact that $0 \leqq F(t, W_{m-1}(t)) \leqq F(t, W(t))$ for $t \in [a, c)$, it follows that if $[a, b]$ is a compact subinterval of $[a, c)$, then the family of matrix functions $\{W_m(t)\}$, $t \in [a, b]$, is uniformly bounded and equi-continuous, and consequently the convergence of $\{W_m(t)\}$ to $\hat{W}(t)$ is uniform on each such $[a, b] \subset [a, c]$. Consequently, $\hat{W}(t)$ is a solution of (5.1) on $[a, c)$ satisfying the initial condition $\hat{W}(a) = W_o(a) + Q$, and hence conclusion (5.7b) holds, thus completing the proof of the theorem.

Finally, we shall establish the following continuation theorem.

THEOREM 5.4. *Suppose that hypotheses* $\mathfrak{H}_1$-a, b, c, d, e, f *and* $\mathfrak{H}_2(I)$ *hold, and that* $W = W_o(t)$ *is a solution of* (2.1) *with* $W_o(a) \geqq 0$. *If* $Q \geqq 0$ *and* $W = W(t)$ *is the solution of* (5.1) *satisfying the initial condition* $W(a) = W_o(a) + Q$, *then the right maximal interval of existence of* $W(t)$ *is* $I^+(a)$.

Since $B(t) \geqq 0$ for $t$ a.e. on $I$ it follows from the differential equation (5.1) that if the right maximal interval of existence of $W(t)$ is $[a, c)$ then

$$0 \leqq W(t) \leqq W(a) + \int_a^t [C(s) - W(s)A(s) - A^*(s)W(s) + F(s, W(s))]\, ds$$

for $t \in [a, c)$, and as $v[C(s)]E \geqq C(s)$, $(\mu_1(s) + \mu_2(s)v[W(s)])E \geqq F(s, W(s))$ and $2v[W(s)]v[A(s)]E \geqq -W(s)A(s) - A^*(s)W(s)$, it follows that

$$v[W(t)] \leqq v[W(a)] + \int_a^t \{\mu_o(s) + \mu_3(s)v[W(s)]\}\, ds \quad \text{for } t \in [a, c),$$

where $\mu_o(t) = \mu_1(t) + v[C(t)]$, $\mu_3(t) = \mu_2(t) + 2v[A(t)]$. By the Gronwall inequality it then follows that

$$v[W(t)] \leqq \left\{ v[W(a)] + \int_a^t \mu_o(s)\, ds \right\} \exp \left\{ \int_a^t \mu_3(s)\, ds \right\} \quad \text{for } t \in [a, c).$$

Consequently, if $c$ were an interior point of $I$ it would follow that $W(t)$ remains bounded as $t \to c^-$, so that $c$ could not be the endpoint of the right maximal interval of existence of $W(t)$, and hence the right maximal interval of existence of $W(t)$ is $I^+(a)$.

**6. Systems disconjugate for large *t*.** When $I$ is an interval of the form $[a, \infty)$, and (2.2) is disconjugate for large $t$, then one has the existence of a *principal solution* $Y_\infty(t) = (U_\infty(t); V_\infty(t))$ of ($2.2_M$) at $\infty$ (see Hartman [5], Reid [7]) and the corresponding *distinguished solution* at $\infty$ of (2.1) given by $W_\infty(t) = V_\infty(t)U_\infty^{-1}(t)$ for $t$ in a neighborhood of $\infty$ (see Sandor [13], Reid [8], [9]). As in the case of the results of Theorems 4.1 and 4.2, these results are relatively easy to establish when (2.2) is identically normal, but considerably more complicated to prove when no assumptions of normality are imposed.

We shall proceed to discuss here two important cases involving assumptions of normality of intermediate strength. For brevity, the notations $\mathfrak{N}^+(I)$ and $\mathfrak{N}^-(I)$ are introduced for the following conditions:

$\mathfrak{N}^+(I)$ $I$ is an open interval, and for $s \in I$ there exists a $b(s) \in I$ such that $s < b(s)$ and (2.2) is normal on $[s, b(s)]$.

$\mathfrak{N}^-(I)$ $I$ is an open interval, and for $s \in I$ there exists an $a(s) \in I$ such that $a(s) < s$ and (2.2) is normal on $[a(s), s]$.

An equivalent formulation of $\mathfrak{N}^+(I)$ is that $d(I^+(s)) = 0$ for arbitrary $s \in I$. In terms of the preferred reducing transformation introduced at the end of § 3, if $B(t) \geqq 0$ for $t$ a.e. on $I$ the condition $\mathfrak{N}^+(I)$ might also be phrased as the condition that for $s \in I$ there exists a $b(s) \in I$ such that $s < b(s)$ and the matrix $B^\circ(t)$ of (3.4) belonging to a preferred reducing transformation is such that $\int_s^b B^\circ(t)\, dt > 0$ if $b \in I$ and $b \geqq b(s)$. The corresponding equivalent formulations of $\mathfrak{N}^-(I)$ will not be stated specifically, as they should be obvious to the reader. In particular, the following theorem generalizes the result of Theorem 5.1 of Reid [8], and is a ready consequence of the argument used to establish Theorem 5.3 of Reid [9].

THEOREM 6.1. *Suppose that* (2.2) *satisfies hypothesis* $\mathfrak{H}$ *on* $I = (-\infty, \infty)$, $B(t) \geqq 0$ *for* $t$ *a.e. on* $I$, *and* (2.2) *is disconjugate on* $I$. *For* $r \in I$ *let* $Y_r(t) = (U_r(t); V_r(t))$ *be the conjoined basis for* (2.2) *satisfying* $U_r(r) = 0$, $V_r(r) = E$. *If condition* $\mathfrak{N}^+(I)$ *holds then for* $s \in I$ *and* $r > b(s)$ *the matrix* $U_r(t)$ *is nonsingular for* $t \in I^-(s) = \{t | t \in I, t \leqq s\}$ *and* $W_r(t) = V_r(t)U_r^{-1}(t)$ *tends to a limit* $W_\infty(t)$ *as* $r \to \infty$ *for* $t \in I$, *and* $W_\infty(t)$ *is the distinguished solution of* (2.1) *at* $\infty$. *Correspondingly, if* $\mathfrak{N}^-(I)$ *holds then for* $s \in I$ *and* $r < a(s)$ *the matrix* $U_r(t)$ *is nonsingular for* $t \in I^+(s) = \{t | t \in I, t \geqq s\}$ *and*

$W_r(t) = V_r(t)U_r^{-1}(t)$ *tends to a limit* $W_{-\infty}(t)$ *as* $r \to -\infty$ *for* $t \in I$, *and* $W_{-\infty}(t)$ *is the distinguished solution of* (2.1) *at* $-\infty$.

For the further discussion of solutions of Riccati matrix differential equations (2.1) we shall suppose that the following hypothesis of intermediate strength is satisfied.

$\mathfrak{H}_{\mathfrak{N}}(-\infty, \infty)$ The $n \times n$ matrix functions $A(t)$, $B(t)$, $C(t)$ satisfy hypothesis $\mathfrak{H}$ on $I = (-\infty, \infty)$, $B(t) \geqq 0$ for $t$ a.e. on $I$, and there exists a nonnegative integer $d$ such that $d[a, b] = d$ for arbitrary $[a, b] \subset (-\infty, \infty)$.

In all cases the argument will be presented for the case $d > 0$, since in the alternate case of identical normality the results are well known, and arguments simplify through the nonexistence of certain matrices. For such systems there is an $n \times d$ matrix $V_{do}(t)$ which satisfies $V_{do}'(t) + A^*(t)V_{do}(t) = 0$, $B(t)V_{do}(t) = 0$ for $t \in (-\infty, \infty)$, and such that the column vectors of $V_{do}(t)$ form a basis for $\Lambda(I_o)$, where $I_o$ is any nondegenerate subinterval of $I$.

For $s \in I$, let $\Delta(s) = V_{do}(s)[V_{do}^*(s)V_{do}(s)]^{-1/2}$, where $[V_{do}^*(s)V_{do}(s)]^{-1/2}$ denotes the inverse of the positive definite square root matrix $[V_{do}^*(s)V_{do}(s)]^{1/2}$ of the positive definite $d \times d$ Hermitian matrix $V_{do}^*(s)V_{do}(s)$ (see, for example, [12, pp. 263–265]). Then $V_{ds}(t) = V_{do}(t)[V_{do}^*(s)V_{do}(s)]^{-1/2}$ is an $n \times d$ matrix function whose column vectors form a basis for $\Lambda(I_o)$ on arbitrary nondegenerate subintervals $I_o$ of $I$, and $V_{ds}(s) = \Delta(s)$ where $\Delta^*(s)\Delta(s) = E_d$. Moreover, let $Q(s)$ be an $n \times (n-d)$ matrix such that $\Delta^*(s)Q(s) = 0$ and $Q^*(s)Q(s) = E_{n-d}$, so that for $s \in (a, \infty)$ the $n \times n$ matrix $[Q \quad \Delta(s)]$ is unitary; in particular, the matrix $Q(s)$ may be chosen to be locally a.c. on $[a, \infty)$, although this property will not be used in the following discussion.

Let $Y_{2s}(t) = (U_{2s}(t); V_{2s}(t))$ be the solution of $(2.2_M)$ determined by the initial conditions $U_{2s}(s) = \Delta(s)$, $V_{2s}(s) = 0$; as $V_{ds}^*(t)U_{2s}(t)$ is constant on $I$, the value of this matrix function is $V_{ds}^*(s)U_{2s}(s) = \Delta^*(s)\Delta(s) = E_d$. Now if $s$ and $r$ are distinct points on $I$, the hypothesis that (2.2) is disconjugate on this interval implies that there is a unique solution $Y_{sr}(t) = (U_{sr}(t); V_{sr}(t))$ of $(2.2_M)$ which satisfies the initial conditions

$$(6.1) \qquad U_{sr}(s) = Q(s), \qquad U_{sr}(r) = 0, \qquad \Delta^*(s)V_{sr}(s) = 0.$$

Moreover, as in the proof of Theorem 5.3 of Reid [9], if $a < r_1 < s < r_2 < r_3 < \infty$ it follows that $U_{sr_1}^*(s)V_{sr_1}(s) > U_{sr_3}^*(s)V_{sr_3}(s) > U_{sr_2}^*(s)V_{sr_2}(s)$. Consequently, $U_{sr}^*(s)V_{sr}(s) = Q^*(s)V_{sr}(s)$ is a monotone nondecreasing bounded family of Hermitian matrices for $r \in (s, \infty)$, and hence there exists an Hermitian matrix $H$ such that $Q^*(s)V_{sr}(s) \to H$ as $r \to \infty$. Moreover, as $\Delta^*(s)V_{sr}(s) = 0$ and $[Q(s) \quad \Delta(s)]$ is nonsingular, it follows that $V_{s\infty} = \lim_{r \to \infty} V_{sr}(s)$ exists, $\Delta^*(s)V_{s\infty} = 0$, $Q^*(s)V_{s\infty}$ is Hermitian, and $Q^*(s)V_{sr_1}(s) > Q^*(s)V_{s\infty} > Q^*V_{sr}(s)$ for $a < r_1 < s < r < \infty$.

THEOREM 6.2. *Suppose that* (2.2) *satisfies hypothesis* $\mathfrak{H}_{\mathfrak{N}}(-\infty, \infty)$ *and is disconjugate on the interval* $I = (a, \infty)$, *and for* $s \in I$ *the* $n \times (n-d)$ *matrix* $V_{s\infty}$ *has been determined as indicated above. If* $Y_{s\infty}^\circ(t) = (U_{s\infty}^\circ(t); V_{s\infty}^\circ(t))$ *is the solution of* $(2.2_M)$ *satisfying the initial condition* $Y_{s\infty}^\circ(s) = (Q(s); V_{s\infty})$, *and* $Y_{s\infty}(t) = (U_{s\infty}(t); V_{s\infty}(t))$ *with* $U_{s\infty}(t) = [U_{s\infty}^\circ(t) \quad U_{2s}(t)]$, $V_{s\infty}(t) = [V_{s\infty}^\circ(t) \quad V_{2s}(t)]$, *then*:

(i) $Y_{s\infty}(t)$ *is a conjoined basis for* (2.2) *with* $U_{s\infty}(t)$ *nonsingular on* $I$, *and* $V_{ds}^*(t)U_{s\infty}^\circ(t) = 0$ *for* $t \in I$; *correspondingly,* $W_{s\infty}(t) = V_{s\infty}(t)U_{s\infty}^{-1}(t)$ *is an Hermitian solution of* (2.1) *satisfying* $\Delta^*(s)W_{s\infty}(s) = 0$.

(ii) *$Y_{s\infty}(t)$ is a principal solution of* (2.2$_M$) *at* $\infty$, *and* $W_{s\infty}(t)$ *is a distinguished solution of* (2.1) *at* $\infty$, *in the sense of Reid* [9]; *that is, if*

$$S(t,s;U_{s\infty}) = \int_s^t U_{s\infty}^{-1}(r)B(r)U_{s\infty}^{*-1}(r)\,dr, \tag{6.2}$$

*and*

$$\Theta(t,s|W_{s\infty}) = U_{s\infty}(s)S(t,s;U_{s\infty})U_{s\infty}^*(s), \tag{6.3}$$

*then the E. H. Moore generalized inverse* $\Theta^{\#}(t,s|W_{s\infty})$ *of* $\Theta(t,s|W_{s\infty})$ *tends to* 0 *as* $t \to \infty$.

As $V_{ds}^*(t)U_{s\infty}^{\circ}(t) = 0$ and $B(t)V_{ds}(t) = 0$ for $t \in I$, there exists an $n \times (n-d)$ matrix $\Phi_s(t)$ such that

$$U_{s\infty}^{-1}(t) = \begin{bmatrix} \Phi_s^*(t) \\ V_{ds}^*(t) \end{bmatrix} \quad \text{for } t \in I; \tag{6.4}$$

consequently $U_{s\infty}^{-1}(r)B(r)U_{s\infty}^{*-1}(r) = \operatorname{diag}\{\Phi_s^*(r)B(r)\Phi_s(r);0\}$, and

$$S(t,s;U_{s\infty}) = \operatorname{diag}\{\hat{S}(t,s;U_{s\infty});0\}, \tag{6.5}$$

where $\hat{S}(t,s|U_{s\infty})$ is the $(n-d)\times(n-d)$ matrix function

$$\hat{S}(t,s;U_{s\infty}) = \int_s^t \Phi_s^*(r)B(r)\Phi_s(r)\,dr. \tag{6.6}$$

Moreover, if $Y_{3s}^{\circ}(t) = (U_{3s}^{\circ}(t); V_{3s}^{\circ}(t))$ is the solution of (2.2$_M$) determined by the initial condition $Y_{3s}^{\circ}(s) = (0;Q(s))$, and $Y_{3s}(t) = (U_{3s}(t);V_{3s}(t))$ is the solution of (2.2$_M$) with $U_{3s}(t) = [U_{3s}^{\circ}(t) \quad U_{2s}(t)]$, $V_{3s}(t) = [V_{3s}^{\circ}(t) \quad V_{2s}(t)]$, then by argument as in Reid [7, § 3] it follows that

$$U_{3s}(t) = U_{s\infty}(t)[\operatorname{diag}\{0;E_d\} - S(t,s;U_{s\infty})\{U_{3s};V_{3s}|U_{s\infty};V_{s\infty}\}].$$

By direct computation we have $\{U_{3s};V_{3s}|U_{s\infty};V_{s\infty}\} = \operatorname{diag}\{-Q^*Q;0\}$ and hence

$$U_{3s}^{\circ}(t) = U_{s\infty}^{\circ}(t)\hat{S}(t,s;U_{s\infty})Q^*Q.$$

As (2.2) is disconjugate on $I$ it then follows that $U_{3s}^{\circ}(t)$ is of rank $n-d$ for $t \neq s$, and consequently the $(n-d)\times(n-d)$ matrix function $\hat{S}(t,s;U_{s\infty})$ is nonsingular for $t \neq s$.

Now, in general, if $K$ is an $n \times n$ Hermitian matrix of rank $n-d$, and $\psi$ is an $n \times (n-d)$ matrix whose column vectors form an orthonormal basis for the linear subspace of $\mathfrak{C}_n$ spanned by the column vectors of $K$, then there exists a nonsingular $(n-d)\times(n-d)$ matrix $\kappa$ such that $K = \psi\kappa\psi^*$. In terms of these component matrices $\psi$, $\kappa$ it follows readily that $K^{\#} = \psi\kappa^{-1}\psi^*$. Also, if $N$ is a nonsingular $n \times n$ matrix and $K_1 = NKN^*$, then $K_1 = \psi_1\kappa_1\psi_1^*$, where $\psi_1 = N\psi\lambda^{-1/2}$, $\kappa_1 = \lambda^{1/2}\kappa\lambda^{1/2}$ with $\lambda = \psi^*N^*N\psi$, satisfy the conditions specified above for $\psi$, $\kappa$, and hence $K_1^{\#} = \psi_1\kappa_1^{-1}\psi_1^* = N\psi\lambda^{-1}\kappa^{-1}\lambda^{-1}\psi^*N^*$. In particular, if $\kappa = \kappa(t)$ for $t \in I$ and $\psi$ is independent of $t$, then $K^{\#}(t) = \psi\kappa^{-1}(t)\psi^* \to 0$ as $t \to \infty$ if and only if $\kappa^{-1}(t) \to 0$ as $t \to \infty$; moreover, if $N$ is also independent of $t$ then $K_1^{\#}(t) \to 0$ as $t \to \infty$ if and only if $K^{\#}(t) \to 0$ as $t \to \infty$.

From these remarks and the discussion at the end of § 3 we have the following additional results.

COROLLARY 1. *$\hat{S}(t, s; U_{s\infty})$ is nonsingular for $t \neq s$, and $\hat{S}^{-1}(t, s; U_{s\infty}) \to 0$ as $t \to \infty$.*

COROLLARY 2. *If system (2.2°) is related to system (2.2) by the transformation (3.1) and $\hat{Y}_{s\infty}(t) = (\hat{U}_{s\infty}(t); \hat{V}_{s\infty}(t)) = (T^{-1}(t)U_{s\infty}(t); T^*(t)V_{s\infty}(t))$, $\hat{W}_{s\infty}(t) = \hat{V}_{s\infty}(t)\hat{U}_{s\infty}^{-1}(t) = T^*(t)W_{s\infty}(t)T(t)$, then $\Theta^{\circ}(t, s|\hat{W}_{s\infty}) = T^{-1}(s)\Theta(t, s|W_{s\infty})T^{*-1}(s)$ and $\Theta^{\circ\#}(t, s|\hat{W}_{s\infty}) \to 0$ as $t \to \infty$.*

In particular, if $T(t) = Z^{*-1}(t)$, where $Z(t)$ is a fundamental matrix solution of $Z' + A^*(t)Z = 0$ such that the last $d$ column vectors of $Z(t)$ form a basis for $\Lambda(-\infty, \infty)$, then the resulting truncated preferred reduced system (3.9) is disconjugate and identically normal on $(-\infty, \infty)$, so that the results of Reid [8, § 6] may be applied directly to this system. These results, together with the method of Reid [7, § 8], yield for this truncated preferred reduced system the following conclusions.

THEOREM 6.3. *Suppose that (2.2) satisfies hypothesis $\mathfrak{H}_{\mathfrak{N}}(-\infty, \infty)$ and is disconjugate on $(-\infty, \infty)$, and that for $r \in (-\infty, \infty)$ the solution of a corresponding truncated preferred reduced matrix system ($3.9_{\mathrm{M}}$) satisfying the initial conditions $H(r) = 0$, $Z(r) = E_{n-d}$ is denoted by $H_r(t)$, $Z_r(t)$. Then $H_r(t)$ is nonsingular for $t \neq r$, $\Omega_r(t) = Z_r(t)H_r^{-1}(t)$ is such that $\Omega_{\infty}(t) = \lim_{r\to\infty} \Omega_r(t)$ and $\Omega_{-\infty}(t) = \lim_{r\to-\infty} \Omega_r(t)$ exist, are the distinguished solutions of (3.10) at $\infty$ and $-\infty$, respectively, and possess the following properties:*

(i) *If $\Omega(t)$ is an Hermitian solution of (3.10) which exists on $(-\infty, \infty)$ then $\Omega(t) - \Omega_{\infty}(t) \geqq 0$ and $\Omega_{-\infty}(t) - \Omega(t) \geqq 0$ throughout $(-\infty, \infty)$, while if $\Omega(t)$ is an Hermitian solution of (3.10) for which at some value $s$ the matrix $\Omega(s) - \Omega_{\infty}(s)$ ($\Omega_{-\infty}(s) - \Omega(s)$) fails to be nonnegative definite then $\Omega(t)$ does not exist throughout the interval $[s, \infty)$ ($(-\infty, s]$).*

(ii) *If also $\hat{C}_{11}(t) \geqq 0$ for $t$ a.e. on $(-\infty, \infty)$, then $\Omega_{\infty}(t) \leqq 0$ and $\Omega_{-\infty}(t) \geqq 0$ for $t \in (-\infty, \infty)$.*

**7. Comments.** Among the many occurrences of Riccati matrix differential equations are extensive applications in the theory of filtering and control. The concluding remarks of this section will be limited to comments on relationships that exist between the results of the preceding sections and recent work of Bucy [2] (this paper has been reproduced in essentially verbatim form in Bucy and Joseph [3; Chap. V]) and Wonham [14].

In each of these applications the Riccati matrix differential equation is of the form (2.1) with real-valued coefficient matrix functions, and with the real symmetric matrices $B(t)$, $C(t)$ in factored forms. In essentially the notation of Bucy [2],

$$B(t) = H^*(t)R^{-1}(t)H(t), \qquad C(t) = G(t)Q(t)G^*(t), \tag{7.1}$$

where $R(t)$ is a positive definite $s \times s$ matrix, $Q(t)$ is a nonnegative definite $r \times r$ matrix, while $H(t)$ and $G(t)$ are of respective dimensions $s \times n$ and $n \times r$. In particular, for such equations the matrix functions $B(t)$ and $C(t)$ are both nonnegative definite, for arbitrary compact subintervals the functional (4.1) is positive definite on $\mathscr{D}_0[a, b]$, and hence by Theorem 4.1 the corresponding linear Hamiltonian system (2.2) is disconjugate on the interval of consideration. In particular, the monotoneity properties presented in the corollary to Theorem 4.2, and in Theorem 4.4, hold for these equations.

In view of the result presented in Lemma 3.1, the concept of $R$-complete observability of Bucy [2] is the condition $\mathfrak{N}^{-}(I)$ for the corresponding system (2.2), and the concept of $Q$-complete controllability is the condition $\mathfrak{N}^{+}(I)$ for the associated obverse system (2.11). In particular, the result of Theorem 1 of Bucy [2] follows from his comment on $Q$-complete controllability and the corollary to Theorem 4.2. The result of Bucy's Theorem 2 and its corollaries are consequences of conclusion (b) of Theorem 2.1 for autonomous linear Hamiltonian systems, together with certain of the general results presented in § 2. The result of his Theorem 3 follows from conclusion (b) of Theorem 2.1, and Theorem 6.1, together with the fact that the limit matrix defined in his Theorem 3 must define a constant solution of the Riccati matrix differential equation, and hence must be a solution of the corresponding algebraic quadratic matrix equation. In this connection, the reader is also referred to Reid ([8, § 7] and [10]).

For the matrix differential system (2.2a, b) of Wonham [14], the results of the above § 5 imply, in particular, conclusions (i), (ii) of his Theorem 2.1. Various portions of the results proved by him in the course of establishing conclusion (iv) of his Theorem 2.1 also appear as special instances of the monotoneity properties presented in §§ 2, 4 and 5 of the present paper.

## REFERENCES


[1] G. A. Bliss, *Lectures on the Calculus of Variations*, University of Chicago Press, Chicago, 1946.

[2] R. S. Bucy, *Global theory of the Riccati equation*, J. Comput. System Sci., 1 (1967), pp. 349–361.

[3] R. S. Bucy and P. D. Joseph, *Filtering for Stochastic Processes with Applications to Guidance*, Interscience, New York, 1968.

[4] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[5] P. Hartman, *Self-adjoint, non-oscillatory systems of ordinary, second order linear differential equations*, Duke Math. J., 24 (1957), pp. 25–35.

[6] W. T. Reid, *Oscillation criteria for linear differential systems with complex coefficients*, Pacific J. Math., 6 (1956), pp. 733–751.

[7] ———, *Principal solutions of non-oscillatory self-adjoint linear differential systems*, Ibid., 8 (1958), pp. 147–169.

[8] ———, *Riccati matrix differential equations and non-oscillation criteria for associated linear differential systems*, Ibid., 13 (1963), pp. 665–685.

[9] ———, *Principal solutions of nonoscillatory linear differential systems*, J. Math. Anal. Appl., 9 (1964), pp. 397–423.

[10] ———, *A matrix equation related to a non-oscillation criterion and Liapunov stability*, Quart. Appl. Math., 23 (1965), pp. 83–87.

[11] ———, *Generalized linear differential systems and related Riccati matrix integral equations*, Illinois J. Math., 10 (1966), pp. 701–722.

[12] F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Frederic Ungar, New York, 1955.

[13] S. Sandor, *Sur l'equation differentielle matricielle de type Riccati*, Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine, (N.S.), 3 (1959), (51), pp. 229–249.

[14] W. M. Wonham, *On a matrix Riccati equation of stochastic control*, SIAM J. Control, 6 (1968), pp. 681–697.

SIAM J. MATH. ANAL.
Vol. 1, No. 2, May 1970

# WELL-POSED PROBLEMS FOR A PARTIAL DIFFERENTIAL EQUATION OF ORDER $2m + 1$*

R. E. SHOWALTER†

We are concerned here with well-posed problems for the partial differential equation

$$u_t(x, t) + \gamma M u_t(x, t) + Lu(x, t) = f(x, t)$$

containing the elliptic differential operator $M$ of order $2m$ and the differential operator $L$ of order $\leqq 2m$. Hilbert space methods are used to formulate and solve an abstract form of the problem and to discuss existence, uniqueness, asymptotic behavior and boundary conditions of a solution.

The formulation of a generalized problem is the objective of § 1, and we shall have reason to consider two types of solutions, called weak and strong. Sufficient conditions on the operator $M$ are given for the existence and uniqueness of a weak solution to the generalized problem. These conditions constitute elliptic hypotheses on $M$ and are discussed briefly in § 3. Similar assumptions on $L$ lead to results on the asymptotic behavior of a weak solution. The case in which $M$ and $L$ are equal and self-adjoint is discussed in § 2, and it is here that the role of the coefficient $\gamma$ of the equation appears first. Special as it is, this is a situation that often arises in applications, and there has been considerable interest in this coefficient $\gamma$ [4], [25]. The weak and strong solutions are distinguished not only by regularity conditions but also by their associated boundary conditions. It first appears in § 5 that it is possible to prescribe too many (independent) boundary conditions on a strong solution, but in the applications it is seen that the interdependence of these conditions is built into the assumptions on the domains of the operators. Two examples of applications appear in § 6 with a discussion of the types of boundary conditions that are appropriate.

**1. The generalized problem.** Let $G$ be a nonempty open set in the $n$-dimensional real Euclidean space, $\mathbb{R}^n$, whose boundary $\partial G$ is an $(n-1)$-dimensional manifold with $G$ lying on one side of it. $C^\infty(G)$ is the space of infinitely differentiable functions on $G$, and $C_0^\infty(G)$ is the linear subspace of $C^\infty(G)$ consisting of functions with compact support in $G$. The Sobolev space $H^m(G) = H^m$ is the Hilbert space of (equivalence classes of) functions in $L^2(G)$, all of whose distributional derivatives through order $m$ belong to $L^2(G)$. The inner product and norm are given, respectively, by

$$(f, g)_m = \sum\left\{\int_G D^\alpha f \overline{D^\alpha g}\, dx : |\alpha| \leqq m\right\}$$

and $\|f\|_m = \sqrt{(f,f)_m}$, where $\alpha = (\alpha_1, \cdots, \alpha_n)$ denotes an $n$-tuple of nonnegative integers, and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_2^{\alpha_2}}$$

is a derivative of order $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

* Received by the editors September 18, 1969.

† Department of Mathematics, The University of Texas, Austin, Texas 78712.

$H_0^m(G) = H_0^m$ is the closure of $C_0^\infty(G)$ in $H^m$; it is known that if $\partial G$ is $m$ times continuously differentiable and $\phi$ is in $C^{m-1}(\mathrm{cl}\,(G))$ then $\phi$ is in $H_0^m(G)$ if and only if it is in $H^m(G)$ and vanishes on $\partial G$ together with all derivatives of order $\leqq m - 1$. Hence, $\phi \in H_0^m$ is a weak Dirichlet boundary condition. In order to determine other boundary conditions, we let $V$ be a closed subspace of $H^m$ that contains $C_0^\infty(G)$ and define the norm on $V$ by $\|\phi\|_V = \|\phi\|_m$ for $\phi \in V$.

We shall consider the equation

$$u'(t) + \gamma \mathscr{M} u'(t) + \mathscr{L} u(t) = f(t) \tag{1.1}$$

containing the indicated vector-valued functions and the partial differential operators of order $2m$ in the divergence forms

$$\mathscr{M} = \sum \{(-1)^{|\rho|} D^\rho m^{\rho\sigma}(x) D^\sigma : |\rho|, |\sigma| \leqq m\}, \tag{1.2}$$

$$\mathscr{L} = \sum \{(-1)^{|\rho|} D^\rho l^{\rho\sigma}(x) D^\sigma : |\rho|, |\sigma| \leqq m\}. \tag{1.3}$$

Since we are concerned with *weak* solutions, it suffices to require only that the coefficients in (1.2) and (1.3) be bounded and measurable on $G$. This implies that the sesquilinear forms

$$m(\phi, \psi) = \sum \{(m^{\rho\sigma} D^\sigma \phi, D^\rho \psi)_0 : |\rho|, |\sigma| \leqq m\}, \tag{1.4}$$

$$l(\phi, \psi) = \sum \{(l^{\rho\sigma} D^\sigma \phi, D^\rho \psi)_0 : |\rho|, |\sigma| \leqq m\} \tag{1.5}$$

are bounded on $V$; in particular, for all $\phi$ and $\psi$ in $V$ we have

$$|m(\phi, \psi)| \leqq K_m \|\phi\|_V \|\psi\|_V, \tag{1.6}$$

$$|l(\phi, \psi)| \leqq K_l \|\phi\|_V \|\psi\|_V, \tag{1.7}$$

where $K_m = \sup \{\|m^{\rho\sigma}\|_\infty\}$ and $K_l = \sup \{\|l^{\rho\sigma}\|_\infty\}$. These sesquilinear forms can be used to specify solutions of (1.1) in $V$, since for any $u$ in $V$ the conjugate linear maps $\phi \mapsto m(u, \phi)$ and $\phi \mapsto l(u, \phi)$ are continuous from $\mathscr{D}(G)$ into $\mathbb{C}$, where $\mathscr{D}(G)$ is the linear space $C_0^\infty(G)$ with the topology of L. Schwartz [11], [19]. These maps determine elements of $\mathscr{D}'(G)$, the space of distributions, and they satisfy

$$m(u, \phi) = \langle \mathscr{M} u, \bar{\phi} \rangle, \tag{1.8}$$

$$l(u, \phi) = \langle \mathscr{L} u, \bar{\phi} \rangle \tag{1.9}$$

for all $\phi$ in $\mathscr{D}(G)$. The operators $\mathscr{M}$ and $\mathscr{L}$ map $V$ into $\mathscr{D}'(G)$.

Let $H$ be the Hilbert space $L^2(G)$. Define linear subsets of $H$ by $D(M) = \{u \in V : \mathscr{M}(u) \in H$ and (1.8) holds for all $\phi \in V\}$ and $D(L) = \{u \in V : \mathscr{L}(u) \in H$ and (1.9) holds for all $\phi$ in $V\}$, and let $M$ and $L$ denote the restrictions of $\mathscr{M}$ and $\mathscr{L}$ to $D(M)$ and $D(L)$, respectively. Then $M$ and $L$ are unbounded operators on $H$ whose domains are contained in $V$ [3], [10]. Furthermore, for any $u$ in $D(M)$,

$$m(u, v) = (Mu, v)_H \tag{1.10}$$

for all $v$ in $V$ and

$$l(u, v) = (Lu, v)_H \tag{1.11}$$

for all $u$ in $D(L)$ and $v$ in $V$.

The *generalized problem* is the following: Let $V$ and $H$ be Hilbert spaces for which the injections $\mathscr{D}(G) \hookrightarrow V \hookrightarrow H$ are continuous and $\mathscr{D}(G)$ is dense in $H$. Let $m$ and $l$ be sesquilinear forms on $V$ which satisfy (1.6) and (1.7). Let $u_0$ belong to $V$, and let $f$ be a continuous map of $\mathbb{R}$ into $H$. Find a continuously differentiable function $u$ of $\mathbb{R}$ into $V$ such that $u(0) = u_0$ and

$$(u'(t), v)_H + \gamma m(u'(t), v) + l(u(t), v) = (f(t), v)_H \tag{1.12}$$

for all $v$ in $V$ and $t$ in $\mathbb{R}$.

A solution of the generalized problem is a *weak solution* of (1.1), since for all $\phi$ in $C_0^\infty(G)$ it follows from (1.8) and (1.9) that

$$\langle u'(t), \bar{\phi} \rangle + \gamma \langle \mathscr{M} u'(t), \bar{\phi} \rangle + \langle \mathscr{L} u(t), \bar{\phi} \rangle = \langle f(t), \bar{\phi} \rangle,$$

hence (1.1) holds in $\mathscr{D}'(G)$. Furthermore, if $u(t)$ belongs to $D(L)$ and $u'(t)$ to $D(M)$ for all $t$ in $\mathbb{R}$, then

$$u'(t) + \gamma M u'(t) + L u(t) = f(t)$$

in $H$, and $u(t)$ is called a *strong solution* of (1.12).

We shall hereafter assume, with no loss of generality, that

$$\|\phi\|_V \geqq \|\phi\|_H \tag{1.13}$$

for $\phi$ in $V$.

**2. A special case with $L = M = L^*$.** We first use the method of eigenfunction expansions to obtain a rather precise description of solutions of the generalized problem with $m \equiv l$. Assume that

$$l(u, v) = \overline{l(v, u)} \quad \text{for all } u, v \text{ in } V, \tag{2.1}$$

$$l(u, u) \geqq k_l \|u\|_V^2 \quad \text{for all } u \text{ in } V, k_l > 0, \tag{2.2}$$

and the injection

$$V \hookrightarrow H \quad \text{is completely continuous.} \tag{2.3}$$

The condition (2.1) implies that $L$ is symmetric, while (2.2) implies that $L$ is a bijection of $D(L)$ onto $H$ [11], [12], [15], [16]. In fact, (2.2) and (1.11) imply that for any $\phi$ in $H$,

$$k_l \|L^{-1}\phi\|_V^2 \leqq (\phi, L^{-1}\phi)_H \leqq \|\phi\|_H \|L^{-1}\phi\|_H,$$

so $L^{-1}$ is continuous from $H$ into $V$ and satisfies

$$\|L^{-1}\phi\|_V \leqq k_l^{-1} \|\phi\|_H$$

for all $\phi$ in $H$. The condition (2.3) will be satisfied if $G$ is bounded and either $V = H_0^m(G)$ or $\partial G$ is sufficiently smooth [1], [5], [16].

From (1.7), (2.1) and (2.2) it follows that the sesquilinear form $[u, v] \equiv l(u, v)$ is an inner product on $V$ for which the associated norm $\|u\|_l \equiv [u, u]^{1/2}$ is equivalent to the norm $\|u\|_V$. Letting $K$ be the restriction of $L^{-1}$ to $V$, we see that

$$[Ku, v] = (u, v)_H \tag{2.4}$$

for all $u$ and $v$ in $V$, and from (2.1) it follows that $K$ is symmetric on $V$ with respect to

$[\cdot,\cdot]$. Also, $K$ is the composition of the continuous operator $L^{-1}: H \to V$ and the completely continuous injection, so $K$ is a completely continuous operator on $V$.

The spectral resolution of completely continuous and symmetric operators is well known [18]: there is a complete orthonormal sequence $\{\phi_n\}$ of eigenvectors of $K$ in $V$ and associated eigenvalues $\{\rho_n\}$ such that

(2.5a) $$K\phi_n = \rho_n\phi_n \qquad \text{for all } n \geqq 1,$$

(2.5b) $$[\phi_m, \phi_n] = \delta_{m,n} \qquad \text{for all } m, n \geqq 1,$$

(2.5c) $$\rho_1 \geqq \rho_2 \geqq \rho_3 \geqq \cdots \rho_n \to 0 \quad \text{as } n \to \infty,$$

and every $v$ in $V$ can be written as

(2.5d) $$v = \sum_{n \geqq 1} [v, \phi_n]\phi_n.$$

Let $\lambda_n = (\rho_n)^{-1}$; the sequence $\{\lambda_n\}$ is nondecreasing and unbounded by (2.5c), and $L\phi_n = \lambda_n\phi_n$ for $n \geqq 1$. Since $(\phi_m, \phi_n)_H = [K\phi_m, \phi_n] = \rho_m\delta_{m,n}$ for $m, n \geqq 1$, the sequence $\{\lambda_n^{1/2}\phi_n\}$ is orthonormal in $H$. It is also complete, for if $f$ is in $H$ there is a $u$ in $V$ with $Lu = f$. The sequence $u_n = \sum_{k=1}^{n} [u, \phi_k]\phi_k = \sum_{k=1}^{n} (f, \phi_k)_H\phi_k$ converges in $V$ to $u$, hence $u_n \to u$ in $H$. The sequence $Lu_n = \sum_{k=1}^{n} (f, \lambda_k^{1/2}\phi_k)_H\lambda_k^{1/2}\phi_k$ converges in $H$, since it is the Fourier expansion of $f$ by $\{\lambda_k^{1/2}\phi_k\}$, so $L$ being closed implies $Lu = f = \sum_{k=1}^{\infty} (f, \lambda_k^{1/2}\phi_k)_H\lambda_k^{1/2}\phi_k$.

Let $u(t)$ be a solution of the generalized problem. For each $t$ in $\mathbb{R}$ there is a unique sequence $\{u_n(t)\}$ of complex numbers for which

(2.6) $$u(t) = \sum_{n \geqq 1} u_n(t)\phi_n.$$

These Fourier coefficients are given by $u_n(t) = [u(t), \phi_n]$, so each is a continuously differentiable function which satisfies the initial condition

(2.7) $$u_n(0) = [u_0, \phi_n].$$

If $s_n(t)$ denotes the $n$th partial sum of the series (2.6), then $s_n(t)$ converges to $u(t)$ in $V$. The continuity of $u(t)$ implies that this convergence is uniform on compact subsets of $\mathbb{R}$. To verify this, let $g_n(t) = \|u(t) - s_n(t)\|_l^2$. Then each $g_n$ is continuous, the sequence $g_n(t)$ converges to zero for each $t$, and from

$$g_n(t) = \sum_{k=n+1}^{\infty} |u_k(t)|^2$$

it follows that the sequence is monotone, so the convergence is uniform on each compact subset of $\mathbb{R}$ by a well-known theorem of Dini [12].

Furthermore, the sequence of formal derivatives $\{s_n'(t)\}$ converges to $u'(t)$ in $V$. This follows by obtaining the Fourier expansion of $u'(t)$, which converges uniformly on compact subsets of $R$ as above, and integrating this series termwise to obtain $u(t)$. Since $s_n(t) \to u(t)$ and $s_n'(t) \to u'(t)$ in $V$, we have for any $v$ in $V$, $l(s_n(t), v) \to l(u(t), v)$ and $\gamma l(s_n'(t), v) + (s_n'(t), v)_H \to \gamma l(u'(t), v) + (u'(t), v)_H$. The sequence

$\{\lambda_n^{1/2}\phi_n\}$ is orthonormal and complete in $H$, so

$$(f(t), v)_H = \Big(\sum_{n \geqq 1} (f(t), \lambda_n^{1/2}\phi_n)_H \lambda_n^{1/2}\phi_n, v\Big)_H$$

$$= \sum_{n \geqq 1} (f(t), \phi_n)_H[\phi_n, v].$$

Thus, for each $t$ in $\mathbb{R}$ and $v$ in $V$ we have, by (2.4) and (2.5a),

$$\sum_{n \geqq 1} \{(\rho_n + \gamma)u_n'(t) + u_n(t) - (f(t), \phi_n)_H\}[\phi_n, v] = 0,$$

and this yields the necessary condition

$$(\rho_n + \gamma)u_n'(t) + u_n(t) = (f(t), \phi_n)_H, \qquad n \geqq 1, \tag{2.8}$$

for $u(t)$ to be a solution of (1.12).

Let $M$ be the (finite) set of integers $m$ for which $\gamma + \rho_m = 0$, and $N$ the set of integers $n \geqq 1$ for which $\gamma + \rho_n \neq 0$. It follows from (2.7) and (2.8) that for all $n$ in $N$,

$$u_n(t) = [u_0, \phi_n] \exp(-(\gamma + \rho_n)^{-1}t)$$
$$+ (\gamma + \rho_n)^{-1} \int_0^t \exp((\gamma + \rho_n)^{-1}(\tau - t))(f(\tau), \phi_n)_H \, d\tau,$$

and for $m$ in $M$ we must have $u_m(t) = (f(t), \phi_m)_H$. In particular, the initial function must satisfy the compatibility condition $[u_0, \phi_m] = (f(0), \phi_m)_H$ for all $m$ in $M$. That is, $\lambda_m u_0 - f(0)$ is orthogonal in $H$ to $\phi_m$ whenever $\gamma + \rho_m = 0$. These remarks verify the uniqueness and representation statements of the following result.

THEOREM 1. *With the assumptions* (2.1), (2.2) *and* (2.3), *the generalized problem of* § 1 *with* $m \equiv l$ *has at most one solution. A solution exists if and only if for each integer in* $M = \{m : \gamma + \rho_m = 0\}$, *the compatibility condition*

$$l(u_0, \phi_m) = (f(0), \phi_m)_H$$

*holds and the function* $t \mapsto (f(t), \phi_m)$ *is continuously differentiable. This solution is given by the expansion*

$$u(t) = \sum_{n \in N} [u_0, \phi_n] \exp(-(\gamma + \rho_n)^{-1}t)\phi_n$$
$$+ \sum_{n \in N} \left\{ \int_0^t (\gamma + \rho_n)^{-1} \exp((\gamma + \rho_n)^{-1}(\tau - t))(f(\tau), \phi_n)_H \, d\tau \right\} \phi_n \tag{2.9}$$
$$+ \sum_{m \in M} (f(t), \phi_m)_H \phi_m,$$

*where* $N$ *is the set of integers* $n \geqq 1$ *with* $\gamma + \rho_n \neq 0$.

We need only to verify that the function defined by (2.9) is a solution of the problem. Since the sequence $\{\rho_n\}$ converges, the sequence $\{(\gamma + \rho_n)^{-1}\}$ is uniformly bounded for $n$ in $N$. If $K$ is a compact subset of $\mathbb{R}$ and $m \geqq n > \sup(M)$, then from the estimate

$$\left\| \sum_{k=n}^{m} [u_0, \phi_k] \exp(-(\gamma + \rho_n)^{-1} t) \phi_k \right\|_l^2 = \sum_{k=n}^{m} |[u_0, \phi_k] \exp(-(\gamma + \rho_n)^{-1} t)|^2$$
$$\leqq \sup \{\exp(-(\gamma + \rho_n)^{-1} t) : n \in N, t \in K\}^2 \cdot \sum_{k=n}^{m} |[u_0, \phi_k]|^2$$

for all $t$ in $K$ and from the convergence of the expansion of $u_0$ by $\{\phi_k\}$, it follows that the first series in (2.9) converges uniformly on each compact $K$ in $\mathbb{R}$. A similar estimate shows that all derivatives of this series converge uniformly on compact subsets of $\mathbb{R}$, so these can be integrated term-by-term to show that the sum of this series is infinitely differentiable with respect to $t$ in the $V$-norm (equivalently, the $l$-norm) and its derivatives are obtained by differentiating the series term-by-term.

In order to discuss the second term in (2.9), let $T > 0$ and $0 \leqq \tau \leqq T$. The continuity of $f: \mathbb{R} \to H$ and of $L^{-1}: H \to V$ imply that the function $L^{-1} f: \mathbb{R} \to V$ is continuous; hence, the series

$$\sum_{n=1}^{\infty} [L^{-1} f(\tau), \phi_n] \phi_n = \sum_{n=1}^{\infty} (f(\tau), \phi_n)_H \phi_n \tag{2.10}$$

converges to $L^{-1} f(\tau)$ for each $\tau$ in $\mathbb{R}$, and the convergence is uniform on $[0, T]$ by an argument as above which depends on the theorem of Dini. Letting $\eta$ denote the supremum of the numbers

$$|(\gamma + \rho_n)^{-1} \exp(\gamma + \rho_n)^{-1} (\tau - t)|$$

over all $n$ in $N$ and $\tau$ in $[0, T]$, we obtain the estimate

$$\begin{aligned} &\left\| \sum_{k=n}^{m} (\gamma + \rho_k)^{-1} \exp((\gamma + \rho_k)^{-1} (\tau - t)) (f(\tau), \phi_k)_H \phi_k \right\|_l^2 \\ &\quad = \sum_{k=n}^{m} |(\gamma + \rho_k)^{-1} \exp((\gamma + \rho_k)^{-1} (\tau - t)) (f(\tau), \phi_k)_H|^2 \\ &\quad \leqq \eta^2 \sum_{k=n}^{m} |(f(\tau), \phi_k)_H|^2 \\ &\quad = \eta^2 \left\| \sum_{k=n}^{m} (f(\tau), \phi_k)_H \phi_k \right\|_l^2 \end{aligned} \tag{2.11}$$

for $\tau$ in $[0, T]$ and $m \geqq n > \sup(M)$. But we have shown that the series (2.10) is uniformly convergent on $[0, T]$, hence uniformly Cauchy, so this shows that the series appearing in the first term of (2.11) is uniformly Cauchy on $[0, T]$. We may then integrate this series termwise with respect to $\tau$ over the interval $[0, t]$, and this integrated series converges uniformly for all $t$ in $[0, T]$. Thus, the second series in (2.9) converges uniformly on compact subsets of $\mathbb{R}$ to a continuous function from $\mathbb{R}$ into $V$. Application of a similar argument to the termwise derivative of this series shows that the sum of this series is continuously differentiable in $V$, and its derivative is the limit in $V$ of the termwise derivative of the series. The convergence is uniform on compact subsets of $\mathbb{R}$.

That the continuously differentiable $V$-valued function defined by (2.9) is the solution of the generalized problem follows easily by a routine computation similar to that which led to (2.8) above.

**3. Existence of a solution.** The objective in this section is to develop sufficient conditions to guarantee the existence of a solution to the generalized problem of § 1. This development depends on the Lax–Milgram theorem, which gives sufficient conditions on a sesquilinear form in the situation of § 1 for the associated unbounded operator to be onto [11], [15], [16], and the calculus of functions taking values in a Banach space [3], [9]. The major result is Theorem 2, and the two following corollaries give sufficient conditions on the parameter $\gamma$ in order that the hypothesis of Theorem 2 be fulfilled for the case in which the operator $\mathscr{M}$ is elliptic.

Let the Hilbert spaces $H$ and $V$ and sesquilinear forms $m$ and $l$ be as specified in the generalized problem, and assume further that there is a constant $k > 0$ for which

$$|\gamma m(\phi, \phi) + \|\phi\|_H^2| \geqq k\|\phi\|_V^2 \tag{3.1}$$

for all $\phi$ in $V$. This implies that the operator $\gamma M + I$ is a bijection of $D(M)$ onto $H$ and that $D(M)$ is dense in $V$. It follows then from (1.10), (1.11) and (1.7) that for any $\phi$ in $D(L)$

$$\begin{aligned} k\|(\gamma M + I)^{-1}L\phi\|_V^2 &\leqq |(L\phi, (\gamma M + I)^{-1}L\phi)_H| \\ &= |l(\phi, (\gamma M + I)^{-1}L\phi)| \\ &\leqq K_l\|\phi\|_V\|(\gamma M + I)^{-1}L\phi\|_V, \end{aligned}$$

and hence the estimate

$$\|(\gamma M + I)^{-1}L\phi\|_V \leqq (K_l/k)\|\phi\|_V \tag{3.2}$$

for all $\phi$ in $D(L)$. If $D(L)$ is dense in $V$, it follows from (3.2) that $(\gamma M + I)^{-1}L$ has a unique bounded extension from $V$ into $V$ which we shall hereafter denote by $B$.

Since $B$ belongs to the Banach algebra $\mathscr{L}(V)$ of continuous linear operators on $V$, we may define by a power series a one-parameter group of bounded operators on $V$ by

$$\exp(-Bt) = \sum_{n \geqq 0} ((-1)^n/n!)(tB)^n$$

for $t$ in $\mathbb{R}$ [9]. The operator-valued function $t \mapsto \exp(-Bt)$ is differentiable in the uniform operator topology of $\mathscr{L}(V)$ and satisfies

$$\frac{d}{dt}\exp(-Bt) = -B \cdot \exp(-Bt).$$

We now define a $V$-valued function as follows. Since $(\gamma M + I)^{-1}$ is a bounded map of $H$ into $V$ as a consequence of (3.1), and since $f:\mathbb{R} \to H$ is continuous, it follows that $(\gamma M + I)^{-1}f$ is continuous from $\mathbb{R}$ into $V$, and so also is the function

$$\tau \mapsto \exp(B(\tau - t))(\gamma M + I)^{-1}f(\tau)$$

for each $t$ in $\mathbb{R}$. Hence we can define for each $t$ in $\mathbb{R}$ an element of $V$ by the formula

$$u(t) = \exp(-Bt)\cdot u_0 + \int_0^t \exp(B(\tau - t))\cdot(\gamma M + I)^{-1}\cdot f(\tau)\,d\tau, \tag{3.3}$$

where $u_0$ is the initial condition specified in $V$. Then $u:\mathbb{R} \to V$ is continuously differentiable and satisfies the equations

$$u'(t) + Bu(t) = (\gamma M + I)^{-1}f(t), \qquad u(0) = u_0 \tag{3.4}$$

in $V$. From (3.4) we can show that $u$ is a solution of the generalized problem. Let $t \in \mathbb{R}$ and let $\{\phi_n\}$ be a sequence in $D(L)$ for which $\phi_n \to u(t)$ in $V$. The continuity of $B$ implies by (3.4) that $\{B\phi_n\}$ converges in $V$ to $-u'(t) + (\gamma M + I)^{-1}f(t)$. Since $B$ is an extension of $(\gamma M + I)^{-1}L$ and each $\phi_n$ is in $D(L)$, it follows that each $B\phi_n$ is in $D(M)$ and

$$(\gamma M + I)(-B\phi_n) + L\phi_n = 0.$$

Thus for each $n \geqq 1$ and each $v$ in $V$ we obtain, by (1.10) and (1.11),

$$\gamma m(-B\phi_n, v) + (-B\phi_n, v)_H + l(\phi_n, v) = 0,$$

and taking the limit in this equation as $n \to \infty$ we obtain (1.12).

The requirement that $D(L)$ be dense in $V$ (which was used twice in the above arguments) is not essential for existence or uniqueness. In particular, $u(t)$ is a solution if and only if $w(t) = e^{-\lambda t}u(t)$ is a solution of the problem with initial data $u_0$, nonhomogeneous term $F(t) = e^{-\lambda t}f(t)$, and the equations (1.12) with $l$ replaced by

$$\lambda((u, v)_H + \gamma m(u, v)) + l(u, v).$$

By taking $\lambda$ sufficiently large, say, $\lambda = (K_l + k_l)/k$, it follows from (1.7) and (3.1) that we may assume without loss of generality that

$$|l(\phi, \phi)| \geqq k_l\|\phi\|_V^2 \tag{3.5}$$

for all $\phi$ in $V$. From the Lax–Milgram theorem and (3.5) it follows that $D(L)$ is dense in $V$, and we obtain the following theorem.

THEOREM 2. *If the sesquilinear form $m$ of the generalized problem satisfies* (3.1), *then there exists a solution of this problem, and it is given by the formula* (3.3).

Coercive inequalities like (3.1) and (3.5) are known to hold for the sesquilinear forms associated with strongly elliptic partial differential operators. Garding has verified the following result [8]:

> Let $\mathscr{M}$ be the operator specified by (1.2); if all the coefficients are bounded and measurable, the principal coefficients $\{m^{\rho\sigma}:|\rho| = |\sigma| = m\}$ are uniformly continuous on cl $(G)$, and if
>
> $$\operatorname{Re}\left\{\sum_{|\rho|=|\sigma|=m} \xi^\rho m^{\rho\sigma}(x)\xi^\sigma\right\} \geqq c_0|\xi|^{2m}, \qquad c_0 > 0,$$
>
> for all real vectors $\xi$ in $\mathbb{R}^n$, then there exist real numbers $c_1 > 0$ and $c_2$ such that
>
> $$\operatorname{Re}\{m(\phi, \phi)\} + c_2\|\phi\|_0^2 \geqq c_1\|\phi\|_m^2$$
>
> for all $\phi$ in $H_0^m(G)$, where $G$ is bounded and open in $\mathbb{R}^n$.

Similar results will be established for some particular examples in § 6 for spaces $V$ other than $H_0^m(G)$. For some other coerciveness results see [13], [16], [17]. We shall make the following assumption: there is a number $\alpha$ such that for every $\varepsilon > 0$ there is a $\beta(\alpha, \varepsilon) > 0$ for which

$$\text{Re}\, m(\phi, \phi) + (\alpha + \varepsilon)\|\phi\|_H^2 \geqq \beta\|\phi\|_V^2 \tag{3.6}$$

for all $\phi$ in $V$.

Consider first the generalized problem with $\gamma > 0$. From (3.6) it follows that

$$\text{Re}\, \gamma m(\phi, \phi) + \|\phi\|_H^2 \geqq \gamma\beta\|\phi\|_V^2 \tag{3.7}$$

for all $\phi$ in $V$ if $\gamma(\alpha + \varepsilon) \leqq 1$ for some $\varepsilon > 0$ in the case $\alpha \geqq 0$ and for any $\gamma$ if $\alpha < 0$. Since

$$\text{Re}\, \gamma m(\phi, \phi) + \|\phi\|_H^2 \leqq |\gamma m(\phi, \phi) + \|\phi\|_H^2|,$$

we obtain from Theorem 2 the following corollary.

COROLLARY 1. *Assume that the sesquilinear form m satisfies* (3.6). *Then the generalized problem has a solution if* $\alpha > 0$ *and* $0 < \gamma < \alpha^{-1}$ *or if* $\alpha \leqq 0$ *and* $0 < \gamma$.

For the case of $\gamma < 0$ the above method is applicable only if (3.6) holds for some $\alpha < 0$, for if

$$-\gamma\, \text{Re}\, m(\phi, \phi) - \|\phi\|_H^2 \geqq k\|\phi\|_V^2$$

for all $\phi$ in $V$, then

$$\text{Re}\, m(\phi, \phi) - (-\gamma)^{-1}\|\phi\|_H^2 \geqq (-\gamma)^{-1}k\|\phi\|_V^2,$$

so (3.6) holds with $\alpha = \gamma^{-1} < 0$. Conversely, if (3.6) holds for some $\alpha < 0$, then

$$\begin{aligned} \text{Re}\,\{-\gamma m(\phi, \phi) - \|\phi\|_H^2\} &\geqq (-\gamma)\beta\|\phi\|_V^2 - [(\alpha + \varepsilon)(-\gamma) + 1]\|\phi\|_H^2 \\ &\geqq (-\gamma)\beta\|\phi\|_V^2 \end{aligned} \tag{3.8}$$

for all $\phi$ in $V$ if, for some $\varepsilon > 0$, $(\alpha + \varepsilon)(-\gamma) + 1 \leqq 0$, and this is true if $-\gamma > (-\alpha)^{-1}$.

COROLLARY 2. *Assume that the sesquilinear form m satisfies* (3.6) *with* $\alpha < 0$. *Then the generalized problem has a solution if* $\gamma < \alpha^{-1}$.

**4. Uniqueness and boundedness.** The solution of the generalized problem constructed in § 3 is the only solution. In particular, we shall show that (3.1) yields estimates on the growth of a solution and dependence on the initial data and nonhomogeneous term of (1.12). Estimates of the type (3.6) for $m$ and $l$ and symmetry of $m$ imply that the solution of the homogeneous equation is asymptotically stable, since all such solutions decay exponentially to zero.

Consider the sesquilinear forms $m$ and $l$ introduced above on $V \times V$. For each $\phi$ in $V$, the conjugate linear functional $\psi \mapsto m(\phi, \psi)$ on $V$ is bounded by (1.6), so the Riesz–Fréchet theorem [11] implies the existence of a unique $m_0(\phi)$ in $V$ for which

$$m(\phi, \psi) = (m_0(\phi), \psi)_V \tag{4.1}$$

for all $\psi$ in $V$. This determines a bounded operator $m_0 : V \to V$ whose norm in $\mathscr{L}(V)$ satisfies $\|m_0\| \leqq K_m$ by (1.6). Similarly, there is a unique operator $l_0$ in $\mathscr{L}(V)$ for which

$$l(\phi, \psi) = (l_0(\phi), \psi)_V \tag{4.2}$$

for all $\phi$ and $\psi$ in $V$ with $\mathscr{L}(V)$-norm $\|l_0\| \leqq K_l$ by (1.7). The continuity of the injection $V \hookrightarrow H$ suggests the construction of an operator $J: H \to V$ as follows. For each $\phi$ in $H$, the conjugate linear form $\psi \mapsto (\phi, \psi)_H$ is continuous on $V$, so there is a unique $J(\phi)$ in $V$ for which

$$(\phi, \psi)_H = (J\phi, \psi)_V \tag{4.3}$$

for all $\psi$ in $V$. This operator $J$ maps $H$ into $V$, and it follows from (1.13) that the $\mathscr{L}(H, V)$-norm of $J$ satisfies

$$\|J\| \leqq 1.$$

Let $v(t)$ be any solution of the generalized problem. It follows from (1.12), (4.1), (4.2) and (4.3) that

$$(J + \gamma m_0)v'(t) + l_0 v(t) = Jf(t) \tag{4.4}$$

in $V$. That is, $v(t)$ satisfies (4.4) in $V$ with bounded operator coefficients. From the estimate (3.1) and the Lax-Milgram theorem it follows that the bounded operator $\gamma m_0 + J$ on $V$ associated with the $V$-coercive sesquilinear form $\gamma m(\phi, \psi) + (\phi, \psi)_H$ is a topological isomorphism of $V$ onto $V$ for which the $\mathscr{L}(V)$-norm of the inverse satisfies $\|(\gamma m_0 + J)^{-1}\| \leqq k^{-1}$. Hence the function $v(t)$ satisfies the equation

$$v'(t) + (J + \gamma m_0)^{-1} l_0 \cdot v(t) = (J + \gamma m_0)^{-1} Jf(t). \tag{4.5}$$

Since $v: \mathbb{R} \to V$ is continuously differentiable, the real-valued function

$$\sigma(t) \equiv \|v(t)\|_V^2$$

is continuously differentiable and by (4.5) satisfies

$$\begin{aligned} \sigma'(t) &= 2 \operatorname{Re}(v'(t), v(t))_V \\ &= 2 \operatorname{Re}\{-((J + \gamma m_0)^{-1} l_0 v(t), v(t))_V + ((J + \gamma m_0)^{-1} J \cdot f(t), v(t))_V\} \end{aligned}$$

and this in turn implies

$$\begin{aligned} |\sigma'(t)| &\leqq 2k^{-1} K_l \|v(t)\|_V^2 + 2k^{-1} \|f(t)\|_H \|v(t)\|_V \\ &\leqq k^{-1}(2K_l + 1)\sigma(t) + k^{-1} \|f(t)\|_H^2 \end{aligned} \tag{4.6}$$

for all $t$ in $\mathbb{R}$. From (4.6) we obtain the estimates

$$\begin{aligned} \sigma(t) &\leqq \sigma(0) \exp(k^{-1}(2K_l + 1)|t|) \\ &\quad + k^{-1} \left| \int_0^t \exp(k^{-1}(2K_l + 1)|t - \tau|) \cdot \|f(\tau)\|_H^2 \, d\tau \right| \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} \sigma(t) &\geqq \sigma(0) \exp(-k^{-1}(2K_l + 1)|t|) \\ &\quad - k^{-1} \left| \int_0^t \exp(-k^{-1}(2K_l + 1)|t - \tau|) \|f(\tau)\|_H^2 \, d\tau \right| \end{aligned} \tag{4.8}$$

for all $t$ in $\mathbb{R}$.

The linearity of the problem and the preceding remarks yield the following result.

THEOREM 3. *Let the sesquilinear form $m$ of the generalized problem satisfy* (3.1) *for all $\phi$ in $V$. If $u_i(t)$, $i = 1, 2$, are solutions of the generalized problem with initial* *data* $u_1(0)$ *and* $u_2(0)$ *and nonhomogeneous terms* $f_1(t)$ *and* $f_2(t)$, *respectively, then* $\sigma(t) = \|u_1(t) - u_2(t)\|_V^2$ *satisfies the growth and decay estimates* (4.7) *and* (4.8) *with* $f = f_1 - f_2$. *In particular, the generalized problem has at most one solution.*

Stronger estimates on the solution can be obtained when the sesquilinear form $m$ is symmetric and satisfies estimates of the form obtained for the corollaries of § 3. Let us assume then that

$$m(\phi, \psi) = \overline{m(\psi, \phi)} \qquad \text{for all } \phi, \psi \text{ in } V, \tag{4.9}$$

and that

$$|\gamma| m(\phi, \phi) + \operatorname{sgn}(\gamma)\|\phi\|_H^2 \geqq |\gamma|\beta\|\phi\|_V^2 \tag{4.10}$$

for all $\phi$ in $V$. The condition (4.9) implies that $m(\phi, \phi) = \overline{m(\phi, \phi)}$ is real for each $\phi$ in $V$, and (4.10) is equivalent to (3.7) when $\gamma > 0$ and to (3.8) when $\gamma < 0$. Thus (4.10) follows from the coercive estimate (3.6) for certain values of $\gamma$.

Let $u(t)$ be a solution of the generalized problem. Then the real-valued function

$$\Sigma(t) \equiv |\gamma| m(u(t), u(t)) + \operatorname{sgn}(\gamma)\|u(t)\|_H^2$$

is continuously differentiable, and from (4.9) we obtain

$$\begin{aligned}\Sigma'(t) &= 2 \operatorname{Re}\{|\gamma| m(u'(t), u(t)) + \operatorname{sgn}(\gamma)(u'(t), u(t))_H\} \\ &= 2 \operatorname{sgn}(\gamma) \operatorname{Re}\{\gamma m(u'(t), u(t)) + (u'(t), u(t))_H\}.\end{aligned}$$

If (1.12) is homogeneous, then

$$\Sigma'(t) = 2 \operatorname{sgn}(\gamma) \operatorname{Re}\{-l(u(t), u(t))\},$$

and if $l$ satisfies the coercive estimate

$$\operatorname{sgn}(\gamma) \operatorname{Re} l(\phi, \phi) \geqq k_l\|\phi\|_V^2 \tag{4.11}$$

for some $k_l > 0$ and all $\phi$ in $V$, then we have from this and (1.6) the estimate

$$\Sigma'(t) \leqq -2k_l\|u(t)\|_V^2 \leqq -2k_l(|\gamma|K_m + 1)^{-1}\Sigma(t).$$

But this implies that for all $t \geqq 0$,

$$\Sigma(t) \leqq \exp(-2k_l(|\gamma|K_m + 1)^{-1}t)\Sigma(0).$$

We summarize these results in the following theorem.

THEOREM 4. *Assume that the sesquilinear forms of the generalized problem satisfy* (4.9), (4.10) *and* (4.11). *Then there exists a unique solution to the generalized problem, and if* $f \equiv 0$ *in* (1.12), *then this solution satisfies the estimate*

$$\|u(t)\|_V \leqq (\beta^{-1}K_m + |\gamma|^{-1})\|u_0\|_V \exp(-k_l(|\gamma|K_m + 1)^{-1}t) \tag{4.12}$$

*for all* $t \geqq 0$.

This last inequality follows from the estimate on $\Sigma(t)$ together with (1.6) and (4.10). Also, (4.10) implies (3.1). By the usual linearity arguments, one may obtain estimates for the solution of the nonhomogeneous equation (1.12) by adding (4.8) with $\sigma(0) = \|u_0\|_V^2 = 0$ and (4.12). The same argument shows that if (4.11) is replaced by the estimate

$$-\operatorname{sgn}(\gamma)\operatorname{Re} l(\phi, \phi) \geqq k_l\|\phi\|_V^2, \tag{4.13}$$

then one obtains an estimate like (4.12) with the inequality reversed, so the solution *grows* at least exponentially in norm. Finally, we remark that with all the hypotheses above except $f \equiv 0$, the difference of two solutions with different initial data satisfies (4.12), so the effect of initial data is "transient".

**5. Weak and strong solutions.** The objective of this section is to show that if $M$ and $L$ satisfy elliptic hypotheses and if $M$ is "stronger" than $L$, then the weak solution of the problem is a strong solution if and only if the initial function $u_0$ is in the domain of $L$.

THEOREM 5. *Assume that the sesquilinear forms $m$ and $l$ of the generalized problem satisfy the estimates* (3.1) *and* (3.5), *and that $D(M) \subseteq D(L)$. If $u_0$ belongs to $D(L)$, then the weak solution* (3.3) *of the generalized problem is a strong solution* (§ 1).

*Proof.* From the estimate (3.5) it follows that $L^{-1}$ is a continuous injection of $H$ into $V$. Hence we can define by

$$\|\phi\|_L = \|L\phi\|_H \tag{5.1}$$

a norm on $D(L)$ for which the injection $D(L) \hookrightarrow V$ is continuous. The completeness of $H$ shows that $D(L)$ is complete in the norm (5.1). The bounded extension $B$ of $(\gamma M + I)^{-1}L$ maps $D(L)$ into $D(M)$, and the assumption above that $D(M) \subseteq D(L)$ implies that $B$ maps $D(L)$ into $D(L)$. Thus $B$ is a continuous linear operator from $V$ into $V$, and the space $D(L)$ is invariant under $B$. This implies by the closed graph theorem that $B$ is continuous from $D(L)$ into itself with the norm (5.1). To see this, let $\{\phi_n\}$ be a sequence in $D(L)$ for which $\|\phi_n - x_0\|_L \to 0$ and $\|B\phi_n - y_0\|_L \to 0$ as $n \to \infty$, where $y_0$ and $x_0$ are in $D(L)$. Then

$$\begin{aligned}\|y_0 - Bx_0\|_V &\leqq \|y_0 - B\phi_n\|_V + \|B(\phi_n - x_0)\|_V \\ &\leqq \|y_0 - B\phi_n\|_V + \|B\|_{\mathscr{L}(V)}\|\phi_n - x_0\|_V,\end{aligned}$$

and the continuity of the injection $D(L) \hookrightarrow V$ implies that each of these terms converges to zero, so $y_0 = Bx_0$. Thus $B$ is a closed and everywhere-defined linear operator and is hence continuous on $D(L)$ [9], [18].

The significance of the continuity of $B$ on $D(L)$ is that the restrictions of the operators

$$\{\exp(-Bt) : t \text{ in } \mathbb{R}\}$$

are bounded on $D(L)$, and hence the function $t \mapsto \exp(-Bt)u_0$ is in $C^1(D(L))$. Finally, each $(\gamma M + I)^{-1}f(t)$ belongs to $D(M)$, hence also $D(L)$, and an argument like that above shows that $(\gamma M + I)^{-1}$ is continuous from $H$ into $D(L)$, so $f: \mathbb{R} \to H$ being continuous implies that the function

$$t \mapsto \int_0^t \exp(B(\tau - t))(\gamma M + I)^{-1}f(\tau)\,d\tau$$

is in $C^1(D(L))$. Hence the (weak) solution of the generalized problem given by (3.3) is in $C^1(D(L))$, and differentiating this function shows that $u'(t)$ belongs to $D(M)$ for each $t$ in $\mathbb{R}$, so $u(t)$ is a strong solution of the problem.

Theorem 5 is really a regularity result, for the domain of an elliptic operator consists of functions which are "smooth". In particular, the global regularity results for elliptic operators can be used to show that $B$ leaves invariant the subspaces $V \cap H^p(G)$, and an argument like that above shows that $u(t)$ belongs to $V \cap H^p(G)$, where the integer $p$ depends on the coefficients in $M$ and $L$ and the boundary of $G$. The details for the case $V = H_0^1(G)$ for Dirichlet boundary conditions on an equation of order 3 appear in [22].

The interesting distinction between weak and strong solutions is the type of boundary conditions they carry. If $u(t)$ is a strong solution of the generalized problem, then $u(t)$ and $u'(t)$ belong to $D(L)$ and $D(M)$, respectively, and from (1.8) and (1.9) it follows that

$$m(u'(t), v) = (Mu'(t), v)_H \tag{5.2}$$

and

$$l(u(t), v) = (Lu(t), v)_H \tag{5.3}$$

for all $v$ in $V$. These constitute independent boundary conditions on $u'(t)$ and $u(t)$, respectively, if $V$ *properly* contains $H_0^m(G)$. Also, the conditions that $u(t)$ and $u'(t)$ belong to $V$ constitute boundary conditions if $V$ is properly contained in $H^m(G)$. The conditions (5.2) and (5.3) will be called *strong boundary conditions.*

Suppose $u(t)$ is a weak solution of the generalized problem. Then the identities (1.8), (1.9) and (1.12) imply that

$$u'(t) + \gamma\mathscr{M}u'(t) + \mathscr{L}u(t) = f(t) \tag{5.4}$$

in $\mathscr{D}'(G)$. From (1.12) and (5.4), we obtain the identity

$$(\gamma\mathscr{M}u'(t) + \mathscr{L}u(t), v)_H = \gamma m(u'(t), v) + l(u(t), v) \tag{5.5}$$

for all $v$ in $V$. This will be called a *weak boundary condition,* since it is certainly implied by the strong boundary conditions.

**6. Applications.** We shall discuss the implications of our above results in two examples. The first originates in the flow of second order fluids as discussed in [4] and [25], and our results contain most of those in these references. The second example includes the above as well as problems in consolidation of clay [24] and homogeneous fluid flow in fissured rocks [2]. Our results are adequate to discuss all of the boundary value problems associated with these theories as well as many for which no physical applications are known to this writer.

For the first example, let $G$ be the interval $(0, T)$, $T > 0$, and define

$$l(u, v) = \int_0^T u_x \overline{v_x}\, dx$$

for $u$ and $v$ in $H^1(G)$. For functions in $H^1(G)$ we have, for $x$, $y$ in $G$,

$$|u(x) - u(y)| = \left| \int_x^y u'(s)\, ds \right| \leqq |x - y|^{1/2} \|u\|_1, \tag{6.1}$$

so $H^1(G)$ contains only continuous functions. Suppose $V$ is a closed subspace of

$H^1(G)$ which contains only functions which vanish at $x = T$. Then for such $\phi$ in $V$,

(6.2) $$\sup\{|\phi(x)| : 0 \leqq x \leqq T\} \leqq (T)^{1/2}\|\phi\|_1.$$

From (6.1) and (6.2) it follows that any sequence $\{\phi_n\}$ of elements of $V$ for which $\|\phi_n\| \leqq 1$ for all $n$ is a sequence of equicontinuous and uniformly bounded functions. By the Ascoli–Arzelà theorem [11], [18], such a sequence has a uniformly convergent subsequence which then converges in the mean-square norm. That is, the injection of $V$ into $H \equiv L^2(G)$ is completely continuous.

For any $\phi$ in $V$, we have

$$\int_0^T \left\{|\phi(x)|^2 + x\frac{d}{dx}|\phi(x)|^2\right\} dx = x|\phi(x)|^2\Big|_0^T = 0$$

since $\phi(T) = 0$, so

$$\int_0^T |\phi^2(x)|\, dx \leqq 2\int_0^T x \cdot |\phi(x)| \cdot |\phi'(x)|\, dx.$$

From the inequality $2\alpha\beta \leqq 2\alpha^2 + \beta^2/2$, we obtain

$$\int_0^T |\phi(x)|^2\, dx \leqq \frac{1}{2}\int_0^T |\phi(x)|^2\, dx + 2T^2\int_0^T |\phi'(x)|^2\, dx,$$

and hence the inequality

(6.3) $$\|\phi\|_0 \leqq 2T\|\phi_x\|_0$$

for $\phi$ in $V$. From (6.3) we have for all $u$ in $V$,

$$l(u,u) = \int_0^T |u_x|^2\, dx$$
$$\geqq \frac{1}{2}\int_0^T |u_x|^2\, dx + \frac{1}{8T^2}\int_0^T |u|^2\, dx \geqq k_l\|u\|_1^2,$$

where $k_l = \min[1/2, 1/(8T^2)] > 0$. Thus the conditions (2.1), (2.2) and (2.3) are satisfied. By Theorem 1 there is a unique solution of the generalized problem of § 1 for certain values of $\gamma$ which is then a solution of the equation $u'(t) + \gamma\mathscr{L}u'(t) + \mathscr{L}u(t) = f(t)$, where $\mathscr{L}$ is the distributional derivative $-d^2/dx^2$. Furthermore, the inequality (6.2) shows that for each $x \in (0, T)$, the "evaluation" functional $e_x : u \to u(x)$ from $V$ into $\mathbb{C}$ is continuous, so $u'(t)(x) = \partial[u(t)(x)]/\partial t$ in the equation. If $\gamma$ is not equal to any of the eigenvalues $\{\rho_n\}$, then the initial data and nonhomogeneous term are prescribed arbitrarily. For the exceptional values, a compatibility condition is necessary and sufficient for the existence of the solution which is given by (2.9). We shall discuss two choices for $V$ and the associated problem.

If $V = \{\phi \text{ in } H^1(G) : \phi(T) = 0\}$ then the sequence of eigenvalues is given by $\rho_n = (2T/((2n-1)\pi))^2$, $n \geqq 1$, and the eigenfunctions are $\cos(\rho_n^{-1/2}x)$.

For $u$ and $v$ in $V$, we obtain by integrating by parts

(6.4) $$l(u,v) = (\mathscr{L}u, v)_0 - u_x \cdot v|_0^T,$$

so $u$ is in $D(L)$ if and only if $u_x \cdot v|_0^T = 0$ for all $v$ in $V$. That is, $u_x(0) = 0$. The condition that $u$ belongs to $V$ implies $u(T) = 0$. Thus the solution $u(t)$ of the generalized problem satisfies the weak boundary conditions

$$(\gamma u_x'(t) + u_x(t))|_{x=0} = 0,$$
$$u(t)|_{x=T} = 0$$

from (5.5) and $u(t) \in V$, respectively. The condition $u'(t)|_{x=T} = 0$ follows from $u'(t) \in V$, but is redundant since it can be obtained from the second condition above by differentiation. If $\gamma$ is chosen such that (3.1) holds and if $u_0$ is in $D(L)$ (see above), then the solution satisfies the strong boundary condition

$$u_x(t)|_{x=0} = 0, \qquad u(t)|_{x=T} = 0$$

from (5.3) and $u(t) \in V$. (Note that (5.2) leads to a redundant condition $u_x'(t)|_{x=0} = 0$.)

If $V = H_0^1(G) = \{\phi \in H^1(G) : \phi(0) = \phi(T) = 0\}$, then the sequence of eigenvalues and functions is given by $\rho_n = (T/(n\pi))^2$ and $\sin(\rho_n^{-1/2} s)$, $n \geqq 1$. For $u, v$ in $V$, all boundary terms are zero in (6.4), so the identities (5.2), (5.3) and (5.5) do not determine boundary conditions. However $u(t) \in V$ implies the boundary conditions

$$u(t)|_{x=0} = u(t)|_{x=T} = 0.$$

Similar applications hold in spaces of higher dimension. Estimates like (6.3) hold for smooth domains and functions which vanish on a sufficiently large portion of the boundary, and the injection of $V$ into $H = L^2(G)$ is completely continuous if $G$ is bounded and either $V = H_0^m(G)$ or the boundary is $m$ times continuously differentiable [5], [17]. Nonhomogeneous boundary data may be introduced by superposition [22]. The relation between $u'(t)(x)$ and $\partial[u(t)(x)]/\partial t$ is not always so clear as above; see [9, pp. 68–71] for results in this direction.

For a second example, which exhibits more of the "flavor" of these problems, we define the forms

$$m(u, v) = \sum_{i=1}^{n} (u_{x_i}, v_{x_i})_0 + \int_{\partial G} \alpha(s) u(s) \bar{v}(s)\, ds,$$

$$l(u, v) = \sum_{i=1}^{n} (u_{x_i}, v_{x_i})_0 + \int_{\partial G} \beta(s) u(s) \bar{v}(s)\, ds,$$

where $G$ is a bounded open set in $\mathbb{R}^n$ with smooth boundary $\partial G$, and $ds$ denotes Lebesgue measure on $\partial G$. The functions $\alpha, \beta$ are in $L^\infty(\partial G)$ and $\alpha(s) \geqq 0$. By elementary results on "traces" [17], $m$ and $l$ are bounded on $H^1(G)$. Since $\alpha(s) \geqq 0$, it follows that for each $\varepsilon > 0$

$$m(\phi, \phi) + \varepsilon \|\phi\|_0^2 \geqq \min(1, \varepsilon) \|\phi\|_1^2,$$

so (3.6) is satisfied with $V \leqq H^1(G)$, $H = L^2(G)$ and $\alpha = 0$. Hence the generalized problem has a unique solution for each $\gamma > 0$.

If the elements of $V$ satisfy the estimate

$$\|\phi\|_0^2 \leqq K \sum_{i=1}^{n} \|\phi_{x_i}\|_0^2, \tag{6.5}$$

then (3.6) holds for $\alpha$ small but negative, so the generalized problem has a unique solution for all $\gamma < (\alpha)^{-1}$. Furthermore, if (6.5) holds and $\beta(s) \geqq 0$, then the form $l(u, v)$ satisfies an estimate of the form (4.11) with $\gamma > 0$, so the solution is asymptotically stable if $f \equiv 0$ in (1.12). Similarly, if $\gamma < (\alpha)^{-1} < 0$ then (6.5) implies (4.13), and the solution grows exponentially as $t \to \infty$ by the remarks at the end of § 4.

In any case, there exists a unique solution of the generalized problem for $\gamma$ satisfying either of the two corollaries, and the $V$-valued function $u(t)$ satisfies in $\mathscr{D}'(G)$ the equation

$$u'(t) - \gamma\Delta_n u'(t) - \Delta_n u(t) = f(t),$$

where $\mathscr{M} = \mathscr{L} = -\Delta_n$ is the Laplace operator in $n$ variables, and the initial condition $u(0) = u_0$. If $u$ and $v$ are sufficiently regular and if $\partial G$ is sufficiently smooth for the divergence theorem to apply [13], then we have (formally)

$$m(u, v) = (-\Delta_n u, v)_0 + \int_{\partial G} \left(\frac{\partial u}{\partial n} + \alpha u\right) \bar{v}\, ds \tag{6.6}$$

and

$$l(u, v) = (-\Delta_n u, v)_0 + \int_{\partial G} \left(\frac{\partial u}{\partial n} + \beta u\right) \bar{v}\, ds \tag{6.7}$$

for all $u$ and $v$ in $V$, where $\partial/\partial n$ denotes the normal derivative. Thus, the weak boundary condition is

$$\int_{\partial G} \left\{ \gamma\left(\frac{\partial u'}{\partial n} + \alpha u'\right) + \left(\frac{\partial u}{\partial n} + \beta u\right)\right\} \bar{v}\, ds = 0 \tag{6.8}$$

for all $v$ in $V$, while the strong boundary conditions are

$$\int_{\partial G} \left(\frac{\partial u'}{\partial n} + \alpha u'\right) \bar{v}\, ds = 0, \qquad \int_{\partial G} \left(\frac{\partial u}{\partial n} + \beta u\right) \bar{v}\, ds = 0 \tag{6.9}$$

for all $v$ in $V$. The condition

$$u(t) \in V \tag{6.10}$$

also holds true.

Let the boundary $\partial G$ be equal to the disjoint union of $\Gamma_1$ and $\Gamma_2$. Let $V$ be the closure in $H^1(G)$ of the space of restrictions to $G$ of those functions in $C_0^\infty(\mathbb{R}^n)$ whose support is disjoint from $\Gamma_1$. The condition (6.10) means that each $u(t)$ vanishes on $\Gamma_1$ while the condition (6.8) implies that

$$\gamma\left(\frac{\partial u'(t)}{\partial n} + \alpha u'(t)\right) + \frac{\partial u(t)}{\partial n} + \beta u = 0 \tag{6.11}$$

on $\Gamma_2$.

If that portion $\Gamma_1$ of $\partial G$ on which the Dirichlet condition is prescribed is sufficiently large, then the estimate (6.5) holds for some $K$ and all $\phi$ in $V$, and if $\beta \geqq 0$, then one can obtain the estimate (3.5) for $l$. Finally, the identities (6.6) and

(6.7) show that $D(M) \subseteq D(L)$ if $\alpha \equiv \beta$. From Theorem 5 it follows that if $u_0$ belongs to $D(L)$, that is, if

$$\frac{\partial u_0}{\partial n} + \alpha u_0 = 0$$

on $\Gamma_2$, then the solution of the generalized problem, $u(t)$, satisfies the strong boundary condition

$$\frac{\partial u(t)}{\partial n} + \alpha u(t) = 0 \tag{6.12}$$

on $\Gamma_2$ for all $t$ in $\mathbb{R}$. Note that in order to obtain the condition $D(M) \subseteq D(L)$, we had to choose $\alpha \equiv \beta$, and this makes the two conditions in (6.9) dependent, for the first can be obtained from the second by differentiation. Although the boundary conditions obtained by combining (6.10) with (6.11) or (6.12) for various choices of $\Gamma_1$, $\alpha$ and $\beta$ include all cases of physical interest, many other types can be introduced by adding more boundary integrals to the sesquilinear forms.

## REFERENCES

[1] S. AGMON, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand, New York, 1965.

[2] G. BARENBLAT, I. ZHELTOV AND I. KOCHIVA, *Basic concepts in the theory of seepage of homogeneous liquids in fissured rocks*, J. Appl. Math. Mech., 24 (1960), pp. 1286–1303.

[3] R. CARROLL, *Partial Differential Equations*, Harper and Row, New York, 1969.

[4] B. D. COLEMAN, R. J. DUFFIN AND V. J. MIZEL, *Instability, uniqueness and nonexistence theorems for the equation* $u_t = u_{xx} - u_{xtx}$ *on a strip*, Arch. Rational Mech. Anal., 19 (1965), pp. 100–116.

[5] J. DENY AND J. L. LIONS, *Sur les espaces de Beppo-Levi*, Ann. Inst. Fourier, 5 (1955), pp. 305–370.

[6] A. FRIEDMAN, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, N.J., 1964.

[7] C. GALPERN, *Cauchy problem for systems of linear partial differential equations*, Dokl. Akad. Nauk SSSR, 119 (1958), pp. 640–643.

[8] L. GARDING, *Dirichlet's problem for linear elliptic partial differential equations*, Math. Scand., 1 (1953), pp. 55–72.

[9] E. HILLE AND R. PHILLIPS, *Functional Analysis and Semi-Groups*, Colloquium Publications, vol. 31, American Mathematical Society, New York, 1957.

[10] L. HORMANDER, *On the theory of general partial differential operators*, Acta. Math., 94 (1955), pp. 161–248.

[11] J. HORVATH, *Topological Vector Spaces and Distributions*, Addison-Wesley, Reading, Mass., 1966.

[12] J. L. KELLEY, *General Topology*, Van Nostrand, New York, 1955.

[13] O. KELLOG, *Foundations of Potential Theory*, Dover, New York, 1953.

[14] A. KOSTACHENKO AND G. ESKIN, *Cauchy problem for an equation of Sobolev-Galpern type*, Trudy Moscov. Mat. Obshch., 10 (1961), pp. 273–284.

[15] P. D. LAX AND A. MILGRAM, *Parabolic equations*, Contributions to the Theory of Partial Differential Equations, Annals of Mathematics Studies, no. 33, Princeton University Press, Princeton, 1954, pp. 167–190.

[16] J. L. LIONS, *Lectures on Elliptic Partial Differential Equations*, Tata Institute, Bombay, 1957.

[17] ———, *Problemes aux limites dans les equations aux derivees partialles*, University of Montreal, Montreal, 1962.

[18] F. RIESZ AND B. SZ. NAGY, *Functional Analysis*, Frederick Ungar, New York, 1955.

[19] L. SCHWARTZ, *Theorie des distributions*, Hermann, Paris, 1966.

[20] R. E. SHOWALTER, *Partial differential equations of Sobolev–Galpern type*, Pacific J. Math., 31 (1969), pp. 787–794.

[21] ———, *Pseudo-parabolic partial differential equations*, Doctoral thesis, University of Illinois, Urbana, 1968.

[22] R. E. SHOWALTER AND T. W. TING, *Pseudo-parabolic partial differential equations*, this Journal, 1 (1970), pp. 1–26.

[23] S. SOBOLEV, *Some new problems in mathematical physics*, Izv. Akad. Nauk SSSR Ser. Mat., 18 (1954), pp. 3–50.

[24] D. W. TAYLOR, *Research on Consolidation of Clays*, Massachusetts Institute of Technology Press, Cambridge, 1952.

[25] T. W. TING, *Certain non-steady flows of second-order fluids*, Arch. Rational Mech. Anal., 14 (1963), pp. 1–26.

[26] ———, *Parabolic and pseudo-parabolic partial differential equations*, J. Math. Soc. Japan, 21 (1969), pp. 440–453.

[27] T. W. TING AND R. E. SHOWALTER, *Asymptotic behavior of solutions of pseudo-parabolic partial differential equations*, Proc. Amer. Math. Soc., to appear.

SIAM J. MATH. ANAL.
Vol. 1, No. 2, May 1970

# SOME EXTENSIONS OF LARDNER'S RELATIONS BETWEEN $_0F_3$ AND BESSEL FUNCTIONS*

B. C. CARLSON†

**Abstract.** The even and odd parts of a hypergeometric $_pF_q$-series are expressed as $_{2p}F_{2q+1}$-series, and inversely, a restricted $_{2p}F_{2q+1}$-series is expressed in terms of a pair of $_pF_q$-series. Generalizations are obtained by decomposing Meijer's $G$-function into two $G$-functions which are not, however, always even or odd. More fundamentally, it is shown that any solution of a generalized hypergeometric differential equation with restricted parameters can be expressed in terms of solutions of an equation with order half as large. The results are illustrated by applications to Bessel functions, Kelvin functions, generalized Fresnel integrals, and restricted $_4F_3$-series with unit argument. A number of restricted $G$-functions are expressed in terms of more familiar functions. A reducibility criterion is used to identify cases in which fourth order differential equations governing vibrations of beams and deformations of shells can be solved in terms of Bessel functions or $_1F_1$-functions.

**1. Introduction.** In a recent note T. J. Lardner [4] gave some connections between Bessel functions and hypergeometric $_0F_3$-series, in particular,

$$(1)\qquad {}_0F_3(\tfrac{1}{2},\tfrac{1}{2},1;z) = \tfrac{1}{2}J_0(4z^{1/4}) + \tfrac{1}{2}I_0(4z^{1/4})$$

and

$$(2)\qquad \begin{aligned} \operatorname{ber}(x) &= {}_0F_3(\tfrac{1}{2},\tfrac{1}{2},1;-x^4/256),\\ \operatorname{bei}(x) &= \frac{x^2}{4}\,{}_0F_3(\tfrac{3}{2},\tfrac{3}{2},1;-x^4/256). \end{aligned}$$

In the present note it is observed that (1) and (2) can be obtained by decomposing a $_0F_1$-series into even and odd parts and that this procedure leads to similar relations for the functions $J_\nu$, $I_\nu$, $\operatorname{ber}_\nu$, $\operatorname{bei}_\nu$. Additional results could be deduced by differentiation by following Lardner [4]. The separation into even and odd parts is then applied to a $_pF_q$-series and illustrated by several examples. The relations for $_pF_q$-series are in turn shown to be special cases of a decomposition of Meijer's $G$-function, which is then used to enlarge substantially the list of restricted $G$-functions known to be expressible in terms of simpler functions. A theorem about differential equations of hypergeometric type provides a final generalization as well as a better insight into the origin of the decompositions.

The results bear on some problems in elasticity, e.g., vibration of beams and deformation of certain types of shells, which are governed by fourth order differential equations of hypergeometric type. A number of such problems are listed, and for three of them it is determined what special values of the parameters allow them to be solved in terms of Bessel functions or other confluent hypergeometric functions.

**2. Even and odd parts of $_0F_1$.** We keep to the notations used in [2]. For any complex $c$ except 0 or a negative integer and for any finite complex $z$, the series $_0F_1(c;z)$ converges absolutely and can therefore be separated into two series

* Received by the editors May 8, 1969, and in revised form November 26, 1969.

† Physics and Mathematics Departments, Iowa State University, Ames, Iowa 50010. This research was performed at the Ames Laboratory of the United States Atomic Energy Commission.

consisting of even and odd terms:

$$
\begin{aligned}
{}_0F_1(c;z) &= \sum_{m=0}^{\infty} \frac{z^m}{(c)_m m!} \\
&= \sum_{n=0}^{\infty} \frac{z^{2n}}{(c)_{2n}(1)_{2n}} + \frac{z}{c}\sum_{n=0}^{\infty} \frac{z^{2n}}{(c+1)_{2n}(2)_{2n}} \\
&= \sum_{n=0}^{\infty} \frac{(z^2/16)^n}{(\frac{1}{2}c)_n(\frac{1}{2}c+\frac{1}{2})_n(\frac{1}{2})_n n!} + \frac{z}{c}\sum_{n=0}^{\infty} \frac{(z^2/16)^n}{(\frac{1}{2}c+\frac{1}{2})_n(\frac{1}{2}c+1)_n(\frac{3}{2})_n n!}.
\end{aligned}
$$

Hence

(3)
$$
{}_0F_1(c;z) = {}_0F_3(\tfrac{1}{2}, \tfrac{1}{2}c, \tfrac{1}{2}c+\tfrac{1}{2}; z^2/16) + \frac{z}{c}\,{}_0F_3(\tfrac{3}{2}, \tfrac{1}{2}c+\tfrac{1}{2}, \tfrac{1}{2}c+1; z^2/16),
$$

where $c \neq 0, -1, -2, \cdots$. Adding or subtracting the same equation with $z$ replaced by $-z$ and making suitable changes of notation, we find

(4)
$$
\begin{aligned}
{}_0F_3(\tfrac{1}{2}, c, c+\tfrac{1}{2}; z) &= \tfrac{1}{2}\,{}_0F_1(2c; 4z^{1/2}) + \tfrac{1}{2}\,{}_0F_1(2c; -4z^{1/2}), \\
{}_0F_3(\tfrac{3}{2}, c, c+\tfrac{1}{2}; z) &= \frac{2c-1}{8z^{1/2}}[{}_0F_1(2c-1; 4z^{1/2}) - {}_0F_1(2c-1; -4z^{1/2})],
\end{aligned}
$$

where $c \neq 0, -\frac{1}{2}, -1, -\frac{3}{2}, \cdots$. The right-hand side of the second equation has singularities at $z = 0$ and at $c = \frac{1}{2}$ which are removed by requiring continuity at these values.

For the Bessel functions

(5)
$$
\begin{aligned}
J_\nu(z) &= \frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1(\nu+1; -z^2/4), \\
I_\nu(z) &= \frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_1(\nu+1; z^2/4),
\end{aligned}
$$

we obtain from (3) the representations

(6)
$$
\begin{aligned}
J_\nu(z) &= \frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_3(\tfrac{1}{2}, \tfrac{1}{2}\nu+\tfrac{1}{2}, \tfrac{1}{2}\nu+1; z^4/256) \\
&\quad - \frac{(z/2)^{\nu+2}}{\Gamma(\nu+2)}\,{}_0F_3(\tfrac{3}{2}, \tfrac{1}{2}\nu+1, \tfrac{1}{2}\nu+\tfrac{3}{2}; z^4/256), \\
I_\nu(z) &= \frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_3(\tfrac{1}{2}, \tfrac{1}{2}\nu+\tfrac{1}{2}, \tfrac{1}{2}\nu+1; z^4/256) \\
&\quad + \frac{(z/2)^{\nu+2}}{\Gamma(\nu+2)}\,{}_0F_3(\tfrac{3}{2}, \tfrac{1}{2}\nu+1, \tfrac{1}{2}\nu+\tfrac{3}{2}; z^4/256).
\end{aligned}
$$

Adding and subtracting these equations, we find the inverse relations

(7)
$$
\begin{aligned}
\frac{1}{\Gamma(2c)}\,{}_0F_3(\tfrac{1}{2}, c, c+\tfrac{1}{2}; z) &= \tfrac{1}{2}(2z^{1/4})^{1-2c}[I_{2c-1}(4z^{1/4}) + J_{2c-1}(4z^{1/4})], \\
\frac{1}{\Gamma(2c)}\,{}_0F_3(\tfrac{3}{2}, c, c+\tfrac{1}{2}; z) &= \tfrac{1}{2}(2z^{1/4})^{-2c}[I_{2c-2}(4z^{1/4}) - J_{2c-2}(4z^{1/4})].
\end{aligned}
$$

Although exceptional values of the parameter $\nu$ or $c$ were excluded in the proof, (6) and (7) hold without restriction because both sides of each equation are entire functions of the parameter. The first equation of (7) reduces to (1) if $c = \frac{1}{2}$.

The Kelvin functions of general order are defined by

$$\operatorname{ber}_\nu(z) \pm i \operatorname{bei}_\nu(z) = J_\nu(z\, e^{\pm i3\pi/4}). \tag{8}$$

Solving these two equations for $\operatorname{ber}_\nu$ and $\operatorname{bei}_\nu$ and using the first equation of (6), we find

$$\begin{aligned}
\operatorname{ber}_\nu(z) &= \cos\frac{3\nu\pi}{4}\frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_3(\tfrac{1}{2}, \tfrac{1}{2}\nu + \tfrac{1}{2}, \tfrac{1}{2}\nu + 1; -z^4/256) \\
&\quad - \sin\frac{3\nu\pi}{4}\frac{(z/2)^{\nu+2}}{\Gamma(\nu+2)}\,{}_0F_3(\tfrac{3}{2}, \tfrac{1}{2}\nu + 1, \tfrac{1}{2}\nu + \tfrac{3}{2}; -z^4/256), \\
\operatorname{bei}_\nu(z) &= \sin\frac{3\nu\pi}{4}\frac{(z/2)^\nu}{\Gamma(\nu+1)}\,{}_0F_3(\tfrac{1}{2}, \tfrac{1}{2}\nu + \tfrac{1}{2}, \tfrac{1}{2}\nu + 1; -z^4/256) \\
&\quad + \cos\frac{3\nu\pi}{4}\frac{(z/2)^{\nu+2}}{\Gamma(\nu+2)}\,{}_0F_3(\tfrac{3}{2}, \tfrac{1}{2}\nu + 1, \tfrac{1}{2}\nu + \tfrac{3}{2}; -z^4/256).
\end{aligned} \tag{9}$$

This representation is valid for all complex $\nu$ and $z$ except the possibly singular points $z = 0$ and $z = \infty$. One of the two terms in each equation vanishes if $3\nu/2$ is an integer, and (2) is the case $\nu = 0$.

**3. Even and odd parts of ${}_pF_q$.** The separation into even and odd parts which led to (3) can easily be effected for any ${}_pF_q$-series ($p \leqq q + 1$) within its circle of convergence. Using $(a)$ to denote the array of parameters $a_1, \cdots, a_p$ and $(c)$ to denote $c_1, \cdots, c_q$, we find[1]

$$\begin{aligned}
{}_pF_q[(a);(c);z] &= {}_{2p}F_{2q+1}\left[\begin{matrix}(\tfrac{1}{2}a), (\tfrac{1}{2}a + \tfrac{1}{2}); \\ \tfrac{1}{2}, (\tfrac{1}{2}c), (\tfrac{1}{2}c + \tfrac{1}{2});\end{matrix}\; 4^{p-q-1}z^2\right] \\
&\quad + \frac{a_1 \cdots a_p}{c_1 \cdots c_q} z\, {}_{2p}F_{2q+1}\left[\begin{matrix}(\tfrac{1}{2}a + \tfrac{1}{2}), (\tfrac{1}{2}a + 1); \\ \tfrac{3}{2}, (\tfrac{1}{2}c + \tfrac{1}{2}), (\tfrac{1}{2}c + 1);\end{matrix}\; 4^{p-q-1}z^2\right],
\end{aligned} \tag{10}$$

where $c_j \neq 0, -1, -2, \cdots, j = 1, 2, \cdots, q$. Adding or subtracting the same equation with $z$ replaced by $-z$ and modifying the notation slightly, we have

$$\begin{aligned}
{}_{2p}F_{2q+1}\left[\begin{matrix}(a), (a + \tfrac{1}{2}); \\ \tfrac{1}{2}, (c), (c + \tfrac{1}{2});\end{matrix}\; z\right] &= \tfrac{1}{2}\,{}_pF_q[(2a);(2c);2^{1+q-p}z^{1/2}] \\
&\quad + \tfrac{1}{2}\,{}_pF_q[(2a);(2c);-2^{1+q-p}z^{1/2}]; \\
{}_{2p}F_{2q+1}\left[\begin{matrix}(a + \tfrac{1}{2}), (a + 1); \\ \tfrac{3}{2}, (c + \tfrac{1}{2}), (c + 1);\end{matrix}\; z\right] &= \frac{c_1 \cdots c_q}{a_1 \cdots a_p}\frac{z^{-1/2}}{4}\{{}_pF_q[(2a);(2c);2^{1+q-p}z^{1/2}] \\
&\quad - {}_pF_q[(2a);(2c);-2^{1+q-p}z^{1/2}]\},
\end{aligned} \tag{11}$$

[1] After the revision of this paper was completed, it was learned that (10) was stated in 1954 by MacRobert [13, (8)] in terms of his $E$-function. Since the formula was incidental to evaluation of a definite integral, he did not proceed to (11).

where $c_j \neq -\frac{1}{2}, -1, -\frac{3}{2}, \cdots, j = 1, 2, \cdots, q$. We assume also that $c_j \neq 0$ in the first equation, but the singularities of the right-hand side of the second equation at $c_j = 0$ and $a_i = 0$, $i = 1, 2, \cdots, p$, and $z = 0$ can all be removed by requiring continuity at these points.

If $p = q = 0$, (11) is the decomposition of the cosine and sine functions into a sum and difference of exponentials. If $p = 1, q = 0$, it is a pair of elementary cases of the Gauss hypergeometric function [1, (15.1.9) and (15.1.10)]. The case $p = 0, q = 1$ is (4). If $p = q = 1$ one can put $c = 2a$ to get two special cases of a known relation between ${}_2F_3$ and a product of Bessel functions [2, Eq. 7.2(49)]. A more interesting result follows from applying Kummer's transformation [2, Eq. 6.3(7)] to the ${}_1F_1$-series and then substituting (10) with $p = q = 1$ to obtain

$$(12)\quad \begin{aligned} {}_2F_3\left[\begin{matrix} a, a+\frac{1}{2}; \\ \frac{1}{2}, c, c+\frac{1}{2}; \end{matrix}\; z\right] &= \cosh(2z^{1/2})\,{}_2F_3\left[\begin{matrix} c-a, c-a+\frac{1}{2}; \\ \frac{1}{2}, c, c+\frac{1}{2}; \end{matrix}\; z\right] \\ &\quad + \frac{a-c}{c}\,2z^{1/2}\sinh(2z^{1/2})\,{}_2F_3\left[\begin{matrix} c-a+\frac{1}{2}, c-a+1; \\ \frac{3}{2}, c+\frac{1}{2}, c+1; \end{matrix}\; z\right], \\ {}_2F_3\left[\begin{matrix} a+\frac{1}{2}, a+1; \\ \frac{3}{2}, c+\frac{1}{2}, c+1; \end{matrix}\; z\right] &= \frac{c}{a}\,\frac{\sinh(2z^{1/2})}{2z^{1/2}}\,{}_2F_3\left[\begin{matrix} c-a, c-a+\frac{1}{2}; \\ \frac{1}{2}, c, c+\frac{1}{2}; \end{matrix}\; z\right] \\ &\quad + \frac{a-c}{a}\cosh(2z^{1/2})\,{}_2F_3\left[\begin{matrix} c-a+\frac{1}{2}, c-a+1; \\ \frac{3}{2}, c+\frac{1}{2}, c+1; \end{matrix}\; z\right], \end{aligned}$$

where $c \neq -\frac{1}{2}, -1, -\frac{3}{2}, \cdots$. We assume also that $c \neq 0$ in the first equation, but the singularities in the second equation at $a = 0$ and $c = 0$ are removable by continuity.

Specialization of (11), e.g., by putting $c_1 = a_1 + \frac{1}{2}$, gives decompositions for series with an odd number of numerator parameters and an even number of denominator parameters. Likewise, putting $c = a + \frac{1}{2}$ in (12) gives similar formulas for the generalized Fresnel integrals

$$(13)\quad \begin{aligned} \int_0^z t^{2a-1}\cos t\,dt &= \frac{z^{2a}}{2a}\,{}_1F_2(a; \tfrac{1}{2}, a+1; -z^2/4) \\ &= \frac{z^{2a}\cos z}{2a}\,{}_1F_2(1; a+\tfrac{1}{2}, a+1; -z^2/4) \\ &\quad + \frac{z^{2a+1}\sin z}{2a(2a+1)}\,{}_1F_2(1; a+1, a+\tfrac{3}{2}; -z^2/4), \\ \int_0^z t^{2a-1}\sin t\,dt &= \frac{z^{2a+1}}{2a+1}\,{}_1F_2(a+\tfrac{1}{2}; \tfrac{3}{2}, a+\tfrac{3}{2}; -z^2/4) \\ &= \frac{z^{2a}\sin z}{2a}\,{}_1F_2(1; a+\tfrac{1}{2}, a+1; -z^2/4) \\ &\quad - \frac{z^{2a+1}\cos z}{2a(2a+1)}\,{}_1F_2(1; a+1, a+\tfrac{3}{2}; -z^2/4), \end{aligned}$$

where Re $a > 0$ in the first integral and Re $a > -\frac{1}{2}$ in the second. The expressions in which $a$ appears in both denominator parameters of the ${}_1F_2$-series are suitable for numerical computation of the integrals if $|a| \gtrsim |z|$.

If $p = 2$, $q = 1$, application of Euler's transformation [2, Eq. 2.1(23) or 2.1(22)] to the ${}_2F_1$-series on the right-hand side of (11), followed by substitution of (10), leads to relations analogous to (12) between three ${}_4F_3$-series. Another type of result is obtained by putting $z = 1$ in (11); one ${}_2F_1$-series on the right-hand side can be summed by Gauss' theorem and the other series by Kummer's theorem [2, Eq. 2.8(47)] provided $c = a - b + \frac{1}{2}$. Thus we obtain the sums of two well-poised ${}_4F_3$-series with unit argument,

$$(14)\quad \begin{aligned} {}_4F_3\left[\begin{matrix} a, a+\frac{1}{2}, b, b+\frac{1}{2}; \\ \frac{1}{2}, a-b+1, a-b+\frac{1}{2}; \end{matrix}\ 1\right] &= 2^{-2a-1}\frac{\Gamma(1+2a-2b)}{\Gamma(1+a-2b)}\left[\frac{\Gamma(\frac{1}{2}-2b)}{\Gamma(\frac{1}{2}+a-2b)} + \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+a)}\right], \\ {}_4F_3\left[\begin{matrix} a+1, a+\frac{1}{2}, b+1, b+\frac{1}{2}; \\ \frac{3}{2}, a-b+1, a-b+\frac{3}{2}; \end{matrix}\ 1\right] &= \frac{2^{-2a-3}\Gamma(2+2a-2b)}{ab\Gamma(1+a-2b)}\left[\frac{\Gamma(\frac{1}{2}-2b)}{\Gamma(\frac{1}{2}+a-2b)} - \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+a)}\right], \end{aligned}$$

where Re $b < \frac{1}{4}$ and $a - b \neq -\frac{1}{2}, -1, -\frac{3}{2}, \cdots$. In the second equation the singularities at $a = 0$ and $b = 0$ are removable by continuity.

**4. Decomposition of Meijer's $G$-function.** The ${}_pF_q$-series is equivalent to a special case of Meijer's $G$-function, viz., $G^{1,p}_{p,q+1}$. In the general case the $G$-function may have a branch point at $z = 0$ and so may not have well-defined even and odd parts. However, it can always be decomposed into two $G$-functions which coincide with its even and odd parts when these are well-defined. We consider first

$$(15)\quad G^{m+2,n}_{p,q+2}\left(z\,\middle|\,\begin{matrix}(a)\\ 0,\frac{1}{2},(c)\end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\Gamma(-s)\Gamma(\frac{1}{2}-s)\prod_{j=1}^{m}\Gamma(c_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-c_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)} z^s\,ds,$$

where $L$ is a suitable contour [2, p. 207]. The poles of $\Gamma(-s)\Gamma(\frac{1}{2}-s)$ can be separated into integers and half-odd-integers according to the identity

$$(16)\quad \Gamma(-s)\Gamma(\tfrac{1}{2}-s) = \pi\, e^{\pm i\pi s}\left[\frac{\Gamma(-s)}{\Gamma(\frac{1}{2}+s)} \pm i\frac{\Gamma(\frac{1}{2}-s)}{\Gamma(1+s)}\right],$$

which is easily verified by using Euler's functional relation [2, Eq. 1.2(6)]. It follows that

$$(17)\quad \begin{aligned} G^{m+2,n}_{p,q+2}\left(z\,\middle|\,\begin{matrix}(a)\\ 0,\frac{1}{2},(c)\end{matrix}\right) &= \pi G^{m+1,n}_{p,q+2}\left(e^{\pm i\pi}z\,\middle|\,\begin{matrix}(a)\\ 0,(c),\frac{1}{2}\end{matrix}\right) \\ &\quad \pm i\pi G^{m+1,n}_{p,q+2}\left(e^{\pm i\pi}z\,\middle|\,\begin{matrix}(a)\\ \frac{1}{2},(c),0\end{matrix}\right). \end{aligned}$$

We shall use the following abbreviations:

(18)
$p, q, m, n$ are nonnegative integers with $n \leqq p$ and $m \leqq q$,

$(a, a+\frac{1}{2})$ is the ordered array $a_1, a_1+\frac{1}{2}, \cdots, a_n, a_n+\frac{1}{2}, \cdots, a_p, a_p+\frac{1}{2}$,

$(c, c+\frac{1}{2})$ is the ordered array $c_1, c_1+\frac{1}{2}, \cdots, c_m, c_m+\frac{1}{2}, \cdots, c_q, c_q+\frac{1}{2}$,

$$\rho = \tfrac{1}{2}(p+q+1) - m - n, \qquad \delta = q+1-p, \qquad \varepsilon = q-p-1 = \delta - 2,$$

$$\sigma = \sum_{j=1}^{q} c_j - \sum_{j=1}^{p} a_j + p - m - n,$$

$$\tau = 2\sum_{j=1}^{q} c_j - 2\sum_{j=1}^{p} a_j + p - m - n + 1.$$

By [2, Eq. 5.3(8)] there is no loss of generality in taking one parameter in the $G$-function to be zero. By [2, Eq. 5.3(10)], in which $x$ should be replaced by $x^2$ on the right-hand side, we have

$$G_{p,q+1}^{m+1,n}\left(z\,\middle|\,\begin{matrix}(a)\\0,(c)\end{matrix}\right) = 2^{\sigma}\pi^{\rho-1}G_{2p,2q+2}^{2m+2,2n}\left(\frac{z^2}{4^{\delta}}\,\middle|\,\begin{matrix}(\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2})\\0,\frac{1}{2},(\frac{1}{2}c,\frac{1}{2}c+\frac{1}{2})\end{matrix}\right). \tag{19}$$

Using (17) and [2, Eq. 5.3(8)], we find a generalization of (10), viz.,

$$\begin{aligned} G_{p,q+1}^{m+1,n}\left(z\,\middle|\,\begin{matrix}(a)\\0,(c)\end{matrix}\right) &= 2^{\sigma}\pi^{\rho}G_{2p,2q+2}^{2m+1,2n}\left(\frac{e^{\pm i\pi}z^2}{4^{\delta}}\,\middle|\,\begin{matrix}(\frac{1}{2}a,\frac{1}{2}a+\frac{1}{2})\\0,(\frac{1}{2}c,\frac{1}{2}c+\frac{1}{2}),\frac{1}{2}\end{matrix}\right)\\ &\quad - 2^{\sigma-\delta}\pi^{\rho}zG_{2p,2q+2}^{2m+1,2n}\left(\frac{e^{\pm i\pi}z^2}{4^{\delta}}\,\middle|\,\begin{matrix}(\frac{1}{2}a,\frac{1}{2}a-\frac{1}{2})\\0,(\frac{1}{2}c,\frac{1}{2}c-\frac{1}{2}),-\frac{1}{2}\end{matrix}\right). \end{aligned} \tag{20}$$

We now choose the lower (or upper) signs and replace $z$ by $e^{i\pi}z$ (or $e^{-i\pi}z$) to get a similar equation in which the arguments on the right side have the upper (or lower) signs. Adding or subtracting two equations with the same signs and modifying the notation suitably with the help of [2, Eq. 5.3(8)], we have

$$\begin{aligned} G_{2p,2q+2}^{2m+1,2n}\left(z\,\middle|\,\begin{matrix}(a,a+\frac{1}{2})\\0,(c,c+\frac{1}{2}),\frac{1}{2}\end{matrix}\right) &= 2^{-\tau}\pi^{-\rho}G_{p,q+1}^{m+1,n}\left(i2^{\delta}z^{1/2}\,\middle|\,\begin{matrix}(2a)\\0,(2c)\end{matrix}\right)\\ &\quad + 2^{-\tau}\pi^{-\rho}G_{p,q+1}^{m+1,n}\left(-i2^{\delta}z^{1/2}\,\middle|\,\begin{matrix}(2a)\\0,(2c)\end{matrix}\right),\\ G_{2p,2q+2}^{2m+1,2n}\left(z\,\middle|\,\begin{matrix}(a,a+\frac{1}{2})\\\frac{1}{2},(c,c+\frac{1}{2}),0\end{matrix}\right) &= i2^{-\tau}\pi^{-\rho}G_{p,q+1}^{m+1,n}\left(i2^{\delta}z^{1/2}\,\middle|\,\begin{matrix}(2a)\\0,(2c)\end{matrix}\right)\\ &\quad - i2^{-\tau}\pi^{-\rho}G_{p,q+1}^{m+1,n}\left(-i2^{\delta}z^{1/2}\,\middle|\,\begin{matrix}(2a)\\0,(2c)\end{matrix}\right), \end{aligned} \tag{21}$$

where $i$ stands for $e^{i\pi/2}$ and $-i$ stands for $e^{-i\pi/2}$.

These last equations permit several restricted $G$-functions not listed in [3, pp. 434–439] nor in [5, vol. 1, pp. 230–234] to be expressed in terms of more familiar functions. For present purposes Tricomi's $\Psi$ is more convenient than Whittaker's $W$, to which it is equivalent [2, Eq. 6.9(4)]. Omitting cases of the form $G^{1,n}_{p,q+1}$ with $p \leqq q + 1$, for which the ${}_pF_q$ notation seems preferable, we have

(22)
$$G^{3,0}_{0,4}(z|0, c, c + \tfrac{1}{2}, \tfrac{1}{2}) = 4z^{c/2}\left[\cos\frac{3c\pi}{2}\operatorname{ker}_{2c}(4z^{1/4}) - \sin\frac{3c\pi}{2}\operatorname{kei}_{2c}(4z^{1/4})\right]$$

(23)
$$G^{3,0}_{0,4}(z|\tfrac{1}{2}, c, c + \tfrac{1}{2}, 0) = -4z^{c/2}\left[\sin\frac{3c\pi}{2}\operatorname{ker}_{2c}(4z^{1/4}) + \cos\frac{3c\pi}{2}\operatorname{kei}_{2c}(4z^{1/4})\right],$$

(24)
$$G^{3,0}_{2,4}\left(z\,\middle|\,\begin{matrix} a, a + \frac{1}{2} \\ 0, c, c + \frac{1}{2}, \frac{1}{2}\end{matrix}\right) = 2^{2a-2c-1}\pi^{-1/2}[\exp(-i2z^{1/2})\Psi(2a - 2c, 1 - 2c; i2z^{1/2}) + \exp(i2z^{1/2})\Psi(2a - 2c, 1 - 2c; -i2z^{1/2})],$$

(25)
$$G^{3,0}_{2,4}\left(z\,\middle|\,\begin{matrix} a, a + \frac{1}{2} \\ \frac{1}{2}, c, c + \frac{1}{2}, 0\end{matrix}\right) = i2^{2a-2c-1}\pi^{-1/2}[\exp(-i2z^{1/2})\Psi(2a - 2c, 1 - 2c; i2z^{1/2}) - \exp(i2z^{1/2})\Psi(2a - 2c, 1 - 2c; -i2z^{1/2})],$$

(26)
$$G^{3,2}_{2,4}\left(z\,\middle|\,\begin{matrix} a, a + \frac{1}{2} \\ 0, c, c + \frac{1}{2}, \frac{1}{2}\end{matrix}\right) = 2^{2a-2c}\pi^{1/2}\Gamma(1 - 2a)\Gamma(1 + 2c - 2a) \cdot[\Psi(1 - 2a, 1 - 2c; 2iz^{1/2}) + \Psi(1 - 2a, 1 - 2c; -2iz^{1/2})],$$

(27)
$$G^{3,2}_{2,4}\left(z\,\middle|\,\begin{matrix} a, a + \frac{1}{2} \\ \frac{1}{2}, c, c + \frac{1}{2}, 0\end{matrix}\right) = i2^{2a-2c}\pi^{1/2}\Gamma(1 - 2a)\Gamma(1 + 2c - 2a) \cdot[\Psi(1 - 2a, 1 - 2c; 2iz^{1/2}) - \Psi(1 - 2a, 1 - 2c: -2iz^{1/2})].$$

Relations similar to (20) and (21) but involving functions of the type $G^{2m,2n+1}_{2p+2,2q}$ can be deduced either by use of [2, Eq. 5.3(9)] or by a separate proof entirely similar to the one above. Again taking one parameter to be zero without loss of generality, we find

(28)
$$G^{m,n+1}_{p+1,q}\left(z\,\middle|\,\begin{matrix} 0, (a) \\ (c)\end{matrix}\right) = 2^{\sigma+1}\pi^{\rho}G^{2m,2n+1}_{2p+2,2q}\left(\frac{e^{\pm i\pi}z^2}{4^{\varepsilon}}\,\middle|\,\begin{matrix} 0, (\frac{1}{2}a, \frac{1}{2}a + \frac{1}{2}), \frac{1}{2} \\ (\frac{1}{2}c, \frac{1}{2}c + \frac{1}{2})\end{matrix}\right) - 2^{\sigma+1-\varepsilon}\pi^{\rho}zG^{2m,2n+1}_{2p+2,2q}\left(\frac{e^{\pm i\pi}z^2}{4^{\varepsilon}}\,\middle|\,\begin{matrix} 0, (\frac{1}{2}a, \frac{1}{2}a - \frac{1}{2}), -\frac{1}{2} \\ (\frac{1}{2}c, \frac{1}{2}c - \frac{1}{2})\end{matrix}\right),$$

where $\varepsilon$, $\rho$, $\sigma$ are defined in (18). Inversely we have

$$G^{2m,2n+1}_{2p+2,2q}\left(z\left|\begin{matrix}0,(a,a+\frac{1}{2}),\frac{1}{2}\\(c,c+\frac{1}{2})\end{matrix}\right.\right) = 2^{-\tau-1}\pi^{-\rho}G^{m,n+1}_{p+1,q}\left(i2^{\varepsilon}z^{1/2}\left|\begin{matrix}0,(2a)\\(2c)\end{matrix}\right.\right) + 2^{-\tau-1}\pi^{-\rho}G^{m,n+1}_{p+1,q}\left(-i2^{\varepsilon}z^{1/2}\left|\begin{matrix}0,(2a)\\(2c)\end{matrix}\right.\right),$$

(29)

$$G^{2m,2n+1}_{2p+2,2q}\left(z\left|\begin{matrix}\frac{1}{2},(a,a+\frac{1}{2}),0\\(c,c+\frac{1}{2})\end{matrix}\right.\right) = i2^{-\tau-1}\pi^{-\rho}G^{m,n+1}_{p+1,q}\left(i2^{\varepsilon}z^{1/2}\left|\begin{matrix}0,(2a)\\(2c)\end{matrix}\right.\right) - i2^{-\tau-1}\pi^{-\rho}G^{m,n+1}_{p+1,q}\left(-i2^{\varepsilon}z^{1/2}\left|\begin{matrix}0,(2a)\\(2c)\end{matrix}\right.\right).$$

To (22)–(27) we can now add

(30)
$$G^{2,1}_{2,4}\left(z\left|\begin{matrix}0,\frac{1}{2}\\c,c+\frac{1}{2},d,d+\frac{1}{2}\end{matrix}\right.\right) = \frac{\Gamma(1+2c)\pi^{-1/2}z^c}{\Gamma(1+2c-2d)2^{2d+1}} \cdot[e^{ic\pi}{}_1F_1(1+2c;1+2c-2d;-i2z^{1/2}) + e^{-ic\pi}{}_1F_1(1+2c;1+2c-2d;i2z^{1/2})],$$

(31)
$$G^{2,1}_{2,4}\left(z\left|\begin{matrix}\frac{1}{2},0\\c,c+\frac{1}{2},d,d+\frac{1}{2}\end{matrix}\right.\right) = \frac{i\Gamma(1+2c)\pi^{-1/2}z^c}{\Gamma(1+2c-2d)2^{2d+1}} \cdot[e^{ic\pi}{}_1F_1(1+2c;1+2c-2d;-i2z^{1/2}) - e^{-ic\pi}{}_1F_1(1+2c;1+2c-2d;i2z^{1/2})],$$

(32)
$$G^{4,1}_{2,4}\left(z\left|\begin{matrix}0,\frac{1}{2}\\c,c+\frac{1}{2},d,d+\frac{1}{2}\end{matrix}\right.\right) = 2^{-2d}\pi^{1/2}\Gamma(2c+1)\Gamma(2d+1)z^c \cdot[e^{ic\pi}\Psi(1+2c,1+2c-2d;i2z^{1/2}) + e^{-ic\pi}\Psi(1+2c,1+2c-2d;-i2z^{1/2})],$$

(33)
$$G^{4,1}_{2,4}\left(z\left|\begin{matrix}\frac{1}{2},0\\c,c+\frac{1}{2},d,d+\frac{1}{2}\end{matrix}\right.\right) = i2^{-2d}\pi^{1/2}\Gamma(2c+1)\Gamma(2d+1)z^c \cdot[e^{ic\pi}\Psi(1+2c,1+2c-2d;i2z^{1/2}) - e^{-ic\pi}\Psi(1+2c,1+2c-2d;-i2z^{1/2})].$$

In the last four equations $G^{2,1}_{2,4}$ or $G^{4,1}_{2,4}$ provides an interpretation for the symbol ${}_4F_1$, as does $\Psi$ for ${}_2F_0$. More generally, $G^{n,1}_{q+1,p}$ and $G^{1,n}_{p,q+1}$ are equivalent to ${}_pF_q$ if $p \leqq q+1$ (see [5, vol. 1, p. 147]) and provide an interpretation of the symbol ${}_pF_q$ if $p > q+1$. With this interpretation (10) and (11) are valid also for $p > q+1$.

By means of reduction formulas such as [5, Eqs. 5.4(1) and 5.4(2)], specialization of the preceding expressions for $G^{3,0}_{2,4}$, $G^{3,2}_{2,4}$, $G^{2,1}_{2,4}$ and $G^{4,1}_{2,4}$ yields some further relations in which $\gamma(a,x)$ and $\Gamma(a,x)$ denote incomplete gamma functions [2, Chap. 9]:

(34) $$G^{2,0}_{1,3}\left(z\left|\begin{matrix}0\\c,c+\frac{1}{2},0\end{matrix}\right.\right) = \pi^{-1/2}z^c\sin(2z^{1/2}-c\pi),$$

(35) $$G^{2,0}_{1,3}\left(z\left|\begin{matrix}0\\c,c+\frac{1}{2},1\end{matrix}\right.\right) = \pi^{-1/2}z^c[z^{1/2}\cos(2z^{1/2}-c\pi) + c\sin(2z^{1/2}-c\pi)],$$

(36) $$G^{2,0}_{1,3}\left(z\left|\begin{matrix}1\\c,c+\frac{1}{2},0\end{matrix}\right.\right) = i2^{-2c}\pi^{-1/2}[\gamma(2c,-i2z^{1/2}) - \gamma(2c,i2z^{1/2})],$$

$$(37)\quad G^{2,0}_{1,3}\left(z\,\middle|\,\begin{matrix} c \\ 0, c+1, \frac{1}{2}\end{matrix}\right) = \pi^{-1/2}(c\cos 2z^{1/2} + z^{1/2}\sin 2z^{1/2}),$$

$$(38)\quad G^{2,0}_{1,3}\left(z\,\middle|\,\begin{matrix} c \\ \frac{1}{2}, c+1, 0\end{matrix}\right) = \pi^{-1/2}(c\sin 2z^{1/2} - z^{1/2}\cos 2z^{1/2}),$$

$$(39)\quad G^{2,0}_{1,3}\left(z\,\middle|\,\begin{matrix} c+1 \\ 0, c, \frac{1}{2}\end{matrix}\right) = 2^{2c}\pi^{-1/2}z^{c}[e^{ic\pi}\Gamma(-2c, i2z^{1/2}) + e^{-ic\pi}\Gamma(-2c, -i2z^{1/2})],$$

$$(40)\quad G^{2,0}_{1,3}\left(z\,\middle|\,\begin{matrix} c+1 \\ \frac{1}{2}, c, 0\end{matrix}\right) = i2^{2c}\pi^{-1/2}z^{c}[e^{ic\pi}\Gamma(-2c, i2z^{1/2}) - e^{-ic\pi}\Gamma(-2c, -i2z^{1/2})],$$

$$(41)\quad G^{3,1}_{1,3}\left(z\,\middle|\,\begin{matrix} 0 \\ 0, c, c+\frac{1}{2}\end{matrix}\right) = \pi^{1/2}\Gamma(1+2c)z^{c}[\exp(i2z^{1/2} + ic\pi)\Gamma(-2c, i2z^{1/2}) + \exp(-i2z^{1/2} - ic\pi)\Gamma(-2c, -i2z^{1/2})],$$

$$(42)\quad G^{3,1}_{1,3}\left(z\,\middle|\,\begin{matrix} 0 \\ 1, c, c+\frac{1}{2}\end{matrix}\right) = i2^{-2c}\pi^{1/2}\Gamma(1+2c)[\Psi(2, 2-2c; i2z^{1/2}) - \Psi(2, 2-2c; -i2z^{1/2})].$$

**5. Reducibility criterion applied to problems in elasticity.** In the past ten years hypergeometric functions of types ${}_0F_3$ and ${}_2F_3$ have been applied to a number of problems in elasticity such as vibration of beams and deformation of shells. The differential equations of these problems are linear fourth order equations, and under suitable assumptions about geometrical and mechanical properties they can be transformed into the type of equation satisfied by ${}_pF_q$, viz.,

$$\left\{\delta \prod_{j=1}^{q}(\delta + c_j - 1) - z\prod_{i=1}^{p}(\delta + a_i)\right\}y = 0 \qquad \left(\delta = z\frac{d}{dz}\right),$$

where an empty product is understood to be unity. The ${}_2F_3$-equation describes the elastic deformations of a thin shallow conical shell [7], as well as an approximate model of a deep conical shell, and more generally the deformations of a class of shallow shells of revolution in which the radius varies as a power of the axial coordinate [9]. The ${}_0F_3$-equation is encountered in the bending vibrations of a tapered beam of which the height and width vary either as powers [8] or as exponential functions [12] of the longitudinal coordinate. Symmetric deformation of a circular cylindrical shell also is governed by the ${}_0F_3$-equation if the wall thickness varies either as an exponential function [10] or a power [11] of the axial coordinate. Other problems of essentially the same nature are the vibration of a circular disk with variable thickness, the bending of a beam with variable properties on an elastic foundation, and the bending and twisting of a thin strip with variable thickness.

Since special cases of some of these problems, e.g., the vibrating wedge-shaped beam, had previously been solved in terms of Bessel functions, a connection between ${}_0F_1$ and ${}_0F_3$ was indicated [8]. When established explicitly, the connection was used to pick out other special cases for which the solutions could be reduced to tabulated Bessel functions. In the deformation problem of a cylindrical shell with wall thickness proportional to $x^n$, where $x$ is the axial distance, Lardner [11] thus showed that the solution can be expressed in terms of Bessel functions if $1/(2 - n)$ is a nonzero integer, i.e., if $n = 1, 3, \frac{3}{2}, \frac{5}{2}, \frac{5}{3}, \cdots$. The results of the present paper suggest that there are additional cases of similar kind.

To determine what cases are reducible, it is convenient to include among the denominator parameters of a ${}_{2p}F_{2q+1}$-series an additional parameter with value unity, since $n! = (1)_n$ is always part of the denominator of the $n$th term. By (11) the series can be reduced to two ${}_pF_q$-series if its denominator parameters, and similarly its numerator parameters, occur in pairs whose members differ by $\frac{1}{2}$. Since differentiation can be used to change a parameter by unity, it is sufficient that the members of each pair differ by half an odd integer.

If this criterion is applied to the problem of the cylindrical shell mentioned above, it is found from [11, (12) and (13)] that reducibility occurs if $1/(2 - n)$ is either a nonzero integer or a half-odd-integer. In the latter case, corresponding to $n = 0, 4, \frac{4}{3}, \frac{8}{3}, \frac{8}{5}, \frac{12}{5}, \cdots$, the solution can be expressed in terms of spherical Bessel functions. Only the case $n = 0$ appears to have been noted previously.

We consider next the transverse vibrations of a nonuniform beam, which also are governed by a ${}_0F_3$-equation. Following the notation used in [8], we assume that the flexural rigidity varies as $x^m$ and the mass per unit length as $x^n$, where $x$ is the axial coordinate. The criterion for reducibility is satisfied if either $1/(4 - m + n)$ or $(2 - m)/(4 - m + n)$ is half an odd integer, and the solution can be expressed in terms of Bessel functions in these cases. Only the uniform beam ($m = n = 0$) and the case $m - n = 2$ seem to have been noted previously.

The reducible cases include some degenerate cases (e.g., $m = 3,\ n = 1$) in which the general solution contains logarithmic terms [8]. When this occurs, it may be useful to know that even the logarithmic solutions of the ${}_0F_3$-equation can be found by solving a ${}_0F_1$-equation, as is shown by the following theorem.

THEOREM. *Let $y = f(z)$ be a solution of*

$$\left\{\prod_{j=1}^{q+1} (\delta + c_j - 1) - z \prod_{i=1}^{p} (\delta + a_i)\right\} y = 0 \qquad \left(\delta = z\frac{d}{dz}\right). \tag{43}$$

*If $u_\pm = f(\pm 2^{1+q-p} z^{1/2})$, then $w = u_+$ and $w = u_-$ are solutions of*

$$\left\{\prod_{j=1}^{q+1}\left(\delta + \frac{c_j}{2} - 1\right)\left(\delta + \frac{c_j + 1}{2} - 1\right) - z \prod_{i=1}^{p}\left(\delta + \frac{a_i}{2}\right)\left(\delta + \frac{a_i + 1}{2}\right)\right\} w = 0. \tag{44}$$

*Proof.* By taking $\pm 2^{1+q-p} z^{1/2}$ as the independent variable, we readily verify that $L_+ u_+ = L_- u_- = 0$, where

$$L_\pm = \prod_{j=1}^{q+1}\left(\delta + \frac{c_j + 1}{2} - 1\right) \mp z^{1/2} \prod_{i=1}^{p}\left(\delta + \frac{a_i}{2}\right). \tag{45}$$

If we define

$$M_\pm = \prod_{j=1}^{q+1}\left(\delta + \frac{c_j}{2} - 1\right) \mp z^{1/2} \prod_{i=1}^{p}\left(\delta + \frac{a_i}{2}\right), \tag{46}$$

then the operator $M_- L_+ = M_+ L_-$ annihilates both $u_+$ and $u_-$ and has the form given in brackets in (44).

It follows from the theorem that the general solution of (44) can be obtained at once from the general solution of (43) except in the very unusual case that (43) has a solution that is even or odd in $z$. The theorem provides a rationale for the existence of the explicit decompositions given in earlier sections and goes beyond them to imply the existence of decompositions for solutions in degenerate cases.

We consider finally a deformation problem governed by a ${}_2F_3$-equation and ask for those special cases in which the solutions can be reduced to solutions of the confluent hypergeometric equation. Let the shape of a shallow shell of revolution be described in cylindrical coordinates by

$$z = \beta\left[\left(\frac{r}{a}\right)^{s+1} - 1\right], \qquad s \neq -1, \tag{47}$$

where $\beta$, $a$ and $s$ are constants, and assume that the deformation varies with azimuth as $\cos n\theta$, where $n$ is a nonnegative integer. The parameters of the ${}_2F_3$-equation are given as functions of $n$ and $s$ by [9, (10) and (11)]. The criterion for reducibility to solutions of a ${}_1F_1$-equation is found by detailed examination to be satisfied only in the following cases: $n = 0$ and $s + 1 = \pm 4, \pm\frac{4}{3}, \pm\frac{4}{5}, \pm\frac{4}{7}, \cdots$; or $n = 4$ and $s = 3$ or $\frac{1}{3}$. It is noteworthy that Lardner [9] has expressed the exact solution of the general case in terms of $G$-functions for convenience in discussing the asymptotic behavior at large values of $r$. Hence (21) and (29) of the present paper may be used when $n$ and $s$ have any of the listed values.

## REFERENCES


[1] M. Abramowitz and I. Stegun, eds., *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series, vol. 55, U.S. Government Printing Office, Washington, D.C., 1964.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, vol. 1, McGraw-Hill, New York, 1953.

[3] ———, *Tables of Integral Transforms*, vol. 2, McGraw-Hill, New York, 1954.

[4] T. J. Lardner, *Relations between ${}_0F_3$ and Bessel functions*, SIAM Rev., 11 (1969), pp. 69–72.

[5] Y. L. Luke, *The Special Functions and their Approximations*, Academic Press, New York, 1969.

[6] L. J. Slater, *Generalized Hypergeometric Functions*, Cambridge University Press, Cambridge, 1966.

[7] M. D. McIlroy, *Linear deformations of conical shells*, J. Aero/Space Sci., 26 (1959), pp. 253–254.

[8] H.-C. Wang, *Generalized hypergeometric function solutions on the transverse vibration of a class of nonuniform beams*, Trans. ASME Ser. E. J. Appl. Mech., 34 (1967), pp. 702–708.

[9] T. J. Lardner, *Exact solution of the equations for shallow shells of revolution*, Quart. Appl. Math., 26 (1968), pp. 445–450.

[10] T. J. Lardner and C. R. Steele, *Symmetric deformations of circular cylindrical elastic shells of exponentially varying thickness*, Trans. ASME Ser. E. J. Appl. Mech., 35 (1968), pp. 169–170.

[11] T. J. Lardner, *Symmetric deformation of a circular cylindrical shell of variable wall thickness*, Z. Angew. Math. Phys., 19 (1968), pp. 270–278.

[12] ———, *Vibration of beams with exponentially varying properties*, Acta Mech., 6 (1968), pp. 197–202.

[13] T. M. MacRobert, *Integrals involving a modified Bessel function of the second kind and an E-function*, Proc. Glasgow Math. Assoc., 2 (1954), pp. 93–96.

SIAM J. MATH. ANAL.
Vol. 1, No. 2, May 1970

# SOME REDUCTION FORMULAS FOR GENERALIZED HYPERGEOMETRIC FUNCTIONS*

L. CARLITZ†

**1.** Lardner [1] has exhibited a relationship between Bessel functions and the generalized hypergeometric function

$$ {}_0F_3(\alpha + 1, \beta + \tfrac{1}{2}, \gamma + \tfrac{1}{2}; x) = \sum_{n=0}^{\infty} \frac{x^n}{n!(\alpha + 1)_n(\beta + \frac{1}{2})_n(\gamma + \frac{1}{2})_n}, $$

where $\alpha$ is a nonnegative integer and $\beta$, $\gamma$ are arbitrary integers. This depends on the identity

$$ {}_0F_3(1, \tfrac{1}{2}, \tfrac{1}{2}; x) = \tfrac{1}{2}[J_0(z) + I_0(z)], \qquad z = 4x^{1/4}, $$

and certain differentiation formulas. For example it is shown that

$$ \begin{aligned} &{}_0F_3(m + 1, m - p + \tfrac{1}{2}, m - p - q + \tfrac{1}{2}; x) \\ &\quad = Cx^{p+q-m+1/2}\left(\frac{d}{dx}\right)^q \left(\frac{d}{dx}x\frac{d}{dx}\right)^p x^{m-1/2}\left(\frac{d}{dx}\right)^m {}_0F_3(1, \tfrac{1}{2}, \tfrac{1}{2}; x), \end{aligned} \tag{1.1} $$

where $m$, $p$, $q$ are nonnegative integers and $C$ is an explicit constant. A special case of interest is

$$ {}_0F_3(1, -p + \tfrac{1}{2}, -p + \tfrac{1}{2}; -x) = \frac{x^{p+1/2}}{(\frac{1}{2})_p(\frac{1}{2})_p}\left(\frac{d}{dx}x\frac{d}{dx}\right)^p x^{-1/2}{}_0F_3(1, \tfrac{1}{2}, \tfrac{1}{2}; -x). \tag{1.2} $$

As an example of a relation like (1.1) with $p$ or $q$ negative, Lardner cites

$$ {}_0F_3(1, N + \tfrac{1}{2}, N + \tfrac{1}{2}; x) = (\tfrac{1}{2})_N(\tfrac{1}{2})_N\left(\frac{d}{dx}x\frac{d}{dx}\right)^N {}_0F_3(1, \tfrac{1}{2}, \tfrac{1}{2}; x). \tag{1.3} $$

It is evident from these results that, in order to obtain explicit formulas, it is necessary to expand the differential operator

$$ (DxD)^n \qquad \left(D = \frac{d}{dx}\right). $$

We show that this can be done quite simply.

**2.** It is easily verified that

$$ \begin{aligned} DxD &= D + xD^2, \\ (DxD)^2 &= 2D^2 + 4xD^3 + x^2D^4, \\ (DxD)^3 &= 6D^3 + 18xD^4 + 9x^2D^5 + x^3D^6. \end{aligned} $$

Generally we may put

$$ (DxD)^n = \sum_{s=0}^{n} A(n, s)x^sD^{n+s}. \tag{2.1} $$

* Received by the editors September 8, 1969.

† Department of Mathematics, Duke University, Durham, North Carolina 27706. This research was supported in part by the National Science Foundation under Grant GP-7855.

Then

$$xD(DxD)^n = \sum_{s=0}^{n} A(n,s)(sx^sD^{n+s} + x^{s+1}D^{n+s+1}),$$

$$(DxD)(DxD)^n = \sum_{s=0}^{n} A(n,s)[s^2x^{s-1}D^{n+s} + (2s+1)x^sD^{n+s+1} + x^{s+1}D^{n+s+2}],$$

so that

$$(DxD)^{n+1} = \sum_{s=0}^{n} [A(n,s-1) + (2s+1)A(n,s) + (s+1)^2A(n,s+1)]x^sD^{n+s+1}.$$

Comparison with (2.1) yields the recurrence

$$A(n+1,s) = A(n,s-1) + (2s+1)A(n,s) + (s+1)^2A(n,s+1). \tag{2.2}$$

By means of (2.2) we can easily compute Table 1 for $A(n, s)$.

TABLE 1

| | | | | |
|---|---|---|---|---|
| 1 | 1 | | | |
| 2 | 4 | 1 | | |
| 6 | 18 | 9 | 1 | |
| 24 | 96 | 72 | 16 | 1 |

It may be of interest to note that (2.2) can be written in the form

$$\begin{aligned} A(n+1,s) &= [E^{-1} + 2s + 1 + (s+1)^2E]A(n,s) \\ &= [E^{-2} + (2s+1)E^{-1} + (s+1)^2]A(n,s+1), \end{aligned}$$

where $Ef(n, s) = f(n, s + 1)$. This implies

$$A(n+1,s) = (E^{-1} + s + 1)^2A(n,s+1), \tag{2.3}$$

and more generally

$$A(n+k,s) = (E^{-1}+s+1)^2(E^{-1}+s+2)^2 \cdots (E^{-1}+s+k)^2A(n,s+k). \tag{2.4}$$

Note that the operators $E^{-1} + s + a$, $E^{-1} + s + b$ commute.

The following formula is easily verified by mathematical induction:

$$A(n,s) = \binom{n}{s}\frac{n!}{s!} = \binom{n}{s}^2 (n-s)!. \tag{2.5}$$

**3.** In order to use (1.1) or (1.2) we require the expansion of the operator

$$(xDx)^nx^a \tag{3.1}$$

with $a$ arbitrary. By (2.1) and (2.5),

$$
\begin{aligned}
(xDx)^n x^a &= \sum_{s=0}^{n} \binom{n}{s}^2 (n-s)!\, x^s D^{n+s} x^a \\
&= \sum_{s=0}^{n} \binom{n}{s}^2 (n-s)! \sum_{j=0}^{n+s} \binom{n+s}{j} a(a-1)\cdots(a-j+1) x^{a+s-j} D^{n+s-j} \\
&= \sum_{s+j\leq n} (-1)^j (-a)_j \frac{n!\,n!\,(n+s+j)!}{(s+j)!\,(s+j)!\,(n-s-j)!\,j!\,(n+s)!} x^{a+s} D^{n+s} \\
&= \sum_{s=0}^{n} \frac{n!\,n!}{s!\,s!\,(n-s)!} x^{a+s} D^{n+s} \sum_{j=0}^{n-s} \frac{(-a)_j(-n+s)_j(n+s+1)_j}{j!\,(s+1)_j(s+1)_j}.
\end{aligned}
$$

Therefore

$$
(3.2) \qquad (xDx)^n x^a = \sum_{s=0}^{n} \binom{n}{s}^2 (n-s)!\, {}_3F_2\left[\begin{matrix} -a, -n+s, n+s+1 \\ s+1, s+1 \end{matrix}\right] x^{a+s} D^{n+s}.
$$

It does not seem possible to sum the ${}_3F_2$.

**4.** We remark that the operator $(DxD)^n$ can be conveniently applied to more general hypergeometric functions. Thus for example it is easily verified that

$$
(4.1) \qquad
\begin{aligned}
&(DxD)^n {}_pF_{q+1}\left[\begin{matrix} \alpha_1, \cdots, \alpha_p; \\ 1, \beta_1, \cdots, \beta_q; \end{matrix}\ x\right] \\
&\qquad = \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} {}_pF_{q+1}\left[\begin{matrix} \alpha_1+n, \cdots, \alpha_p+n; \\ 1, \beta_1+n, \cdots, \beta_q+n; \end{matrix}\ x\right],
\end{aligned}
$$

where as usual

$$
{}_pF_{q+1}\left[\begin{matrix} \alpha_1, \cdots, \alpha_p; \\ 1, \beta_1, \cdots, \beta_q; \end{matrix}\ x\right] = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k}{k!\,k!\,(\beta_1)_k \cdots (\beta_q)_k} x^k.
$$

We remark also that the formula

$$
(4.2) \qquad (DxD)^n = \sum_{s=0}^{n} \binom{n}{s}^2 (n-s)!\, x^s D^{n+s}
$$

has as its inverse

$$
(4.3) \qquad x^n D^{2n} = \sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s}^2 (n-s)!\, (DxD)^s D^{n-s}.
$$

This is an instance of the equivalence of

$$
y_n = \sum_{s=0}^{n} \binom{n}{s}^2 (n-s)!\, x_s, \qquad n = 0, 1, 2, \cdots,
$$

$$
x_n = \sum_{s=0}^{n} (-1)^{n-s} \binom{n}{s}^2 (n-s)!\, y_s, \qquad n = 0, 1, 2, \cdots.
$$

## REFERENCE

[1] T. J. LARDNER, *Relations between ${}_0F_3$ and Bessel functions*, SIAM Rev., 11 (1969), pp. 69–72.

SIAM J. MATH. ANAL.
Vol. 1, No. 2, May 1970

# LIE THEORY AND SPECIAL FUNCTIONS SATISFYING SECOND ORDER NONHOMOGENEOUS DIFFERENTIAL EQUATIONS*

WILLARD MILLER, JR.†

**Abstract.** A Lie algebraic technique is given for the systematic study of families of special functions which satisfy second order nonhomogeneous differential equations such that the solutions of the homogeneous equations are of hypergeometric type. Among the functions obtained by this technique are the functions of Struve, Lommel and various nonhomogeneous hypergeometric, confluent hypergeometric and parabolic cylinder functions.

**Introduction.** Among the classes of functions which are useful enough to be considered "special" there are families which satisfy second order nonhomogeneous differential equations whose homogeneous parts are equations of hypergeometric type. Such families typically also satisfy third order homogeneous equations, two linearly independent solutions of which are functions of hypergeometric type, i.e., hypergeometric, confluent hypergeometric, parabolic cylinder or Bessel functions. Furthermore, the families obey simple recurrence formulas. The best known families are the functions of Struve and Lommel, related to Bessel functions, but there are many more which have been studied [1]–[4]. Recently there has appeared a book on this subject by A. W. Babister [5]. However, it is not clear from Babister's book why certain families are studied and others are not.

Here we shed some light on this problem by adopting a thorough Lie algebraic approach. In [6], the irreducible representations of the Lie algebras $\mathcal{G}(1, 0)$, $\mathcal{G}(0, 1)$ and $\mathcal{G}(0, 0)$ were determined, and models of these representations were constructed in terms of first order differential operators (the operator types $A, B, C', C'', D'$) acting on spaces of functions of two complex variables $z, t$. The basis vectors $f_m(z, t) = g_m(z)t^m$ of such irreducible representations turned out to be such that the $g_m(z)$ were functions of hypergeometric type. This connection between Lie algebras and special functions led to recurrence relations, differential equations, generating functions and addition theorems for the functions of hypergeometric type.

In this paper we use the type $A, \cdots, D'$ operators as building blocks to construct more complicated models of irreducible representations of the Lie algebras $\mathcal{G}(a, b)$. Some of the Lie algebra elements in our models will now be second order differential operators. Furthermore, the models will be constructed in such a way that the basis vectors are functions of the kind mentioned in the title of this paper. In the usual way, this connection between Lie algebras and special functions will lead to recurrence formulas, differential equations and generating functions.

The concept of using models of Lie algebras as building blocks to construct more complicated models is a familiar one in theoretical physics where, for example, the annihilation and creation operators for bosons and fermions are used to construct representations of the semisimple Lie algebras [7]. In this paper we have constructed eight classes of models which are as simple as possible. In

* Received by the editors September 15, 1969.
† School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455.

particular, to obtain our new models we modify only one of the operators in a generating basis for the old model. However, even these very simple models suffice to construct most of the well-known nonhomogeneous functions as well as a number of interesting functions which the author has not seen before in the literature. We could obtain many more such functions by constructing more complicated models, by considering reducible Lie algebra representations or by examining new Lie algebras. At any rate, the analysis would be systematic and purely algebraic.

**1.** We begin with differential operators $J^+, J^-, J^3$ and the identity operator $E$ acting on a space of analytic functions $\mathscr{V}$ of the two complex variables $z, t$ and satisfying the commutation relations

$$
\begin{aligned}
[J^+, J^-] &= 2a^2J^3 - bE, \\
[J^3, J^\pm] &= \pm J^\pm, \\
[E, J^\pm] &= [E, J^3] = 0,
\end{aligned}
\tag{1.1}
$$

where 0 is the zero operator and $a, b$ are fixed complex numbers. Here, $[A, B] = AB - BA$ for linear operators $A$ and $B$ on $\mathscr{V}$. We assume that the $J$-operators take the form

$$J^\pm = t^{\pm 1}\left(j^\pm(z)\frac{\partial}{\partial z} + k^\pm(z)t\frac{\partial}{\partial t} + l^\pm(z)\right), \tag{1.2}$$

$$J^3 = t\frac{\partial}{\partial t}. \tag{1.3}$$

(This is no loss of generality. See [6, Chap. 8].) Clearly, $J^\pm, J^3, E$ form a basis for a four-dimensional Lie algebra $\mathscr{G}(a, b)$. In fact, it is easily shown that all of the algebras $\mathscr{G}(a, b)$ are isomorphic to one of $\mathscr{G}(0, 0)$, $\mathscr{G}(0, 1)$ or $\mathscr{G}(1, 0)$ (see [6]). For each of the algebras $\mathscr{G}(a, b)$ it is straightforward to verify that the invariant operator

$$C_{a,b} = J^+J^- + a^2J^3J^3 - a^2J^3 - bJ^3 \tag{1.4}$$

commutes with all elements of the Lie algebra. It can be shown, for a large class of irreducible representations of $\mathscr{G}(a, b)$, that if the $J$-operators act irreducibly on some proper subspace of $\mathscr{V}$ then $C_{a,b}$ is a multiple of the identity operator on that subspace.

In [6] the possible $J$-operators on $\mathscr{V}$ satisfying (1.1) are derived and used to construct irreducible representations of $\mathscr{G}(0, 0)$, $\mathscr{G}(0, 1)$ and $\mathscr{G}(1, 0)$ on subspaces $\mathscr{W}$ of $\mathscr{V}$. In each case a basis $\{f_m(z, t)\}$ for $\mathscr{W}$ is constructed in the form

$$f_m(z, t) = g_m(z)t^m \tag{1.5}$$

such that $J^3f_m = mf_m$, $C_{a,b}f_m = \lambda f_m$, i.e., the basis vectors are simultaneous eigenvectors of the commuting operators $J^3$, $C_{a,b}$. The constant $\lambda$ is fixed by the representation, while $m$ runs over the eigenvalues of $J^3$ acting on $\mathscr{W}$. The $J$-operators which lead to nontrivial special functions $g_m(z)$ are listed in Table 1. In each case when the trivial $t$-dependence is removed from the equation $C_{a,b}f_m(z, t) = \lambda f_m(z, t)$ there remains a second order homogeneous ordinary differential equation for the special function $g_m(z)$.

TABLE 1

| $J$-operators | Lie algebra | $g_m(z)$ |
|---|---|---|
| type $A$ | $\mathscr{G}(1, 0)$ | hypergeometric functions |
| type $B$ | $\mathscr{G}(1, 0)$ | confluent hypergeometric functions |
| type $C'$ | $\mathscr{G}(0, 1)$ | confluent hypergeometric functions |
| type $C''$ | $\mathscr{G}(0, 0)$ | Bessel functions |
| type $D'$ | $\mathscr{G}(0, 1)$ | parabolic cylinder functions |

In this paper we modify our procedure for constructing special functions from Lie algebra representations so that the equation $C_{a,b}f_m(z, t) = \lambda f_m(z, t)$, when the $t$-dependence is factored out, becomes a third order homogeneous ordinary differential equation for $g_m(z)$. In addition we require that two of the three linearly independent solutions of this third order equation be functions of hypergeometric type listed in Table 1. The remaining solution should be a new function not in the table. We shall proceed by using the $J$-operators of (1.1) as building blocks to construct $K$-operators, $K^+$, $K^-$, $K^3$ which still satisfy the commutation relations (1.1) of $\mathscr{G}(a, b)$ but for which the invariant operator

$$C'_{a,b} = K^+K^- + a^2K^3K^3 - a^2K^3 - bK^3 \tag{1.6}$$

is now a third order partial differential operator in $z$ and $t$. The $K$-operators can then be used to construct the irreducible representation of $\mathscr{G}(a, b)$ listed in [6], hence, to construct new classes of special functions.

We make use of the following algebraic methods for the formation of $K$-operators from $J$-operators.

*Method* 1A. Suppose the first order $J$-operators satisfy the commutation relations (1.1) of $\mathscr{G}(a, b)$ and denote a given differential operator (1.2), (1.3) in the form $J^\pm = \tau_{J^\pm} + t^{\pm 1}l^\pm(z)$. Now set

$$\begin{aligned} K^+ &= J^+ + t\alpha(z)(C_{a,b} - \lambda E), \\ K^- &= J^-, \qquad K^3 = J^3, \end{aligned} \tag{1.7}$$

where $C_{a,b}$ is given by (1.4), and fix the function $\alpha(z)$ by the requirement that $K^\pm, K^3, E$ satisfy the commutation relations of $\mathscr{G}(a, b)$. The only commutation relation which is not identically satisfied is

$$[K^+, K^-] = 2a^2K^3 - bE, \tag{1.8}$$

and an easy computation shows that (1.8) holds if and only if $\alpha(z)$ satisfies the equation

$$\tau_{J^-}(t\alpha(z)) = 0. \tag{1.9}$$

Forming the invariant operator $C'_{a,b}$ we find that

$$C'_{a,b} = C_{a,b} + t\alpha(z)J^-(C_{a,b} - \lambda E). \tag{1.10}$$

Now consider a model of an irreducible representation of $\mathscr{G}(a, b)$ formed by the $K$-operators such that $C'_{a.b} = \lambda E$. Then for every basis vector $f_m(z, t)$ in $\mathscr{W}$ the relation $C'_{ab}f_m = \lambda f_m$ becomes, due to (1.10), (1.11) $$(E + t\alpha(z)J^-)h_m(z, t) = 0,$$

(1.12) $$h_m(z, t) = (C_{a,b} - \lambda E)f_m(z, t).$$

If the basis functions $f_m(z, t)$ satisfy $C_{a,b}f_m = \lambda f_m$, then $h_m \equiv 0$ and the $K$-operators become identical with the $J$-operators on this basis. However, we can solve the first order partial differential equation (1.11) to find nonzero solutions $h_m(z, t)$. Then, the basis functions $f_m(z, t)$ will satisfy the *nonhomogeneous* second order differential equation (1.12). We shall always be able to factor out the $t$-dependence from (1.11), (1.12) and reduce the problem to a solution of ordinary differential equations.

*Method* 1B. This method is a variant of Method 1A where we alter $J^-$ rather than $J^+$. Set

(1.13) $$K^- = J^- + t^{-1}\beta(z)(C_{a,b} - \lambda E),\qquad K^+ = J^+, \qquad K^3 = J^3.$$

It is straightforward to show that the $K$-operators form a basis for $\mathcal{G}(a, b)$ if and only if

(1.14) $$\tau_{J^+}(t^{-1}\beta(z)) = 0.$$

In this case

(1.15) $$C'_{a,b} = C_{a,b} + t^{-1}\beta(z)J^+(C_{a,b} - \lambda E).$$

Now consider a model of an irreducible representation of $\mathcal{G}(a, b)$ formed by the $K$-operators such that $C'_{a,b} = \lambda E$. Then the relation $C'_{a,b}f_m = \lambda f_m$ satisfied by the basis vectors becomes

(1.16) $$(E + t^{-1}\beta(z)J^+)h_m(z, t) = 0,$$

(1.17) $$h_m(z, t) = (C_{a,b} - \lambda E)f_m(z, t).$$

If $h_m \equiv 0$ then the $J$- and $K$-operators agree and we get nothing new, while if $h_m$ is a nonzero solution of (1.16) then the basis vectors $f_m$ satisfy the nonhomogeneous second order equation (1.17).

*Method* 2A. Suppose the $J$-operators form a basis for $\mathcal{G}(0, 0)$, i.e., they satisfy (1.1) with $a = b = 0$. We shall use the $J$-operators to construct $K$-operators which form a basis for $\mathcal{G}(0, 1)$. In particular we set

(1.18) $$K^+ = J^+J^3 + t\alpha(z)(C_{0,0} - E) + \mu J^+,\qquad K^- = J^-, \qquad K^3 = J^3.$$

It is easy to check that the $K$-operators form a basis for $\mathcal{G}(0, 1)$ if and only if

(1.19) $$\tau_{J^-}(t\alpha(z)) = -1.$$

In this case

(1.20) $$C'_{0,1} = (J^3 + t\alpha(z)J^-)(C_{0,0} - E) + (\mu - 1)C_{0,0}.$$

Consider an irreducible representation of $\mathcal{G}(0, 1)$ by the $K$-operators such that $C'_{0,1} = (\mu - 1)E$. Then the relation $C'_{0,1}f_m = (\mu - 1)f_m$ satisfied by the basis vectors can be transformed to

$$(J^3 + t\alpha(z)J^- + (\mu - 1)E)h_m(z,t) = 0, \tag{1.21}$$

$$h_m(z,t) = (C_{0,0} - E)f_m(z,t). \tag{1.22}$$

If the $h_m$ are a nonzero solution of (1.21), then the basis vectors $f_m$ satisfy the nonhomogeneous differential equation (1.22).

*Method* 2B. In this variant of Method 2A we modify $J^-$ rather than $J^+$. Thus we assume that the $J$-operators form a basis for $\mathscr{G}(0,0)$ and construct a basis for $\mathscr{G}(0,1)$ from the $K$-operators:

$$\begin{aligned} K^- &= J^-J^3 + t^{-1}\beta(z)(C_{0,0} - E) + \mu J^-, \\ K^+ &= J^+, \qquad K^3 = J^3, \end{aligned} \tag{1.23}$$

where

$$\tau_{J^+}(t^{-1}\beta(z)) = 1. \tag{1.24}$$

Then

$$C'_{0,1} = (J^3 + t^{-1}\beta(z)J^+ + E)(C_{0,0} - E) + \mu C_{0,0}. \tag{1.25}$$

If we consider an irreducible representation of $\mathscr{G}(0,1)$ by $K$-operators such that $C'_{0,1} = \mu E$, then the relation $C'_{0,1}f_m = \mu f_m$ becomes

$$(J^3 + t^{-1}\beta(z)J^+ + (\mu + 1)E)h_m(z,t) = 0, \tag{1.26}$$

$$h_m(z,t) = (C_{0,0} - E)f_m(z,t). \tag{1.27}$$

*Method* 3A. Suppose the $J$-operators satisfy (1.1) for $a = 0$, $b = 1$ (as in $\mathscr{G}(0,1)$). We shall construct $K$-operators which form a basis for $\mathscr{G}(1,0)$. In particular we set

$$\begin{aligned} K^+ &= -J^+J^3 - t\alpha(z)(C_{0,1} - \lambda E) + \lambda J^+, \\ K^3 &= J^3, \qquad K^- = J^-. \end{aligned} \tag{1.28}$$

These operators satisfy the commutation relations for $\mathscr{G}(1,0)$ if and only if

$$\tau_{J^-}(t\alpha(z)) = -1. \tag{1.29}$$

In this case

$$C'_{1,0} = \{-J^3 - t\alpha(z)J^- + (\lambda + 1)E\}(C_{0,1} - \lambda E) + \lambda(\lambda + 1)E.$$

Now consider an irreducible representation of $\mathscr{G}(1,0)$ by $K$-operators such that $C'_{1,0} = \lambda(\lambda + 1)E$. Then the relation $C'_{1,0}f_m = \lambda(\lambda + 1)f_m$ becomes

$$(-J^3 - t\alpha(z)J^- + (\lambda + 1)E)h_m(z,t) = 0, \tag{1.30}$$

$$h_m(z,t) = (C_{0,1} - \lambda E)f_m(z,t). \tag{1.31}$$

*Method* 3B. This is identical with Method 3A except that the $J^-$-operator is altered rather than $J^+$. Thus,

$$\begin{aligned} K^- &= -J^-J^3 + t^{-1}\beta(z)(C_{0,1} - \lambda E) + (\lambda + 1)J^-, \\ K^+ &= J^+, \qquad K^3 = J^3. \end{aligned} \tag{1.32}$$

These operators form a basis for $\mathscr{G}(1, 0)$ if and only if

$$\tau_{J^+}(t^{-1}\beta(z)) = -1. \tag{1.33}$$

Then

$$C'_{1,0} = (-J^3 + t^{-1}\beta(z)J^+ + \lambda E)(C_{0,1} - \lambda E); \tag{1.34}$$

and if $C'_{1,0}f_m = 0$, then

$$\begin{aligned} &(-J^3 + t^{-1}\beta(z)J^+ + \lambda E)h_m(z, t) = 0, \\ &h_m(z, t) = (C_{0,1} - \lambda E)f_m(z, t). \end{aligned} \tag{1.35}$$

*Method* 4A. We proceed by analogy with Methods 2 and 3. Suppose the $J$-operators form a basis for $\mathscr{G}(1, 0)$. We set

$$\begin{aligned} &K^+ = J^+J^3 + \omega J^+ + t\alpha(z)(C_{1,0} - \lambda E), \\ &K^- = J^-, \qquad K^3 = J^3, \end{aligned} \tag{1.36}$$

and determine $\alpha(z)$ so that $[K^+, K^-]$ is independent of the operators $J^+$ and $J^-$. It is easy to show that this will be the case provided that

$$\tau_{J^-}(t\alpha(z)) = -1. \tag{1.37}$$

In this case $[K^+, K^-] = 3J^3J^3 + (2\omega - 1)J^3 - \lambda E$ and the operators $K^\pm, K^3$ generate an infinite-dimensional Lie algebra $\mathscr{G}^{\omega,\lambda}$. Even though this Lie algebra is not finite-dimensional it is still useful since it has a simple invariant operator

$$C' = (J^3 + t\alpha(z)J^- + (\omega - 1)E)(C_{1,0} - \lambda E) + \lambda(\omega - 1)E. \tag{1.38}$$

Given an irreducible representation of $\mathscr{G}^{\omega,\lambda}$ such that $C'f_m = \lambda(\omega - 1)f_m$ we obtain the equations

$$(J^3 + t\alpha(z)J^- + (\omega - 1)E)h_m(z, t) = 0, \tag{1.39}$$

$$h_m(z, t) = (C_{1,0} - \lambda E)f_m(z, t). \tag{1.40}$$

From the point of view of the factorization method we have constructed a factorization which does not have a realization as a representation of a finite-dimensional Lie algebra [6], [8].

*Method* 4B. We assume that the $J$-operators form a basis for $\mathscr{G}(1, 0)$ and set

$$\begin{aligned} &K^- = J^-J^3 + \omega J^- + t^{-1}\beta(z)(C_{1,0} - \lambda E), \\ &K^+ = J^+, \qquad K^3 = J^3. \end{aligned} \tag{1.41}$$

If $\tau_{J^+}(t^{-1}\beta(z)) = 1$ then $[K^+, K^-] = 3J^3J^3 + (2\omega - 1)J^3 - \lambda E$ and the $K$-operators generate an infinite-dimensional Lie algebra $\mathscr{G}^{\omega,\lambda}$ with the invariant operator

$$C' = (J^3 + t^{-1}\beta(z)J^+ + (\omega + 1)E)(C_{1,0} - \lambda E) + \lambda\omega E. \tag{1.42}$$

If $C'f_m = \lambda\omega f_m$, then

$$(J^3 + t^{-1}\beta(z)J^+ + (\omega + 1)E)h_m(z, t) = 0, \tag{1.43}$$

$$h_m(z, t) = (C_{1,0} - \lambda E)f_m(z, t). \tag{1.44}$$

**2. Special functions related to Bessel's equation.** The type $C''$ operators

$$J^{\pm} = t^{\pm 1}\left(\frac{\pm\partial}{\partial z} - \frac{t}{z}\frac{\partial}{\partial t}\right), \qquad J^3 = \frac{t\partial}{\partial t} \tag{2.1}$$

satisfy the commutation relations of $\mathscr{G}(0,0)$, and these operators can be used to construct the Bessel functions and their basic properties [6], [9]. Here the type $C''$ operators will be used as building blocks to construct more complicated models of Lie algebra representations.

As a first example we apply Method 2A. It follows from (1.19) that

$$-t^{-1}\left(\frac{\partial}{\partial z} + \frac{t}{z}\frac{\partial}{\partial t}\right)(t\alpha(z)) = -1$$

or $\alpha(z) = z/2 + c_1/z$, where $c_1$ is an arbitrary constant. Substituting this result into (1.21) we find

$$\frac{dh_m}{dz} = \left[\frac{-m}{z} \frac{+2(m+\mu-1)z}{z^2+2c_1}\right]h_m,$$

where $h_m(z,t) = h_m(z)t^m$. Thus

$$h_m(z) = c_2 z^{-m}(z^2 + c_1)^{m+\mu-1}, \tag{2.2}$$

where the constants $c_2$ may depend on $m$ but not on $z$.

We can now use the $K$-operators to construct models of the irreducible representations of $\mathscr{G}(0,1)$ listed in [6] and [9]. The general theory guarantees that we can always construct such models. As an example, consider the representation $R(\omega, m_0, 1)$ defined for $\omega, m_0 \in \mathcal{C}$ such that $0 \leqq \operatorname{Re} m_0 < 1$ and $\omega + m_0$ is not an integer. The representation space $\mathscr{W}$ has a basis $\{f_m\}$, $m \in S = \{m_0 + n : n = 0, \pm 1, \pm 2, \cdots\}$ such that the action of $\mathscr{G}(0,1)$ on $\mathscr{W}$ is given by

$$K^3 f_m = m f_m, \qquad K^+ f_m = -(m+\omega+1) f_{m+1}, \qquad K^- f_m = -f_{m-1}, \tag{2.3}$$

$$C'_{0,1} f_m = (K^+K^- - K^3) f_m = \omega f_m, \qquad m \in S. \tag{2.4}$$

The construction of suitable sets of basis vectors $\{f_m\}$ satisfying (2.3), (2.4) is straightforward once it is assumed that the $K$-operators are given by (1.18). Clearly, we must have $\omega = \mu - 1$. Since the equation $C'_{0,1} f_m = \omega f_m$ is of third order we can expect to find three linearly independent sets of basis vectors for each choice of the constant $c_1$. One set of basis vectors is easily seen to be $f_m(z,t) = (-1)mJ_m(z)t^m$ and another set is $f_m(z,t) = (-1)^m Y_m(z)t^m$, where $J_m(z)$, $Y_m(z)$ are the Bessel and Neumann functions, respectively. These solutions correspond to setting $c_2 = 0$ in (2.2) and (1.22). To get a third solution we note that (2.4) is equivalent to the nonhomogeneous equation

$$-g''_m(z) - \frac{1}{z}g'_m(z) + \left(\frac{m^2}{z^2} - 1\right)g_m(z) = c_2 z^{-m}(z^2 + 2c_1)^{m+\omega}, \tag{2.5}$$

where $f_m(z,t) = g_m(z)t^m$. We can use the method of Frobenius to find series solutions of (2.5) and choose the constants in the expansions such that the recurrence relations (2.3) are satisfied.

In case $c_1 = 0, \omega = -\frac{1}{2}$ a solution is given by the Struve functions

$$g_m(z) = H_m(z) = \frac{(z/2)^{m+1}}{\Gamma(3/2)\Gamma(3/2 + m)} \cdot {}_1F_2\left(1; \frac{3}{2}, \frac{3}{2} + m; -\frac{z^2}{4}\right). \tag{2.6}$$

In fact, the recurrence formulas (2.3) reduce to

$$\begin{aligned} H'_m(z) + \frac{m}{z} H_m(z) &= H_{m-1}(z), \\ H'_m(z) - \frac{m}{z} H_m(z) - \frac{(z/2)^{m+1}}{z\Gamma(3/2)\Gamma(m + 3/2)} &= -H_{m+1}(z). \end{aligned} \tag{2.7}$$

Here, $c_2(m) = -2^{-m+1}[\Gamma(1/2)\Gamma(1/2 + m)]^{-1}$. If $c_1 = 0$, $\omega$ arbitrary, a solution is given by the Lommel functions

$$g_m(z) = \frac{s_{m+2\omega+1,m}(z)}{2^m\Gamma(m + \omega + 1)} \quad \text{or} \quad \frac{S_{m+2\omega+1,m}(z)}{2^m\Gamma(m + \omega + 1)}, \tag{2.8}$$

where

$$\begin{aligned} s_{\mu,\nu}(z) &= \frac{z^{\mu+1}}{(\mu - \nu - 1)(\mu + \nu + 1)} \cdot {}_1F_2\left(1; \frac{\mu - \nu + 3}{2}, \frac{\mu + \nu + 3}{2}; -\frac{z^2}{4}\right), \\ S_{\mu,\nu}(z) &= s_{\mu,\nu}(z) + \left\{2^{\mu-1}\Gamma\left[\frac{\mu - \nu + 1}{2}\right]\Gamma\left[\frac{\mu + \nu + 1}{2}\right]\right\} \\ &\quad \cdot \left\{\sin\left[\frac{(\mu - \nu)\pi}{2}\right] J_\nu(z) - \cos\left[\frac{(\mu - \nu)\pi}{2}\right] Y_\nu(z)\right\} \end{aligned} \tag{2.9}$$

(see [1]). In this case (2.5) becomes

$$g''_m(z) + \frac{1}{z} g'_m(z) + \left(1 - \frac{m^2}{z^2}\right) g_m(z) = z^{m+2\omega}. \tag{2.10}$$

It is easy to derive more recurrence relations for the Lommel functions by applying Method 2B to the type $C'$ operators, but this is left to the reader.

We can apply the Lie theory of local transformation groups to derive addition theorems for the basis functions $f_m(z, t)$ (see [6], [9]). (Recall that, in our model, $K^-$ is a first order partial differential operator.) Thus

$$\exp(aK^-) f_m = \sum_{n=0}^{\infty} \frac{(-a)^n}{n!} f_{m-n}, \tag{2.11}$$

where

$$\exp(-aK^-) f_m(z, t) = f_m\left[s\left(1 - \frac{2a}{zt}\right)^{1/2}, t\left(1 - \frac{2a}{zt}\right)^{1/2}\right], \qquad \left|\frac{2a}{zt}\right| < 1.$$

Applying this result to the Lommel functions and simplifying, we obtain

$$s_{\mu,\nu}\left[z\left(1 + \frac{a}{z}\right)^{1/2}\right]\left(1 + \frac{a}{z}\right)^{1/2} = \sum_{n=0}^{\infty} a^n \binom{(\mu + \nu - 1)/2}{n} s_{\mu-n,\nu-n}(z), \qquad \left|\frac{a}{z}\right| < 1. \tag{2.12}$$

As another example of the utility of the Lie algebraic approach we apply Weisner's method [6], [10], [11] to the Lommel functions. The function

$$f_m(z,t) = s_{m+2\omega+1,m}(z)t^m$$

is a simultaneous eigenvector of the commuting operators $K^3$ and $C'_{0,1}$:

$$K^3 f_m = m f_m, \qquad C'_{0,1} f_m = \omega f_m.$$

Similarly, it can be shown that $\exp(aK^-)f_m = h_m$ satisfies

$$(aK^- + K^3)h_m = m h_m, \qquad C'_{0,1} h_m = \omega h_m. \tag{2.13}$$

If $a = -1$ we can write

$$h_m = \left(z^2 + \frac{2z}{t}\right)^{-m/2-\omega-1} s_{m+2\omega+1,m}\left[\left(z^2 + \frac{2z}{t}\right)^{1/2}\right] \cdot (2+tz)^m \left(z^2 + \frac{2z}{t}\right)^{\omega+1} \tag{2.14}$$

From (2.9) we see that $z^{-\mu-1}s_{\mu,\nu}(z)$ is an entire function of $z$. Thus, if $\omega = 0, 1, 2, \cdots$ then the function (2.14) has a Laurent expansion in $t$ which converges in the ring $0 < |t| < |2/z|$:

$$h_m(z,t) = \sum_{n=-\infty}^{\infty} j_n(z)t^n.$$

Furthermore, $C'_{0,1}[j_n(z)t^n] = \omega j_n(z)t^n$ and $K^3[j_n(z)t^n] = n j_n(z)t^n$. From this it is not difficult to show that $j_n(z) = c_n J_n(z)$ for $n \leqq -\omega - 1$ and $j_n(z) = c_n s_{n+2\omega+1,n}(z)$ for $n > -\omega - 1$, $c_n \in \mathbb{C}$. The constants $c_n$ can be computed from the first equation (2.13) or by direct inspection of (2.14). The result is

$$\begin{aligned}
&\left(z^2 + \frac{2z}{t}\right)^{-m/2-\omega-1} s_{m+2\omega+1,m}\left[\left(z^2 + \frac{2z}{t}\right)^{1/2}\right](2+tz)^m\left(z^2 + \frac{2z}{t}\right)^{\omega+1} \\
&\quad = \sum_{n=-\infty}^{-\omega-1} (-1)^{\omega+1} 2^{n+2\omega} \frac{\Gamma(\omega+1)\Gamma(m+\omega+1)}{\Gamma(m-n+1)} J_n(z)t^n \\
&\quad + \sum_{n-\omega-1}^{-1} 2^{m-n} \frac{(n+\omega+1)}{(m+\omega+1)} \frac{\Gamma(\omega+2)}{(-n)!\Gamma(\omega+n+2)} s_{n+2\omega+1,n}(z)t^n \\
&\quad + \sum_{n=0}^{\infty} 2^{m-n} \frac{(n+\omega+1)}{(m+\omega+1)} \frac{\Gamma(m+1)}{n!\Gamma(m+n+1)} s_{n+2\omega+1,n}(z)t^n, \\
&\qquad\qquad 0 < |t| < \left|\frac{2}{z}\right|, \qquad \omega = 0, 1, 2, \cdots.
\end{aligned} \tag{2.15}$$

We can also use our $K$-operators to construct models of the representations $\downarrow_{\omega,-1}$ defined by (2.3), (2.4) except that the spectrum is now $S = \{-\omega - 1 - n : n = 0, 1, 2, \cdots\}$. In the case $c_1 = 0$ the reader can easily verify that the functions (2.8) form a basis for the representation where now $m = -\omega - 1 - n$, $n = 0, 1, 2, \cdots$.

The construction of models of $R(\omega, m_0, 1)$ and $\downarrow_{\omega,-1}$ when $c_1 \neq 0$ in (2.5) is straightforward but more complicated and the work will not be carried out here. Our general Lie algebraic approach guarantees the existence of addition theorems and recurrence formulas for the basis functions. Method 2B leads to the Lommel functions again, but this time the recurrence relation

$$s'_{\mu,\nu}(z) - \left(\frac{\nu}{z}\right)s_{\mu,\nu}(z) = (\mu - \nu - 1)s_{\mu-1,\nu+1}(z)$$

appears.

We now apply Method 1B to the type $C''$ operators. (Method 1A leads to similar results.) Equation (1.14) has the solution $\beta(z) = c_1/z$. Substituting this result into (1.16) and solving for $h_m(z, t) = h_m(z)t^m$ we obtain

$$h_m(z) = c_2 z^m e^{-z^2/(2c_1)}, \tag{2.16}$$

where $c_1 \in \mathcal{C}$ and $c_2$ is a function of $m$ but not of $z$. We shall use the $K$-operators to construct models of the irreducible representations of $\mathcal{G}(0, 0)$ listed in [6] and [9]. In particular we consider the irreducible representations $Q(-1, m_0)$, $0 < \operatorname{Re} m_0 < 1$. The representation space $\mathcal{W}$ has a basis $\{f_m\}$, $m \in S = \{m_0 + n : n = 0, \pm 1, \pm 2, \cdots\}$, such that the action of $\mathcal{G}(0, 1)$ on $\mathcal{W}$ is given by

$$K^3 f_m = m f_m, \quad K^{\pm} f_m = -f_{m\pm 1}, \quad C_{0,0} f_m = J^+ J^- f_m = f_m, \qquad m \in S, \tag{2.17}$$

For the construction of our model we assume that the $K$-operators are given by (1.13) and (2.1) with $\beta(z) = c/z$. Then (1.17) with $\lambda = 1$ reads

$$-g''_m(z) - \frac{1}{z}g'_m(z) + \left(\frac{m^2}{z^2} - 1\right)g_m(z) = kz^m e^{-(z^2/2c)}, \tag{2.18}$$

where $f_m(z, t) = g_m(z)t^m$. Two independent sets of basis vectors are given by $g_m(z) = J_m(z)$ and $g_m(z) = Y_m(z)$. This corresponds to the case when $k = 0$ and the $J$- and $K$-operators coincide. We can find another basis $\{f_m\}$ by expanding the exponential on the right-hand side of (2.18) in a power series and comparing with (2.10):

$$g_m(z) = M_c(m; z) = c^{-m} \sum_{n=0}^{\infty} \left(\frac{-1}{2c}\right)^n \frac{S_{m+2n+1,m}(z)}{\Gamma(m + n + 1)n!}. \tag{2.19}$$

It follows from (2.9) that if $|2/c| < 1$ the series (2.19) converges and defines a function analytic for all $z \neq 0$. (We assume this condition on $c$ in the following.) Relations (2.17) become

$$\begin{aligned} &\left(\frac{d}{dz} - \frac{m}{z}\right)M_c(m; z) = -M_c(m + 1; z), \\ &\left(\frac{d}{dz} + \frac{m}{z}\right)M_c(m; z) - 2\left(\frac{2z}{c}\right)^{m-1} e^{-z^2/(2c)} = M_c(m - 1; z), \\ &M''_c(m; z) + \frac{1}{z}M'_c(m; z) + \left(1 - \frac{m^2}{z^2}\right)M_c(m; z) = -\left(\frac{2z}{c}\right)^m e^{-z^2/(2c)}. \end{aligned} \tag{2.20}$$

In analogy with expression (2.11) we can use the first of the recurrence relations (2.10) and local Lie theory to derive a generating function for the $M_c(m; z)$. The result is

$$M_c\left(m; z\left(1 - \frac{2a}{z}\right)^{1/2}\right)\left(1 - \frac{2a}{z}\right)^{-m/2},$$
$$= \sum_{n=0}^{\infty} \frac{a^n}{n!} M_c(m + n; z), \qquad \left|\frac{2a}{z}\right| < 1.$$

**3. Special functions related to the confluent hypergeometric equation.** The type $C'$ operators

$$J^+ = t\left(\frac{\partial}{\partial z} - 1\right), \qquad J^- = t^{-1}\left(-z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t} + q\right),$$
$$J^3 = t\frac{\partial}{\partial t}, \qquad q \in \mathbb{C}, \tag{3.1}$$

satisfy the commutation relations of $\mathscr{G}(0, 1)$. They were used in [6] to establish recurrence relations and addition theorems for the confluent hypergeometric functions.

We begin by applying Method 3A to the type $C'$ operators. Then (1.29) becomes $z\alpha'(z) + \alpha(z) = 1$ or

$$\alpha(z) = 1 + \frac{c_1}{z}, \qquad c_1 \in \mathbb{C}. \tag{3.2}$$

Furthermore, (1.30) becomes

$$(\lambda + 1 - m)h_m(z) - \left(1 + \frac{c_1}{z}\right)\{-zh'_m(z) + (q - m)h_m(z)\} = 0,$$

where $h_m(z, t) = h_m(z)t^m$, or

$$h_m(z) = c_2(m)z^{q-m}(z + c_1)^{m-\lambda-1}. \tag{3.3}$$

The $K$-operators just constructed satisfy the commutation relations of $\mathscr{G}(1, 0)$. Thus we can use them to construct models of the irreducible representations of $\mathscr{G}(1, 0)$ listed in [6]. In particular, consider the representation $D(u, m_0)$ defined for $u, m_0 \in \mathbb{C}$ such that $m_0 \pm u$ are not integers and $0 \leqq \operatorname{Re} m_0 < 1$. The representation space $\mathscr{W}$ has a basis $\{f_m\}$, $n \in S = \{m_0 + n : n = 0, \pm 1, \pm 2, \cdots\}$ such that the action of $\mathscr{G}(1, 0)$ on $\mathscr{W}$ is given by

$$K^3 f_m = m f_m, \qquad K^{\pm} f_m = (-u \pm m) f_{m \pm 1}, \tag{3.4}$$

$$C_{1,0} f_m = (K^+K^- + K^3K^3 - K^3) f_m = u(u + 1) f_m, \qquad m \in S. \tag{3.5}$$

The representations $D(u, m_0)$ and $D(-u - 1, m_0)$ are isomorphic.

We assume that the $K$-operators are given by (1.28) and (3.1)–(3.3). Clearly, $u(u + 1) = \lambda(\lambda + 1)$ and we chose $u = -\lambda - 1$. The equation $C'_{1,0} f_m = \lambda(\lambda + 1) f_m$ is of third order so we can expect to find three linearly independent sets of basis vectors for each choice of the constant $c_1$. One set of basis vectors is easily shown to be

$$f_m(z, t) = L_{q+\lambda}^{(m-q)}(z)t^m = \frac{\Gamma(m - \lambda)}{\Gamma(m - q + 1)\Gamma(q + \lambda + 1)} \cdot {}_1F_1(-q - \lambda; m - q + 1; z)t^m,$$

where the $L_\nu^{(\alpha)}(z)$ are Laguerre functions and the ${}_1F_1$ are confluent hypergeometric functions [12]. Another set of basis vectors is given by

$$f_m(z,t) = \frac{(-1)^m\Gamma(m-\lambda)}{\Gamma(1+q-m)\Gamma(m+\lambda+1)}\,{}_1F_1(-m-\lambda;1+q-m;z)t^m.$$

Both of these solutions correspond to the case where $c_2 \equiv 0$. To construct solutions for which $c_2 \neq 0$ we assume for simplicity that $c_1 = 0$. Then (3.5) is equivalent to the equation

$$\text{(3.6)}\qquad -zg_m''(z) + (q-m+z-1)g_m'(z) - (q+\lambda)g_m(z) = c_2(m)z^{q-\lambda-1},$$

where $f_m(z,t) = g_m(z)t^m$. We can find a series solution of this equation by the method of Frobenius and adjust the constants in the expansion so that the recurrence relations (3.5) hold. It is not difficult to verify that a solution is given by the functions

$$\text{(3.7)}\qquad g_m(z) = \theta_\sigma\left(a, m+1+\frac{a-\sigma}{2}; z\right),$$

where $\sigma = q-\lambda$, $a = -q-\lambda$ and

$$\text{(3.8)}\qquad \theta_\sigma(a,c;z) = \frac{z^\sigma}{\sigma(\sigma+c-1)}\,{}_2F_2(1,\sigma+a;\sigma+1,\sigma+c;z)$$

is the nonhomogeneous confluent hypergeometric function [5]. This function is analytic in $z$ for all $z \neq 0$. The Lie algebra relations (3.4), (3.5) become

$$z\theta_\sigma'(a,c;z) + (c-1)\theta_\sigma(a,c;z) = (\sigma+c-2)\theta_\sigma(a,c-1;z),$$

$$\text{(3.9)}\qquad (\sigma+c-1)[\theta_\sigma'(a,c;z) - \theta_\sigma(a,c;z)] - z^{\sigma-1} = (a-c)\theta_\sigma(a,c+1;z),$$

$$\text{(3.10)}\qquad z\theta_\sigma'' + (c-z)\theta_\sigma' - a\theta_\sigma = z^{\sigma-1}.$$

Another solution is given by the functions

$$g_m(z) = \Theta_\sigma\left(a, m+1+\frac{a-\sigma}{2}; z\right),$$

where

$$\Theta_\sigma(a,c;z) = \theta_\sigma(a,c;z) - \frac{\Gamma(\sigma)\Gamma(\sigma+c-1)\Gamma(1-\sigma-a)}{\sin \pi c}$$
$$\cdot\left\{\sin[\pi(\sigma+c)]\frac{{}_1F_1(a,c;z)}{\Gamma(c)\Gamma(1-a)} + \sin\pi\sigma\frac{z^{1-c}\,{}_1F_1(a-c+1,2-c;z)}{\Gamma(2-c)\Gamma(c-a)}\right\}.$$

We can apply local Lie theory to the first order operator $K^-$ and derive generating functions for the $\theta_\sigma$. Thus,

$$\theta_\sigma(a,c;z(1-b))(1-b)^{c-1} = \sum_{n=0}^{\infty}\binom{1-c-\sigma+n}{n}b^n\theta_\sigma(a,c-n;z), \qquad |b|<1.$$

Our results can be extended in various ways. We could use the $K$-operators to construct models of other irreducible representations of $\mathcal{G}(1,0)$ and get more information about the $\theta_\sigma$ and $\Theta_\sigma$ functions. Alternatively, we could treat the case where $c_1 \neq 0$ in (3.3). The case where $c_1 \neq 0$ is straightforward but complicated and we omit the computations.

Next we apply Method 1B to the type $C'$ operators. Equation (1.14) has the solution $\beta(z) = c_1$, $c_1 \in \mathbb{C}$. If we substitute this result into (1.16) and solve for $h_m(z, t) = h_m(z)t^m$, we obtain

$$h_m(z) = c_2(m) \exp\left[\frac{c_1 - 1}{xc_1}\right], \qquad c_1 \neq 0. \tag{3.11}$$

Our $K$-operators now satisfy the commutation relations of $\mathcal{G}(0, 1)$, and we can use them to construct models of the irreducible representations $R(\omega, m_0, 1)$, (2.3), (2.4). Here, $\lambda = \omega$ and it turns out that we can set $\lambda = 0$ without loss of generality. As usual, if $c_2 \equiv 0$ then the equation $C'_{0,1} f_m = 0$ reduces to a homogeneous second order equation for $f_m$ and the basis can be expressed in terms of confluent hypergeometric functions [6]. If $c_2 \not\equiv 0$ then (2.4) becomes

$$-zg''_m(z) + (q - m - 1 + z)g'_m(z) - qg_m(z) = c_2(m)\, e^{(c_1 - 1/c_1)z}, \tag{3.12}$$

where $f_m(z, t) = g_m(z)t^m$. For simplicity we assume that $c_1 = 2$. Then it can be shown that there is a basis for the representation $R(0, m_0, 1)$ in the form

$$g_m(z) = \frac{\Omega(-q, m + 1 - q; z)}{\Gamma(m + 1 - q)}, \tag{3.13}$$

where

$$\Omega(a, c; z) = {}_1F_1(a; c; z)$$
$$- \frac{2^{2-c}\Gamma(c)}{\Gamma(a)\Gamma(c - a)} e^{z/2} \int_0^1 e^{-zu/2}(1 + u)^{c-a-1}(1 - u)^{a-1}\, du, \qquad \operatorname{Re} a > 0,$$

is a generalized modified Struve function [5]. Here $\Omega(a, c; z)/\Gamma(c)$ is an entire function of $a$, $c$ and $z$. Relations (2.3) and (2.4) become

$$\Omega'(a, c; z) - \Omega(a, c; z) = -\frac{c - a}{c}\Omega(a, c + 1; z),$$

$$z\Omega'(a, c; z) + (c - 1)\Omega(a, c; z) + \frac{2^{2-c}\Gamma(c)}{\Gamma(a)\Gamma(c - a)}e^{z/2} = (c - 1)\Omega(a, c - 1; z), \tag{3.14}$$

$$z\Omega''(a, c; z) + (c - z)\Omega'(a, c; z) - a\Omega(a, c; z) = \frac{2^{1-c}\Gamma(c)}{\Gamma(a)\Gamma(c - a)}e^{z/2}. \tag{3.15}$$

An application of local Lie theory to the first order operator $K^+$ yields the identity

$$\frac{e^b}{\Gamma(c)}\Omega(a, c; z - b) = \sum_{n=0}^{\infty} \binom{c - a - 1 + n}{n} b^n \frac{\Omega(a, c + n; z)}{\Gamma(c + n)}.$$

Applying Method 1A to the type $C'$ operators we find that (1.9) has the solution $\alpha(z) = c_1/z$, $c_1 \in \mathcal{C}$. Substituting this result into (1.11) we can solve for $h_m(z, t) = h_m(z)t^m$:

$$h_m(z) = c_2(m)z^{q-m}e^{z/c_1}, \qquad c_1 \neq 0. \tag{3.16}$$

Again the $K$-operators satisfy the commutation relations of $\mathcal{G}(0, 1)$ so we can use them to construct models of the irreducible representations $R(\omega, m_0, 1)$, (2.3), (2.4). Again $\lambda = \omega$, and if $c_2 \equiv 0$ then the equation $C'_{0,1}f_m = \lambda f_m$ reduces to a homogeneous second order equation for $f_m$. In this case the basis can be expressed in terms of confluent hypergeometric equations. If $c_2 \not\equiv 0$ then (2.4) becomes

$$-zg''_m(z) + (q - m - 1 + z)g'_m(z) - (\lambda + q)g_m(z) = c_2(m)z^{q-m}e^{z/c_1}, \qquad c_1 \neq 0, \tag{3.17}$$

where $f_m(z, t) = g_m(z)t^m$. Comparing this expression with (3.9) and (3.10) we see that the representation space has a basis of the form

$$g_m(z) = \rho^{-m}\Phi_\rho(a, m - q + 1; z), \tag{3.18}$$

where $\rho = c_1^{-1}$, $a = -\lambda - q$ and

$$\Phi_\rho(a, c; z) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!}\theta_{2-c+n}(a, c; z).$$

This series converges absolutely to an analytic function of $z$ for all $z \neq 0$ and $\rho$. The $\Phi_\rho$ is a special case of the functions $\Lambda_{\rho,\sigma}(a, c; z)$ in [5]. In fact, $\Phi_\rho(a, c; z) = \Lambda_{\rho,2-c}(a, c; z)$. The Lie algebra relations (2.3), (2.4) expressed in terms of the $\Phi_\rho$ become

$$\begin{aligned} &z\Phi'_\rho(a, c; z) + (c - 1)\Phi_\rho(a, c; z) = \rho\Phi_\rho(a, c - 1; z), \\ &\rho\Phi'_\rho(a, c; z) - \rho\Phi_\rho(a, c; z) + e^{\rho z}z^{-c} = (c - a)\Phi_\rho(a, c + 1; z), \end{aligned} \tag{3.19}$$

$$z\Phi''_\rho(a, c; z) + (c - z)\Phi'_\rho(a, c; z) - a\Phi'_\rho(a, c; z) = e^{\rho z}z^{1-c}. \tag{3.20}$$

Application of local Lie theory to the first order operator $K^-$ yields the identity

$$(1 + b)^{c-1}\Phi_\rho(a, c; z(1 + b)) = \sum_{n=0}^{\infty} \frac{(b\rho)^n}{n!}\Phi_\rho(a, c - n; z), \qquad |b| < 1.$$

Finally, Method 3B applied to the type $C'$ operators leads to the expressions

$$\beta(z) = -z + c_1, \qquad h_m(z) = c_2(m)e^z(z - c_1)^{-m+\lambda}.$$

We can use the $K$-operators so obtained to construct irreducible representations of $\mathcal{G}(1, 0)$. However, we omit this since the techniques and results are very similar to those given above.

**4. Special functions related to the parabolic cylinder equation.** The type $D'$ operators

$$J^\pm = t^{\pm 1}\left(\pm\frac{\partial}{\partial z} - \frac{z}{2}\right), \qquad J^3 = t\frac{\partial}{\partial t} \tag{4.1}$$

satisfy the commutation relations of $\mathcal{G}(0, 1)$. In [6] they were used to establish recurrence relations and addition theorems for the parabolic cylinder functions.

Applying Method 3A to these operators, we find that the expression (1.29) becomes $\alpha'(z) = 1$ or $\alpha(z) = z + c_1$. The solution of (1.30) is

$$h_m(z) = c_2(m)(z + c_1)^{m-\lambda-1}e^{-z^2/4}. \tag{4.2}$$

For simplicity we consider only the case $c_1 = 0$. The $K$-operators satisfy the commutation relations of $\mathscr{G}(1,0)$ so we can use them to construct models of the irreducible representations $D(u, m_0)$, (3.4), (3.5). Note that we can choose $u = -\lambda - 1$. As usual we can find three linearly independent sets of basis vectors $\{f_m\}$ corresponding to the equation $C'_{1,0}f_m = \lambda(\lambda + 1)f_m$. One set is

$$f_m(z, t) = \frac{\Gamma(m - \lambda)}{\Gamma(m + \lambda + 1)} D_{m+\lambda}(z)t^m$$

and another is

$$f_m(z, t) = (-1)^m \frac{\Gamma(m - \lambda)}{\Gamma(m + \lambda + 1)} D_{m+\lambda}(-z)t^m,$$

where $D_m(z)$ is a parabolic cylinder function [12]. The solutions correspond to the case $c_2 \equiv 0$. We now look for solutions such that $c_2 \not\equiv 0$. Then (3.5) is equivalent to the equation

$$-g''_m(z) + \left[\frac{z^2}{4} - \left(\lambda + m + \frac{1}{2}\right)\right]g_m(z) = c_2 z^{m-\lambda-1}e^{-z^2/4}, \tag{4.3}$$

where $f_m(z, t) = g_m(z)t^m$. Using the method of Frobenius we find the following basis:

$$g_m(z) = \Delta(\lambda; m - \lambda; z), \tag{4.4}$$

where

$$\Delta(a, c; z) = \frac{e^{-z^2/4}z^{c+1}}{c(c + 1)} {}_2F_2\left(1, a + \frac{1}{2}; \frac{c}{2} + 1; \frac{c}{2} + \frac{3}{2}; \frac{z^2}{2}\right).$$

Here, $\Delta(a, c; z)$ is analytic in $z$ for all $z \neq 0$. The representation formulas (3.4) and (3.5) become

$$\Delta'(a, c; z) + \frac{z}{2}\Delta(a, c; z) = (c - 1)\Delta(a, c - 1; z),$$

$$-c\Delta'(a, c; z) + \frac{z}{2}c\Delta(a, c; z) + z^c e^{-z^2/4} = (2a + c + 1)\Delta(a, c + 1; z), \tag{4.5}$$

$$\Delta''(a, c; z) + \left(2a + c + \frac{1}{2} - \frac{z^2}{4}\right)\Delta(a, c; z) = z^{c-1}e^{-z^2/4}. \tag{4.6}$$

Finally, applying local Lie theory to the first order operator $K^-$ we obtain the identity

$$\exp\left(\frac{b^2}{4} + \frac{zb}{2}\right)\Delta(a, c; z + b) = \sum_{n=0}^{\infty} b^n \binom{c - 1}{n} \Delta(a, c - n; z), \qquad \left|\frac{b}{z}\right| < 1.$$

Next we apply Method 1A to the type $D'$ operators. Equation (1.9) reads $\alpha'(z) = 0$ and has the solution $\alpha(z) = c_1 \in \mathcal{C}$. Substituting this result into (1.11) we find

$$h_m(z) = c_2(m)e^{z/c_1 - z^2/4}, \qquad c_1 \neq 0. \tag{4.7}$$

The $K$-operators now satisfy the commutation relations of $\mathcal{G}(0, 1)$ and we can use them to construct models of the irreducible representations $R(\omega, m_0, 1)$, (2.3), (2.4). Clearly, we must have $\omega = \lambda$. It is easy to construct two linearly independent sets of basis vectors $\{f_m\}$ from the parabolic cylinder functions [6]. These solutions correspond to the case $c_2 \equiv 0$ in (4.7). To find a basis $\{f_m\}$ for which $c_2 \not\equiv 0$ we note that the equation $C'_{0,1} f_m = \omega f_m$ becomes

$$-g''_m(z) + \left[\frac{z^2}{4} - \left(\lambda + m + \frac{1}{2}\right)\right] g_m(z) = c_2 e^{z/c_1 - z^2/4}, \tag{4.8}$$

where $f_m(z, t) = g_m(z)t^m$. Comparing this expression with (4.3)–(4.6) we find that we can construct a basis in the form

$$g_m(z) = \rho^{-m} D_\rho(\lambda + m; z), \tag{4.9}$$

where $\rho = c_1^{-1}$ and

$$D_\rho(c; z) = \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \Delta\left(\frac{c - n - 1}{2}, n + 1; z\right).$$

This series converges absolutely to an analytic function of $z$ for all $z \neq 0$ and $\rho$. In terms of the functions $D_\rho(c; z)$ the defining relations of the Lie algebra representation (2.3), (2.4) read

$$\begin{aligned} &D'_\rho(c; z) + \frac{z}{2} D_\rho(c; z) = \rho D_\rho(c - 1; z), \\ &-\rho D'_\rho(c; z) + \frac{\rho z}{2} D_\rho(c; z) + e^{\rho z - z^2/4} = c D_\rho(c + 1; z), \end{aligned} \tag{4.10}$$

$$D''_\rho(c; z) + \left(c + \frac{1}{2} - \frac{z^2}{4}\right) D_\rho(c; z) = e^{\rho z - z^2/4}. \tag{4.11}$$

From the first order operator $K^-$ we obtain the identity

$$\exp\left[\frac{b^2}{4} + \frac{zb}{2}\right] D_\rho(c; z + b) = \sum_{n=0}^{\infty} \frac{(\rho b)^n}{n!} D_\rho(c - n; z), \qquad \left|\frac{b}{z}\right| < 1.$$

**5. Special functions related to the hypergeometric equation.** The type $A$ operators

$$\begin{aligned} &J^+ = t\left(z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - u\right), \qquad J^3 = t\frac{\partial}{\partial t}, \\ &J^- = t^{-1}\left(z(1 - z)\frac{\partial}{\partial z} - t\frac{\partial}{\partial t} + z(q + u) - u\right), \qquad q, u \in \mathcal{C}, \end{aligned} \tag{5.1}$$

satisfy the commutation relations of $\mathscr{G}(1, 0)$. These operators can be used to obtain the basic properties of the hypergeometric functions [6].

Here, we apply Method 4B to the type $A$ operators to obtain $K$-operators which generate the infinite-dimensional Lie algebra $\mathscr{G}^{\omega,\lambda}$. The equation for $\beta(z)$ is $z\beta' - \beta = 1$ with solution $\beta(z) = -1 + c_1 z$. Substituting this result into (1.43) we find

$$h_m(z) = c_2 z^{u+\omega+1}(1 - c_1 z)^{-m-\omega-1}. \tag{5.2}$$

We now use the $K$-operators to construct models of irreducible representations of $\mathscr{G}^{\omega,\lambda}$. Here we consider only a very special class of irreducible representations which have already appeared in the literature on special function theory. Namely, we write $\lambda = u(u + 1)$, where $2u$ is not an integer, and consider the representations $\mathscr{T}^{\omega,u}(m_0)$, where $m_0 \in \mathcal{C}$, $0 \leqq \operatorname{Re} m_0 < 1$ and $u \pm m_0$, $\omega + m_0$ are not integers. The representation space $\mathscr{W}$ has a basis $\{f_m\}$, $m \in S = \{m_0 - n; n = 0, \pm 1, \pm 2, \cdots\}$. The action of $\mathscr{G}^{\omega,\lambda}$ on $\mathscr{W}$ is given by

$$\begin{aligned} &K^3 f_m = m f_m, \qquad K^+ f_m = (m + \omega + 1) f_{m+1}, \\ &K^- f_m = -(m + u)(m - u - 1) f_{m-1}, \end{aligned} \tag{5.3}$$

$$\begin{aligned} C' f_m &= \{K^+K^- + (K^3)^3 - (1 + u)(K^3)^2 - u^2 K^3\} f_m \\ &= \omega u(u + 1) f_m, \qquad m \in S. \end{aligned} \tag{5.4}$$

*Note.* To classify all irreducible representations of $\mathscr{G}$ which are of interest for special function theory it is easiest to proceed by considering that the operators $J^{\pm}$ define a factorization and using the factorization method to compute the possible ladders of solutions [8], [13]. Here, we are discussing only one class of such representations which proves to be especially useful.

As usual we can find three linearly independent sets of basis vectors $\{f_m\}$ corresponding to the eigenvector equation $C' f_m = \omega u(u + 1) f_m$. One set of solutions is

$$f_m(z, t) = \frac{\Gamma(m - u)}{\Gamma(m + \omega + 1)} {}_2F_1(m - u, -u - q; -2u; z) t^m$$

and another set is

$$\begin{aligned} f_m(z, t) = (-1)^m &\frac{\Gamma(m - u)\Gamma(m + u + 1)}{\Gamma(u - m + 1)\Gamma(m + \omega + 1)} z^{2u+1} \\ &\cdot {}_2F_1(m + u + 1, u - q + 1, 2 + 2u; z). \end{aligned}$$

These solutions correspond to the case $c_2 \equiv 0$ in (5.2). We now look for models of $\mathscr{T}^{u,\omega}(m_0)$ for which $c_2 \not\equiv 0$. In this case the equation $C' f_m = \omega u(u + 1) f_m$ becomes

$$\begin{aligned} &z(1 - z) g_m''(z) + [-2u - z(m - q - 2u + 1)] g_m'(z) - (u - m)(q + u) g_m(z) \\ &\quad = c_2 z^{u+\omega}(1 - c_1 z)^{-m-\omega-1}, \end{aligned} \tag{5.5}$$

where $f_m(z, t) = g_m(z) t^m$.

If $c_1 = 0$ we can solve (5.5) by the method of Frobenius in such a way that the recurrence relations (5.3) hold. It can be shown that a solution is provided by the functions

$$g_m(z) = f_{u+\omega+1}(m-u, -u-q; -2u; z) \tag{5.6}$$

if $\omega + u$ and $\omega - \mu$ are not negative integers, where

$$f_\sigma(a,b;c;z) = \frac{z^\sigma}{\sigma(\sigma+c-1)} {}_3F_2(1, \sigma+a, \sigma+b; \sigma+1, \sigma+c; z)$$

(see [5, p. 201]). This series converges for $|z| < 1$, but the function $f_\sigma(a,b;c;z)$ can be analytically continued to the $z$-plane with a cut from 0 to $\infty$ along the positive real axis. In terms of the $f_\sigma$ the Lie algebra relations (5.3), (5.4) read

$$zf'_\sigma(a,b;c;z) + af_\sigma(a,b;c;z) = (\sigma+a)f_\sigma(a+1,b;c;z),$$

$$(\sigma+a-1)\{(1-z)zf'_\sigma(a,b;c;z) + (c-a-bz)f_\sigma(a,b;c;z)\} - z^\sigma$$
$$= (c-a)(a-1)f_\sigma(a-1,b;c;z), \tag{5.7}$$

$$z(1-z)f''_\sigma(a,b;c;z) + \{c-(a+b+1)z\}f'_\sigma(a,b;c;z) - abf_\sigma(a,b;c;z) = z^{\sigma-1}. \tag{5.8}$$

Applying local Lie theory to the first order operator $K^+$ we obtain the identity

$$(1-t)^{-a}f_\sigma\left(a,b;c;\frac{z}{1-t}\right) = \sum_{n=0}^\infty b^n \binom{a+\sigma+n-1}{n} f_\sigma(a+n,b;c;z), \quad |t|<1. \tag{5.9}$$

Furthermore we can apply Weisner's method, a simple analogy of the computation in [6, p. 210], to obtain the identity

$$\tau^{-\sigma}(1+\tau)^{-a}f_\sigma\left(a,b;c;\frac{z\tau}{1+\tau}\right) = \sum_{n=0}^\infty \binom{-a}{n} f_\sigma(\sigma+c+n,b;c;z)\tau^n, \quad |\tau|<1. \tag{5.10}$$

We now construct a basis for $\mathscr{T}^{\omega,u}(m_0)$ with arbitrary $c_1 = \rho$. For $|\rho z| < 1$ we can expand $(1-\rho z)^{-m-\omega-1}$ in a power series on the right-hand side of (5.5). Making use of relations (5.7), (5.8) we obtain the basis

$$g_m(z) = S_{\rho,u+\omega+1}(m-u, -u-q; -2u; z), \tag{5.11}$$

where

$$S_{\rho,\sigma}(a,b;c;z) = \sum_{n=0}^\infty \binom{a+\sigma+n-1}{n} f_{\sigma+n}(a,b;c;z)\rho^n, \quad |z|<1, \quad |\rho z|<1.$$

Our basis consists of a slightly restricted class of the nonhomogeneous hypergeometric functions $C^{(\tau)}_{\rho,\sigma}(a,b;c;z)$ (see [5]). In fact,

$$S_{\rho,\sigma}(a,b;c;z) = C^{(1-a-\sigma)}_{\rho,\sigma}(a,b;c;z).$$

Relations (5.3), (5.4) become

$$zS'_{\rho,\sigma}(a,b;c;z) + aS_{\rho,\sigma}(a,b;c;z) = (\sigma + a)S_{\rho,\sigma}(a+1,b;c;z),$$

$$(5.12) \quad (\sigma + a - 1)\{(1-z)zS'_{\rho,\sigma}(a,b;c;z) + (c - a - bz)S_{\rho,\sigma}(a,b;c;z)\} - z^{\sigma}(1-\rho z)^{-a-\sigma-1} = (c-a)(a-1)\cdot S_{\rho,\sigma}(a-1,b;c;z),$$

$$(5.13) \quad z(1-z)S''_{\rho,\sigma}(a,b;c;z) + [c - (a+b+1)z]S'_{\rho,\sigma}(a,b;c;z) - abS_{\rho,\sigma}(a,b;c;z) = z^{\sigma-1}(1-\rho z)^{-a-\sigma}.$$

An application of local Lie theory leads to identities similar to (5.9) and (5.10) which we shall not bother to list. Finally, in the case $\omega = -u$, $\rho = \frac{1}{2}$ another basis for the representation space is provided by the inhomogeneous hypergeometric functions $B(a,b;c;z)$:

$$g_m(z) = (u-m)B(m-u, -u-q; -2u; z)$$

(see [5, p. 167]).

At this point we cease the detailed examination of each of our methods and simply list the values of $\alpha(z)$, $\beta(z)$ and $h_m(z)$ for the type $A$ operators

*Method* 4A. $\alpha(z) = (1 + c_1 z)/(1-z)$,
$h_m(z) = c_2 z^{u-\omega+1}(1-z)^{q-m}(1+c_1 z)^{\omega+m-1}$.

*Method* 1A. $\alpha(z) = c_1 z/(1-z)$, $h_m(z) = c_2 z^{m+u}(1-z)^{q-m}e^{1/c_1 z}$.

*Method* 1B. $\beta(z) = c_1 z$, $h_m(z) = c_2 z^{u-m}e^{1/c_1 z}$.

Each of these cases can be treated in ways analogous to Method 4B.

The type $B$ operators

$$(5.14) \quad J^+ = t\left(z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - z + u + 1\right), \qquad J^- = t^{-1}\left(z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t} + u + 1\right), \qquad J^3 = t\frac{\partial}{\partial t}, \qquad u \in \mathcal{C},$$

also satisfy the commutation relations of $\mathcal{G}(1,0)$. A study of these operators leads to recurrence formulas and identities for the confluent hypergeometric functions which are of a different nature from those derived from the type $C'$ operators [6]. We could apply Methods 1A, 1B, 4A, 4B to these operators and derive new classes of nonhomogeneous confluent hypergeometric functions. However, the results so obtained would all turn out to be limiting forms of functions obtained from the type $A$ operators, so we omit the computations and merely list $\alpha(z)$, $\beta(z)$ and $h_m(z)$:

*Method* 1A. $\alpha(z) = c_1 z$, $h_m(z) = c_2 z^{m-u-1}e^{1/c_1 z}$.

*Method* 1B. $\beta(z) = c_1 z$, $h_m(z) = c_2 z^{-m-u-1}e^{1/c_1 z}e^z$.

*Method* 4A. $\alpha(z) = 1 + c_1 z$, $h_m(z) = c_2 z^{-u-\omega}(1+c_1 z)^{m+\omega+1}$.

*Method* 4B. $\beta(z) = c_1 z - 1$, $h_m(z) = c_2 z^{\omega-u}(1-c_1 z)^{-\omega-m-1}e^z$.

Corresponding to many special cases in Methods 4A and 4B the basis functions turn out to be identical with special functions derived in § 3. However, the recurrence relations raise and lower different parameters from what was the case in § 3.

## REFERENCES

[1] G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1944.

[2] H. BUCHHOLZ, *Die Lösungen einer besonderen Whittakerschen inhomogenen Differentialgleichung*, Math. Z., 57 (1953), pp. 167–192.

[3] A. W. BABISTER, *Generalized modified Struve functions*, Quart. J. Math. Oxford Ser. (2), 10 (1959), pp. 214–223.

[4] ———, *An inhomogeneous hypergeometric function*, Math. Z., 76 (1961), pp. 164–173.

[5] ———, *Transcendental Functions Satisfying Nonhomogeneous Differential Equations*, Macmillan, New York, 1967.

[6] W. MILLER, *Lie Theory and Special Functions*, Academic Press, New York, 1968.

[7] H. J. LIPKIN, *Lie Groups for Pedestrians*, North-Holland, Amsterdam, 1966.

[8] W. MILLER, *Lie theory and q-difference equations*, SIAM J. Math. Anal., 1 (1970), pp. 171–188.

[9] ———, *On Lie algebras and some special functions of mathematical physics*, Mem. Amer. Math. Soc., no. 50, Providence, 1964.

[10] L. WEISNER, *Group theoretic origin of certain generating functions*, Pacific J. Math., 5 (1955), pp. 1033–1039.

[11] ———, *Generating functions for Bessel functions*, Canad. J. Math., 11 (1959), pp. 148–155.

[12] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. TRICOMI, *Higher Transcendental Functions*, vols. I, II, McGraw-Hill, New York, 1953.

[13] L. INFELD AND T. HULL, *The factorization method*, Rev. Modern Phys., 23 (1951), pp. 21–68.

SIAM J. MATH. ANAL.
Vol. 1, No. 2, May 1970

# EVALUATION OF THE GAMMA FUNCTION BY MEANS OF PADÉ APPROXIMATIONS*

YUDELL L. LUKE†

**1. Summary and introduction.** In my recent work on the special functions [1, vol. 2, Chap. XIV], it is shown that each of two forms of the incomplete gamma function can be evaluated by sequences of Padé approximations as a function of the variable $z$ for a given value of the parameter $v$. Effective asymptotic error representations are developed. In the present study, we show how to combine these Padé approximations to evaluate $\Gamma(v+1)$ for complex $v$.

**2. Rational approximations for incomplete gamma functions and the gamma function.** It is well known [1, vol. 1, Chap. II] that

$$\Gamma(v) = \int_0^\infty e^{-t}t^{v-1}\,dt, \qquad R(v) > 0, \tag{1}$$

or

$$\Gamma(v) = \int_0^z e^{-t}t^{v-1}\,dt + \int_z^\infty e^{-t}t^{v-1}\,dt, \qquad R(v) > 0, \tag{2}$$

with $z$ suitably restricted. For example, for the purposes at hand, it is sufficient to have $z \neq 0$, $R(z) > 0$. From results in [1, vol. 2, pp. 189–191], we can write

$$v\int_0^z e^{-t}t^{v-1}\,dt = \frac{C_n(v,z)}{D_n(v,z)} + L_n(v,z), \tag{3}$$

$$\begin{aligned} C_n(v,z) = {}& \frac{\Gamma(n+v+1-a)(-1)^n}{\Gamma(v+1)}\left[-\frac{n(n+v)}{z}\right]^a e^{-z}z^{n+v} \\ & \cdot \sum_{k=0}^{n-a} \frac{(a-n)_k(n+v+1)_k}{(v+1)_k(1+a)_k}\, {}_3F_1\left(\begin{matrix}-n+a+k, n+v+1+k, 1\\ 1+a+k\end{matrix}\middle| 1/z\right), \end{aligned} \tag{4}$$

$$\begin{aligned} D_n(v,z) &= \frac{\Gamma(2n+v+1-a)}{\Gamma(v+1)}\, {}_1F_1(-n; -2n+a-v; -z) \\ &= \frac{\Gamma(n+v+1-a)(-1)^n z^n}{\Gamma(v+1)}\, {}_2F_0(-n, n+v+1-a; 1/z), \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{L_n(v,z)}{\Gamma(v+1)} = {}& \frac{(-1)^n z^{2n+v+1-a} n!\,\Gamma(n+v+1-a)}{\Gamma(2n+v+1-a)\Gamma(2n+v+2-a)} \\ & \exp\left\{\frac{z(z-4v+4a)}{4(2n+v+1-a)}\right\}[1+O(n^{-3})], \end{aligned} \tag{6}$$

* Received by the editors July 24, 1969, and in revised form November 28, 1969.

† Midwest Research Institute, Kansas City, Missouri 64110. This research was supported by the United States Atomic Energy Commission under Contract AT(11-1)1619.

where $a = 0$ or 1. The error representation (6) is valid for all fixed $z$ and $v$, $R(v) > 0$. Now

$$e^z z^{-v} C_n(v,z) = \begin{cases} 1 & \text{for } n = 0, \quad a = 0, \\ (v+1)(v+2) + z & \text{for } n = 1, \quad a = 0, \\ (v+1)(v+2)(v+3)(v+4) \\ \qquad - (v+2)(v+3)(v-2)z + 2z^2 & \text{for } n = 2, \quad a = 0; \end{cases}$$

(7)

$$e^z z^{-v} C_n(v,z) = \begin{cases} 0 & \text{for } n = 0, \quad a = 1, \\ v + 1 & \text{for } n = 1, \quad a = 1, \\ (v+2)[(v+1)(v+3) - (v-1)z] & \text{for } n = 2, \quad a = 1; \end{cases}$$

$$D_n(v,z) = \begin{cases} 1 & \text{for } n = 0, \quad a = 0, \\ (v+1)(v+2-z) & \text{for } n = 1, \quad a = 0, \\ (v+1)(v+2)[(v+3)(v+4) - 2z(v+3) + z^2] & \text{for } n = 2, \quad a = 0; \end{cases}$$

(8)

$$D_n(v,z) = \begin{cases} 1/v & \text{for } n = 0, \quad a = 1, \\ v + 1 - z & \text{for } n = 1, \quad a = 1, \\ (v+1)[(v+2)(v+3) - 2z(v+2) + z^2] & \text{for } n = 2, \quad a = 1; \end{cases}$$

and both $C_n(v, z)$ and $D_n(v, z)$ satisfy the same recurrence formula,

$$\begin{aligned} D_{n+1}(v,z) &= \{(2n+v-a)(2n+v+1-a)(2n+v+2-a) - z(v-a)(2n+v+1-a)\} \\ &\quad \cdot \frac{D_n(v,z)}{(2n+v-a)} + \frac{n(n+v-a)(2n+v+2-a)z^2}{(2n+v-a)} D_{n-1}(v,z). \end{aligned} \tag{9}$$

Notice that $e^z z^{-v} C_n(v, z)$ and $D_n(v, z)$ are polynomials in $z$ of degree $n - a$ and $n$, respectively, so that the approximations $e^z z^{-v} C_n(v, z)/D_n(v, z)$ as a function of $z$ are rational and occupy the positions $(n - a, n)$ of the Padé table [1, vol. 2, pp. 75, 188]. Further, from (4) and (5), we deduce that both $e^z z^{-v} C_n(v, z)$ and $D_n(v, z)$ are polynomials in $v$ of degree $2n - a$. Thus the approximations $e^z z^{-v} C_n(v, z)/D_n(v, z)$ are also rational in $v$.

It can be shown that use of (9) in the forward direction to generate $C_n(v, z)$ and $D_n(v, z)$ is stable, that is, the relative error due to arbitrary errors introduced at any stage of the computation process remains bounded as $n \to \infty$. This is a consequence of the fact that both $E_n(v, z)/C_n(v, z)$ and $E_n(v, z)/D_n(v, z)$ go to zero rapidly as $n \to \infty$ where $E_n(v, z) = D_n(v, z)/L_n(v, z)$. This result has been communicated to me by my colleague J. Wimp, and his analysis of this problem for a general difference equation will appear shortly.

Again using data in [1, vol. 2, pp. 198–201, 204] we have

$$v \int_z^\infty e^{-t} t^{v-1}\, dt = \frac{G_m(v,z)}{H_m(v,z)} + U_m(v,z), \tag{10}$$

$$G_m(v,z) = v(2-a-v)_m\left(\frac{mz}{1-v}\right)^a e^{-z}z^v \sum_{k=0}^{m-a}\frac{(a-m)_k(1-v)_k}{(2-v)_k(1+a)_k} \tag{11}$$
$$\cdot\, {}_2F_2\left(\begin{matrix} -m+a+k, 1 \\ 2-v+k, 1+a+k\end{matrix}\middle| -z\right),$$

$$H_m(v,z) = z(2-a-v)_m\, {}_1F_1(-m; 2-a-v; -z), \tag{12}$$

$$\frac{U_m(v,z)}{\Gamma(v+1)} = 2(-1)^{1-a}(\sin v\pi)\exp\left[-2k(2\alpha+\sinh 2\alpha) + \frac{2P_1(\alpha)}{k}\right][1+O(k^{-3})],$$
$$k = m+1-\frac{1}{2}(a+v), \qquad \frac{z}{4k} = \sinh^2\alpha,$$
$$P_1(\alpha) = \frac{(v+a)(v+a-2)}{8}\coth\alpha + \frac{(9\coth\alpha - 6\tanh\alpha + 5\tanh^3\alpha)}{96}, \tag{13}$$

where again $a = 0$ or 1. The error representation (13) is valid for $v$ fixed, $m \to \infty$ and $|\arg z| \leqq \pi - \delta$, $\delta > 0$, uniformly in $z$. If $z$ and $v$ are fixed and $m \to \infty$, then

$$\frac{U_m(v,z)}{\Gamma(v+1)} = 2(-1)^{1-a}(\sin v\pi)\exp\{-4(kz)^{1/2}\}[1+O(k^{-1/2})], \qquad |\arg z| < \pi. \tag{14}$$

Notice that the rational approximation in (10) is exact if $v$ is a positive integer or zero provided $m \geqq v + a - 1$. Further,

$$v^{-1}e^z z^{-v}G_m(v,z) = \begin{cases} 1 & \text{for } m = 0, \quad a = 0, \\ 1+z & \text{for } m = 1, \quad a = 0, \\ 2+(5-v)z+z^2 & \text{for } m = 2, \quad a = 0; \end{cases} \tag{15}$$
$$v^{-1}e^z z^{-v}G_m(v,z) = \begin{cases} 0 & \text{for } m = 0, \quad a = 1, \\ z & \text{for } m = 1, \quad a = 1, \\ (3-v)z+z^2 & \text{for } m = 2, \quad a = 1; \end{cases}$$

$$H_m(v,z) = \begin{cases} z & \text{for } m = 0, \\ z(2-v-a+z) & \text{for } m = 1, \\ z[(2-v-a)(3-v-a)+2z(3-v-a)+z^2] & \text{for } m = 2; \end{cases} \tag{16}$$

and both $G_m(v,z)$ and $H_m(v,z)$ satisfy the same three-term recurrence formula,

$$H_{m+1}(v,z) = (z+2m+2-v-a)H_m(z) - m(m+1-v-a)H_{m-1}(z). \tag{17}$$

The approximations $e^z z^{-v}G_m(v,z)/H_m(v,z)$ are rational in $z$ and as a function of $z$ occupy the positions $(m-a, m)$ of the Padé table [1, vol. 2, pp. 75, 189]. Also from (11) and (12) or from (15)–(17), $e^z z^{-v}G_m(v,z)$ and $H_m(v,z)$ are polynomials in $v$ of degree $m - a$ (provided $m \geqq 1$) and $m$, respectively. Thus the approximations $e^z z^{-v}G_m(v,z)/H_m(v,z)$ are also rational in $v$.

Use of (17) in the forward direction to generate $G_m(v,z)$ and $H_m(v,z)$ is also stable. The discussion is akin to that for the similar use of (9).

Combining our results, we have

$$\Gamma(\nu + 1) = \frac{C_n(\nu, z)}{D_n(\nu, z)} + \frac{G_m(\nu, z)}{H_m(\nu, z)} + L_n(\nu, z) + U_m(\nu, z), \tag{18}$$

and for $\nu$ and $z$ fixed, $R(\nu) \geqq 0$, $R(z) > 0$,

$$\lim_{n\to\infty} L_n(\nu, z) = \lim_{m\to\infty} U_m(\nu, z) = 0. \tag{19}$$

The restrictions on $z$ are sufficient and can be considerably relaxed in view of the comments following (6) and (13).

In connection with the application of (18), the following formulas (see [1, vol. 1, Chap. II]) for the analytic continuation of the gamma function are useful.

$$\Gamma(\nu + 1) = \nu\Gamma(\nu), \tag{20}$$

$$\Gamma(\nu)\Gamma(1 - \nu) = \pi \csc \nu\pi, \tag{21}$$

$$\Gamma(2\nu) = \frac{2^{2\nu-1}\Gamma(\nu)\Gamma(\nu + \frac{1}{2})}{\pi^{1/2}}. \tag{22}$$

In some applications, it is desirable to have $\ln \Gamma(\nu + 1)$. Given $\Gamma(\nu + 1)$, care must be exercised in the evaluation of its natural logarithm to ensure proper specification of its phase as dictated by the known asymptotic expansion of $\ln \Gamma(\nu + 1)$ (see [1, vol. 1, p. 31]). We now illustrate how this can be done. Let

$$\begin{gathered} \Gamma(\nu + 1) = K + iL = (K^2 + L^2)^{1/2} e^{i(\varphi + 2\pi s)}, \\ \nu + 1 = \beta e^{i\xi}, \quad \beta > 0, \quad 0 \leqq \xi < \pi/2, \quad \beta \cos \xi > 1, \end{gathered} \tag{23}$$

where $s$ is a positive integer or zero to be determined. Also let

$$\sigma = \arctan |L/K|, \qquad 0 \leqq \sigma \leqq \pi/2.$$

Then

$$\varphi = \begin{cases} \sigma & \text{if } K \geqq 0, \quad L \geqq 0, \\ \pi - \sigma & \text{if } K < 0, \quad L \geqq 0, \\ \pi + \sigma & \text{if } K < 0, \quad L < 0, \\ 2\pi - \sigma & \text{if } K \geqq 0, \quad L < 0. \end{cases} \tag{24}$$

Now for $|\arg(\nu + 1)| < \pi$,

$$\ln \Gamma(\nu + 1) = \left(\nu + \frac{1}{2}\right) \ln(\nu + 1) - (\nu + 1) + \frac{1}{2} \ln 2\pi + \frac{1}{12(\nu + 1)} + R; \tag{25}$$

and with

$$\nu + 1 = \mu + i\omega, \quad \mu > 1, \quad \omega \geqq 0, \tag{26}$$

we have

$$|R| < \begin{cases} \dfrac{1}{360|v+1|^3} & \text{if } \omega \leqq \mu, \\ \dfrac{1}{720\mu\omega|v+1|} & \text{if } \omega > \mu.^1 \end{cases} \tag{27}$$

Thus for $\mu$ and $\omega$ as defined by (26), $|R| < 1/360$. Now compare the imaginary part of $\Gamma(v+1)$ as defined by (23) with that obtained from (25), and recognize that the magnitude of the imaginary part of $R$ is less than $|R|$. Then we can write

$$|s - \Delta| < \frac{1}{720\pi}, \qquad \Delta = \frac{\beta \sin \xi(\ln \beta - 1) + \xi(\beta \cos \xi - 1/2) - (\sin \xi)/(12\beta) - \varphi}{2\pi}; \tag{28}$$

and since $s$ is a positive integer or zero, the value of $s$ immediately follows from the calculation of $\Delta$.

**3. Analysis of the error terms.** As will be seen, the numerical values of the asymptotic estimates for the error terms $L_n(v, z)/\Gamma(v+1)$ and $U_m(v, z)/\Gamma(v+1)$ are remarkably realistic. Notice from (18) that the sum of these two quantities gives the relative error. It is useful therefore to examine them further so that a priori assessment of the error is readily accomplished.

As previously remarked, the approximation in (3) occupies the positions $(n-a, n)$ of the Padé table. It follows from [1, vol. 2, Chap. XIV, pp. 75, 188] that for $v$ and $n$ fixed $L_n(v, z) = O(z^{2n+1-a})$, which increases as $z$ increases. Again, the approximations in (10) occupy the positions $(m-a, m)$ of the Padé table; and from [1, vol. 2, Chap. XIV, pp. 75, 189] for $v$ and $m$ fixed, $U_m(v, z) = O(z^{a-1-2m})$, which decreases as $z$ increases. Further, for $z$ and $v$ fixed, $L_n(v, z)$ decreases rather rapidly as $n$ increases since

$$\begin{aligned} \frac{L_{n+1}(v, z)}{L_n(v, z)} &= -\frac{z^2(n+1)(n+v+1-a)}{(2n+v+1-a)(2n+v+2-a)^2(2n+v+3-a)} \\ &\quad \cdot \exp\left[-\frac{z(z-4v+4a)}{2(2n+v+1-a)(2n+v+3-a)}\right][1 + O(n^{-3})], \end{aligned} \tag{29}$$

while under the same conditions $U_m(v, z)$ decreases quite slowly since

$$\frac{U_{m+1}(v, z)}{U_m(v, z)} = \exp\left[-2(z/k)^{1/2}\right][1 + O(k^{-1/2})]. \tag{30}$$

Let $n^*$ and $m^*$ be the values of $n$ and $m$, respectively, required in the evaluation of the approximations for $\Gamma(v+1)$ to achieve a specified accuracy. Then, in practice, the free parameter $z$ should be chosen rather large so that the number of machine operations which is a monotonically increasing function of the sum $n^* + m^*$ is reasonably small. Derivation of an optimal choice of $z$ seems formidable. We have found $z = 8$ a convenient value.

[1] The inequalities for $|R|$ are readily deduced from a discussion given by Salzer [2].

The formulas (31)–(33) which follow are useful to estimate changes in the error for changes in $v$. Thus from (6),

$$\frac{L_n(v+h,z)\Gamma(v+1)}{L_n(v,z)\Gamma(v+h+1)}$$

$$= \frac{z^h\Gamma(n+v+1+h-a)\Gamma(2n+v+1-a)\Gamma(2n+v+2-a)}{\Gamma(n+v+1-a)\Gamma(2n+v+1+h-a)\Gamma(2n+v+2+h-a)}$$

$$\cdot \exp\left[-\frac{zh(8n+4+z)}{4(2n+v+1+h-a)(2n+v+1-a)}\right][1+O(n^{-3})] \tag{31}$$

$$= \left[\frac{z(n+v+1-a)}{(2n+v+1-a)(2n+v+2-a)}\right]^h$$

$$\cdot \exp\left[-\frac{zh(8n+4+z)}{4(2n+v+1+h-a)(2n+v+1-a)}\right][1+O(n^{-1})];$$

and from (13) and (14), respectively, we find[2]

$$\frac{U_m(v+h,z)\Gamma(v+1)}{U_m(v,z)\Gamma(v+h+1)} = \left(\frac{\sin(v+1)\pi}{\sin v\pi}\right)\exp[A+B],$$

$$A = 2\alpha h + \frac{h^2}{4k}\tanh\alpha + \frac{h^3(2\cosh^2\alpha+1)}{48k^2\cosh^2\alpha}\tanh\alpha + O(k^{-3}) + O(h^4k^{-3}),$$

$$B = \frac{h}{4k}(2v+2a-2+h)\coth\alpha + \frac{h}{k^2}P_1(\alpha) + \frac{h^2}{8k^2}(2v+2a-2+h)\coth\alpha$$

$$- \frac{h}{64k^2\cosh^2\alpha} \tag{32}$$

$$\cdot\{(2v+2a+2h-1)(2v+2a+2h-3)\coth\alpha + 2\tanh\alpha - 5\tanh^3\alpha\}$$

$$+ O(k^{-3}) + O(h^2k^{-3}),$$

$$\frac{U_m(v+h,z)\Gamma(v+1)}{U_m(v,z)\Gamma(v+h+1)} = \left(\frac{\sin(v+h)\pi}{\sin v\pi}\right)\exp[h(z/k)^{1/2}][1+O(k^{-1/2})]. \tag{33}$$

From (6) and (13), $v$ and $a$ always appear together in the form $(v-a)$ and $(v+a)$, respectively. Thus it is sufficient to know $L_n(v,z)/\Gamma(v+1)$ and $U_m(v,z)/\Gamma(v+1)$ for $a = 0$ only in view of (31)–(33). Further, from (6),

$$\frac{L_n(v+2,z)\Gamma(v+1)}{L_{n+1}(v,z)\Gamma(v+3)} = \frac{(n+v+2-a)}{(n+1)}\exp\left[-\frac{2z}{2n+v+3-a}\right][1+O(n^{-3})]. \tag{34}$$

Also if $v$ is replaced by $v+2$ and $m$ is replaced by $m+1$, the value of $k$ remains the same. Thus, from (13) and (14), respectively,

[2] Equation (32) is a generalization of (22) and (23) given in [1, vol. 2, p. 206]. There, in (23), the sign of $(v^2\tanh\alpha_0)/(4k)$ should be positive. Also in the last line on this page, for 0.672 read 0.677.

$$\frac{U_m(v,z)\Gamma(v+3)}{U_{m+1}(v+2,z)\Gamma(v+1)} = \exp\left[-\frac{(v+a)\coth\alpha}{k}\right][1+O(k^{-3})], \tag{35}$$

$$k = m+1-\frac{1}{2}(a+v),$$

$$\frac{U_m(v,z)\Gamma(v+3)}{U_{m+1}(v+2,z)\Gamma(v+1)} = 1+O(k^{-1/2}). \tag{36}$$

In view of the above analysis, we find it sufficient to prepare tables for the error terms when the real part of $v$ is $\frac{1}{2}$, $\frac{5}{2}$ and $\frac{9}{2}$. See later discussion.

For proof and further developments relating to the material presented herein, see the cited references. There coefficients for the polynomials in (4) and (5) and in (11) and (12) are presented for the more common transcendents of the incomplete gamma function family. Further, tables of $v^{-1}|L_n(v,z)|$ for $v=\frac{1}{2}$ and $v^{-1}|U_n(v,z)|$ for $v=0,\frac{1}{2}$ are presented for extensive values of $n$ and a wide range of complex $z$ values where also in each instance $a=0$.

**4. Numerical examples.** Let

$$v+1=\mu+i\omega, \quad \mu>1, \quad \omega \geqq 0. \tag{37}$$

We therefore consider the complex $v$-plane such that $R(v)>0$ and $I(v)\geqq 0$. Extension to the balance of the complex $v$-plane follows by use of the conjugacy principle, (20) and (21). For $|v|$ very large, $|\arg v|<\pi$, the known asymptotic expansion might suffice. In our approximation scheme, we can allow $v$ to be rather large. Nonetheless, in practice, the formulas (20)–(22) should be used as appropriate.

Equations (38)–(45) given below relate notation to computer output. Let

$$\frac{C_n(v,z)}{D_n(v,z)}=\frac{C(N)}{D(N)}, \qquad \frac{G_m(v,z)}{H_m(v,z)}=\frac{G(M)}{H(M)}, \tag{38}$$

$$X(N)=\frac{C(N)}{D(N)}-\frac{C(N-1)}{D(N-1)}, \qquad Z(M)=\frac{G(M)}{H(M)}-\frac{G(M-1)}{H(M-1)}, \tag{39}$$

$$R(N)=\left|\frac{C(L_1)}{D(L_1)}-\frac{C(N)}{D(N)}\right|, \qquad S(N)=\frac{R(N)}{|\Gamma(v+1)|}, \tag{40}$$

$$W(M)=\left|\frac{G(L_2)}{H(L_2)}-\frac{G(M)}{H(M)}\right|, \qquad Y(M)=\frac{W(M)}{|\Gamma(v+1)|}, \tag{41}$$

where $L_1$ and $L_2$ are defined by the requirements that

$$|X(L_1)|\leqq\varepsilon, \qquad |Z(L_2)|\leqq\varepsilon. \tag{42}$$

Then the approximation for $\Gamma(v+1)$ (we use the same notation) is

$$\Gamma(v+1)=\frac{C(L_1)}{D(L_1)}+\frac{G(L_2)}{H(L_2)}, \tag{43}$$

and the latter is used to evaluate $S(N)$ and $Y(M)$.

Thus for a given $\nu$ and $z$, the machine calculates $C(N)$, $D(N)$ for $N = 0$ and 1 from (7) and (8), and generates these quantities for higher $N$ by use of (9). The ratio $C(N)/D(N)$ is evaluated for each $N$ as is also $X(N)$ and the iteration process ceases for $N = L_1$ as defined by (42). The scheme for $G(M)/H(M)$ is similar. See (15)–(17). To within the obtainable limits of accuracy,

$$(44) \qquad |L_N(\nu,z)| = R(N), \qquad \left|\frac{L_N(\nu, z)}{\Gamma(\nu + 1)}\right| = S(N),$$

$$(45) \qquad |U_M(\nu, z)| = W(M), \qquad \left|\frac{U_M(\nu, z)}{\Gamma(\nu + 1)}\right| = Y(M).$$

Both $e^z z^{-\nu} C(N)$ and $D(N)$ are polynomials in $\nu$ of degree $2n - a$. It calls for remark, however, that one can just as well generate the sequences $sC(N)$ and $sD(N)$ where $s$ is an arbitrary scale factor independent of $N$, since only the ratio $C(N)/D(N)$ is needed. Similarly, $e^z z^{-\nu} G(M)$ and $H(M)$ are polynomials in $\nu$ of degree $m - a$ (provided $m \geqq 1$) and $m$, respectively, and we can generate $tG(M)$ and $tH(M)$ where $t$ is free of $M$.[3] This is important in practice to avoid overflow. Observe that the numerical values assigned to $s$ and $t$ depend on $\varepsilon$ and the particular type of computing equipment used. For guidance, it is known that both $C(N)$ and $D(N)$ are $O(\Gamma(2N + \nu + 1 - a))$,

$$\Gamma(2N + \nu + 1 - a) = (2N - a)!(2N + 1 - a)^\nu(1 + O(N^{-1})),$$

and that both $G(M)$ and $H(M)$ grow like

$$\exp\left\{\frac{(2a + 2\nu - 3)}{4} \ln m + 2(mz)^{1/2}\right\}.$$

More precise data are available in the cited references.

Machine computations for $\Gamma(\nu + 1)$ done on an IBM 360 are illustrated in Table 1. The parameters used are

$$z = 8, \qquad a = A = 0,$$

$$\nu + 1 = \mu + i\omega, \qquad \mu = 7/2, \qquad \omega = 0, 4 \text{ and } 8,$$

$$\varepsilon = 0.1 \cdot 10^{-14}, \qquad s = t = 0.1 \cdot 10^{-20}.$$

Also recorded are data relating to $\ln \Gamma(\nu + 1)$ (see (23)–(28)). The first and second numbers on the line NU = are the real and imaginary parts of $\nu$, respectively, and similarly for data on the line GAMMA (NU + 1) = .

To facilitate a priori estimation of the error, Table 2 gives values of $|L_n(\nu, z)/\Gamma(\nu + 1)|$ and $|U_m(\nu, z)/\Gamma(\nu + 1)|$ for the following values of the parameters:

$$z = 8, \qquad a = A = 0,$$

$$\nu + 1 = \mu + i\omega, \qquad \mu = \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \qquad \omega = 0, 2, 4, \cdots, 10,$$

$$n = m = 6, 7, 8, \cdots, 20.$$

[3] Actually we can permit $s$ and $t$ to depend on $N$ and $M$, respectively. But then the pertinent recursion formulas (9) and (17) must be altered.

TABLE 1
*Typical approximations for* $\Gamma(\nu + 1)$

Z = 0.8000000000000000D +01
NU = 0.2500000000000000D +01
A = 0.0

| N | C(N)/D(N) | R(N) | S(N) |
|---|---|---|---|
| 0 | 0.6072522215027854D −01 | 0.324D +01 | 0.975D +00 |
| 1 | −0.1177325735566625D +00 | 0.342D +01 | 0.103D +01 |
| 2 | 0.1942755925125257D +00 | 0.311D +01 | 0.935D +00 |
| 3 | −0.1499193165428189D +01 | 0.480D +01 | 0.144D +01 |
| 4 | 0.2223005207639932D +01 | 0.108D +01 | 0.324D +00 |
| 5 | 0.3474646102625896D +01 | 0.174D +00 | 0.524D −01 |
| 6 | 0.3287942221865348D +01 | 0.127D −01 | 0.381D −02 |
| 7 | 0.3301359042871663D +01 | 0.753D −03 | 0.227D −03 |
| 8 | 0.3300570264189797D +01 | 0.354D −04 | 0.107D −04 |
| 9 | 0.3300607067913083D +01 | 0.136D −05 | 0.408D −06 |
| 10 | 0.3300605667636696D +01 | 0.431D −07 | 0.130D −07 |
| 11 | 0.3300605711889519D +01 | 0.115D −08 | 0.347D −09 |
| 12 | 0.3300605710709574D +01 | 0.264D −10 | 0.794D −11 |
| 13 | 0.3300605710736471D +01 | 0.520D −12 | 0.157D −12 |
| 14 | 0.3300605710735942D +01 | 0.866D −14 | 0.261D −14 |
| 15 | 0.3300605710735950D +01 | 0.222D −15 | 0.668D −16 |
| 16 | 0.3300605710735950D +01 | 0.0 | 0.0 |

| M | G(M)/H(M) | W(M) | Y(M) |
|---|---|---|---|
| 0 | 0.1897663192196205D −01 | 0.377D −02 | 0.113D −02 |
| 1 | 0.2277195830635446D −01 | 0.267D −04 | 0.803D −05 |
| 2 | 0.2274551003886740D −01 | 0.250D −06 | 0.753D −07 |
| 3 | 0.2274526743130751D −01 | 0.772D −08 | 0.232D −08 |
| 4 | 0.2274526012085870D −01 | 0.409D −09 | 0.123D −09 |
| 5 | 0.2274525974233226D −01 | 0.304D −10 | 0.916D −11 |
| 6 | 0.2274525971477689D −01 | 0.288D −11 | 0.868D −12 |
| 7 | 0.2274525971222103D −01 | 0.328D −12 | 0.987D −13 |
| 8 | 0.2274525971193594D −01 | 0.430D −13 | 0.130D −13 |
| 9 | 0.2274525971189911D −01 | 0.622D −14 | 0.187D −14 |
| 10 | 0.2274525971189375D −01 | 0.860D −15 | 0.259D −15 |
| 11 | 0.2274525971189289D −01 | 0.0 | 0.0 |

GAMMA (NU +1) = 0.3323350970447843D +01
K + IL = GAMMA (NU +1)
0.5*LOG (K*K + L*L) = 0.1200973602347075D +01
PHI = 0.0
BETA = 0.3500000000000000D +01
XI = 0.0
S = 0.0

Z = 0.8000000000000000D 01
NU = 0.2500000000000000D 01 0.4000000000000000D 01
A = 0.0

| N | C(N)/D(N) (REAL) | C(N)/D(N) (IMAGINARY) | R(N) | S(N) |
|---|---|---|---|---|
| 0 | −0.2716458159668867D −01 | 0.5431057090362354D −01 | 0.447D 00 | 0.117D 01 |
| 1 | 0.6897216907222269D −01 | 0.1587113934835240D −01 | 0.366D 00 | 0.962D 00 |
| 2 | −0.2942056165002189D −02 | −0.2806433446668101D −01 | 0.366D 00 | 0.962D 00 |
| 3 | 0.2575026113146132D 00 | 0.1769425342416403D 00 | 0.494D 00 | 0.130D 01 |
| 4 | 0.1561400730657003D 00 | −0.2539804518338708D 00 | 0.909D −01 | 0.239D 00 |
| 5 | 0.2291067503330697D 00 | −0.3282492324241860D 00 | 0.139D −01 | 0.366D −01 |
| 6 | 0.2224770789072401D 00 | −0.3148468610989733D 00 | 0.110D −02 | 0.291D −02 |
| 7 | 0.2225728013388493D 00 | −0.3160125795248319D 00 | 0.691D −04 | 0.182D −03 |
| 8 | 0.2225919163307489D 00 | −0.3159428832869618D 00 | 0.338D −05 | 0.888D −05 |
| 9 | 0.2225899679201291D 00 | −0.3159457946822422D 00 | 0.133D −06 | 0.350D −06 |
| 10 | 0.2225900744755584D 00 | −0.3159457082077062D 00 | 0.432D −08 | 0.114D −07 |
| 11 | 0.2225900703934754D 00 | −0.3159457099461022D 00 | 0.118D −09 | 0.310D −09 |
| 12 | 0.2225900705127446D 00 | −0.3159457099291516D 00 | 0.273D −11 | 0.719D −11 |
| 13 | 0.2225900705099731D 00 | −0.3159457099288608D 00 | 0.546D −13 | 0.144D −12 |
| 14 | 0.2225900705100255D 00 | −0.3159457099288791D 00 | 0.982D −15 | 0.258D −14 |
| 15 | 0.2225900705100247D 00 | −0.3159457099288785D 00 | 0.0 | 0.0 |

| M | G(M)/H(M) (REAL) | G(M)/H(M) (IMAGINARY) | W(M) | Y(M) |
|---|---|---|---|---|
| 0 | −0.3564421720077698D −01 | 0.3389762609038017D −02 | 0.186D −01 | 0.488D −01 |
| 1 | −0.3498984242183827D −01 | −0.1459353416080147D −01 | 0.898D −03 | 0.236D −02 |
| 2 | −0.3560567014818902D −01 | −0.1518615882837604D −01 | 0.636D −04 | 0.167D −03 |
| 3 | −0.3566515018224126D −01 | −0.1518445568924814D −01 | 0.542D −05 | 0.143D −04 |
| 4 | −0.3566890987633954D −01 | −0.1518119151523717D −01 | 0.534D −06 | 0.140D −05 |
| 5 | −0.3566904574412844D −01 | −0.1518073006483716D −01 | 0.599D −07 | 0.157D −06 |
| 6 | −0.3566903617775854D −01 | −0.1518067798901339D −01 | 0.757D −08 | 0.199D −07 |
| 7 | −0.3566903264338116D −01 | −0.1518067245536636D −01 | 0.106D −08 | 0.280D −08 |
| 8 | −0.3566903193779813D −01 | −0.1518067188575171D −01 | 0.165D −09 | 0.433D −09 |
| 9 | −0.3566903181108371D −01 | −0.1518067183136164D −01 | 0.278D −10 | 0.730D −10 |
| 10 | −0.3566903178856681D −01 | −0.1518067182748120D −01 | 0.505D −11 | 0.133D −10 |
| 11 | −0.3566903178448145D −01 | −0.1518067182761661D −01 | 0.984D −12 | 0.259D −11 |
| 12 | −0.3566903178371576D −01 | −0.1518067182778134D −01 | 0.203D −12 | 0.535D −12 |
| 13 | −0.3566903178356695D −01 | −0.1518067182783909D −01 | 0.444D −13 | 0.117D −12 |
| 14 | −0.3566903178353694D −01 | −0.1518067182785593D −01 | 0.101D −13 | 0.265D −13 |
| 15 | −0.3566903178353069D −01 | −0.1518067182786058D −01 | 0.230D −14 | 0.605D −14 |
| 16 | −0.3566903178352934D −01 | −0.1518067182786185D −01 | 0.453D −15 | 0.119D −14 |
| 17 | −0.3566903178352905D −01 | −0.1518067182786219D −01 | 0.0 | 0.0 |

GAMMA (NU +1) = 0.1869210387264956D 00 −0.3311263817567407D 00
K + IL = GAMMA (NU +1)
0.5*LOG(K*K + L*L) = −0.9669467752727471D 00
PHI = 0.5226296879483303D 01
BETA = 0.5315072906367322D 01
XI = 0.8519663271732719D 00
S = 0.0

Z = 0.8000000000000000D 01
NU = 0.2500000000000000D 01 0.8000000000000000D 01
A = 0.0

| N | C(N)/D(N) (REAL) | C(N)/D(N) (IMAGINARY) | R(N) | S(N) |
|---|---|---|---|---|
| 0 | −0.3642182836451304D −01 | −0.4859015356825528D −01 | 0.548D −01 | 0.115D 02 |
| 1 | −0.6000981534478674D −01 | 0.4921224055167995D −02 | 0.114D −01 | 0.238D 01 |
| 2 | −0.5269752720745536D −01 | 0.6465458769242584D −02 | 0.438D −02 | 0.915D 00 |
| 3 | −0.5047283953617965D −01 | 0.4536836782767226D −02 | 0.187D −02 | 0.391D 00 |
| 4 | −0.4843015625317477D −01 | 0.4209479816145428D −02 | 0.676D −03 | 0.141D 00 |
| 5 | −0.4857636950755245D −01 | 0.4928410833344967D −02 | 0.902D −04 | 0.189D −01 |
| 6 | −0.4863727452911832D −01 | 0.4855259599630383D −02 | 0.817D −05 | 0.171D −02 |
| 7 | −0.4862872319009302D −01 | 0.4854872128805095D −02 | 0.554D −06 | 0.116D −03 |
| 8 | −0.4862910131021031D −01 | 0.4855306891462782D −02 | 0.289D −07 | 0.605D −05 |
| 9 | −0.4862910169133455D −01 | 0.4855277044253696D −02 | 0.120D −08 | 0.251D −06 |
| 10 | −0.4862910092059164D −01 | 0.4855278005678841D −02 | 0.406D −10 | 0.849D −08 |
| 11 | −0.4862910096045970D −01 | 0.4855277993931871D −02 | 0.114D −11 | 0.239D −09 |
| 12 | −0.4862910095933553D −01 | 0.4855277993613363D −02 | 0.273D −13 | 0.572D −11 |
| 13 | −0.4862910095935492D −01 | 0.4855277993633314D −02 | 0.572D −15 | 0.120D −12 |
| 14 | −0.4862910955935477D −01 | 0.4855277993632764D −02 | 0.0 | 0.0 |

| M | G(M)/H(M) (REAL) | G(M)/H(M) (IMAGINARY) | W(M) | Y(M) |
|---|---|---|---|---|
| 0 | 0.3720833220434495D −01 | −0.5160625135459283D −01 | 0.505D −01 | 0.106D 02 |
| 1 | 0.5178555111288123D −01 | −0.6689580438438069D −02 | 0.331D −02 | 0.692D 00 |
| 2 | 0.5303403128464572D −01 | −0.4014822340439671D −02 | 0.369D −03 | 0.771D −01 |
| 3 | 0.5323158541461844D −01 | −0.3763890391333904D −02 | 0.524D −04 | 0.110D −01 |
| 4 | 0.5326822413442684D −01 | −0.3738697475901107D −02 | 0.851D −05 | 0.178D −02 |
| 5 | 0.5327509263942108D −01 | −0.3736846936063718D −02 | 0.151D −05 | 0.315D −03 |
| 6 | 0.5327633048837056D −01 | −0.3736955802698447D −02 | 0.284D −06 | 0.595D −04 |
| 7 | 0.5327654059326604D −01 | −0.3737052976158783D −02 | 0.566D −07 | 0.118D −04 |
| 8 | 0.5327657344623811D −01 | −0.3737084419067980D −02 | 0.118D −07 | 0.247D −05 |
| 9 | 0.5327657791189345D −01 | −0.3737092652531001D −02 | 0.257D −08 | 0.539D −06 |
| 10 | 0.5327657832843548D −01 | −0.3737094624859616D −02 | 0.585D −09 | 0.122D −06 |
| 11 | 0.5327657830161350D −01 | −0.3737095076241622D −02 | 0.138D −09 | 0.289D −07 |
| 12 | 0.5327657826977892D −01 | −0.3737095176813370D −02 | 0.339D −10 | 0.709D −08 |
| 13 | 0.5327657825675627D −01 | −0.3737095198788647D −02 | 0.861D −11 | 0.180D −08 |
| 14 | 0.5327657825240628D −01 | −0.3737095203493894D −02 | 0.226D −11 | 0.473D −09 |
| 15 | 0.5327657825106217D −01 | −0.3737095204471811D −02 | 0.613D −12 | 0.128D −09 |
| 16 | 0.5327657825066091D −01 | −0.3737095204664380D −02 | 0.171D −12 | 0.357D −10 |
| 17 | 0.5327657825054283D −01 | −0.3737095204698150D −02 | 0.488D −13 | 0.102D −10 |
| 18 | 0.5327657825050822D −01 | −0.3737095204702363D −02 | 0.141D −13 | 0.295D −11 |
| 19 | 0.5327657825049806D −01 | −0.3737095204702100D −02 | 0.398D −14 | 0.832D −12 |
| 20 | 0.5327657825049506D −01 | −0.3737095204701605D −02 | 0.935D −15 | 0.196D −12 |
| 21 | 0.5327657825049416D −01 | −0.3737095204701337D −02 | 0.0 | 0.0 |

GAMMA (NU +1) = 0.4647477291139390D −02 0.1118182788931428D −02
K + IL = GAMMA (NU +1)
0.5*LOG(K*K + L*L) = −0.5343293332726673D 01
PHI = 0.2361121753612002D 00
BETA = 0.8732124598286488D 01
XI = 0.1158385885197509D 01
S = 0.2000000000000000D 01

TABLE 2

*Numerical values of asymptotic estimates of errors in the approximations for* $\Gamma(\nu + 1)$

$\left|\dfrac{L_n(\nu, z)}{\Gamma(\nu + 1)}\right|, \quad \nu = 1/2 + \omega i, \quad z = 8$

| $n/\omega$ | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| 6 | 0.129(+00) | 0.108(+00) | 0.104(+00) | 0.161(+00) | 0.392(−01) | 0.595(−02) |
| 7 | 0.878(−02) | 0.770(−02) | 0.709(−02) | 0.138(−01) | 0.562(−02) | 0.103(−02) |
| 8 | 0.474(−03) | 0.427(−03) | 0.387(−03) | 0.794(−03) | 0.519(−03) | 0.117(−03) |
| 9 | 0.207(−04) | 0.190(−04) | 0.172(−04) | 0.340(−04) | 0.330(−04) | 0.936(−05) |
| 10 | 0.745(−06) | 0.696(−06) | 0.630(−06) | 0.115(−05) | 0.153(−05) | 0.551(−06) |
| 11 | 0.225(−07) | 0.213(−07) | 0.193(−07) | 0.322(−07) | 0.535(−07) | 0.247(−07) |
| 12 | 0.577(−09) | 0.550(−09) | 0.504(−09) | 0.768(−09) | 0.148(−08) | 0.872(−09) |
| 13 | 0.127(−10) | 0.122(−10) | 0.113(−10) | 0.159(−10) | 0.334(−10) | 0.247(−10) |
| 14 | 0.243(−12) | 0.235(−12) | 0.218(−12) | 0.287(−12) | 0.629(−12) | 0.575(−12) |
| 15 | 0.408(−14) | 0.396(−14) | 0.370(−14) | 0.460(−14) | 0.101(−13) | 0.112(−13) |
| 16 | 0.605(−16) | 0.558(−16) | 0.554(−16) | 0.657(−16) | 0.142(−15) | 0.183(−15) |
| 17 | 0.798(−18) | 0.778(−18) | 0.736(−18) | 0.842(−18) | 0.175(−17) | 0.259(−17) |
| 18 | 0.941(−20) | 0.921(−20) | 0.875(−20) | 0.972(−20) | 0.192(−19) | 0.317(−19) |
| 19 | 0.100(−21) | 0.981(−22) | 0.936(−22) | 0.102(−21) | 0.191(−21) | 0.342(−21) |
| 20 | 0.962(−24) | 0.945(−24) | 0.905(−24) | 0.965(−24) | 0.172(−23) | 0.327(−23) |

$\left|\dfrac{U_m(\nu, z)}{\Gamma(\nu + 1)}\right|, \quad \nu = 1/2 + \omega i, \quad z = 8$

| $m/\omega$ | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| 6 | 0.846(−13) | 0.158(−10) | 0.296(−08) | 0.313(−06) | 0.211(−04) | 0.101(−02) |
| 7 | 0.113(−13) | 0.217(−11) | 0.446(−09) | 0.525(−07) | 0.393(−05) | 0.205(−03) |
| 8 | 0.169(−14) | 0.334(−12) | 0.736(−10) | 0.955(−08) | 0.788(−06) | 0.448(−04) |
| 9 | 0.280(−15) | 0.566(−13) | 0.132(−10) | 0.186(−08) | 0.168(−06) | 0.104(−04) |
| 10 | 0.505(−16) | 0.104(−13) | 0.254(−11) | 0.385(−09) | 0.377(−07) | 0.254(−05) |
| 11 | 0.979(−17) | 0.204(−14) | 0.523(−12) | 0.842(−10) | 0.889(−08) | 0.645(−06) |
| 12 | 0.203(−17) | 0.428(−15) | 0.114(−12) | 0.194(−10) | 0.219(−08) | 0.170(−06) |
| 13 | 0.444(−18) | 0.949(−16) | 0.260(−13) | 0.466(−11) | 0.559(−09) | 0.466(−07) |
| 14 | 0.103(−18) | 0.221(−16) | 0.625(−14) | 0.117(−11) | 0.148(−09) | 0.131(−07) |
| 15 | 0.248(−19) | 0.541(−17) | 0.156(−14) | 0.304(−12) | 0.406(−10) | 0.381(−08) |
| 16 | 0.627(−20) | 0.138(−17) | 0.407(−15) | 0.821(−13) | 0.115(−10) | 0.113(−08) |
| 17 | 0.165(−20) | 0.364(−18) | 0.110(−15) | 0.229(−13) | 0.334(−11) | 0.346(−09) |
| 18 | 0.449(−21) | 0.998(−19) | 0.307(−16) | 0.658(−14) | 0.997(−12) | 0.108(−09) |
| 19 | 0.126(−21) | 0.282(−19) | 0.884(−17) | 0.195(−14) | 0.306(−12) | 0.347(−10) |
| 20 | 0.366(−22) | 0.824(−20) | 0.262(−17) | 0.592(−15) | 0.961(−13) | 0.113(−10) |

$\left|\frac{L_n(v, z)}{\Gamma(v + 1)}\right|, \quad v = 5/2 + \omega i, \quad z = 8$

| $n/\omega$ | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| 6 | 0.380(−02) | 0.348(−02) | 0.323(−02) | 0.531(−02) | 0.228(−02) | 0.458(−03) |
| 7 | 0.226(−03) | 0.210(−03) | 0.195(−03) | 0.340(−03) | 0.225(−03) | 0.545(−04) |
| 8 | 0.106(−04) | 0.100(−04) | 0.933(−05) | 0.159(−04) | 0.152(−04) | 0.455(−05) |
| 9 | 0.407(−06) | 0.389(−06) | 0.363(−06) | 0.584(−06) | 0.744(−06) | 0.279(−06) |
| 10 | 0.129(−07) | 0.124(−07) | 0.117(−07) | 0.175(−07) | 0.275(−07) | 0.130(−07) |
| 11 | 0.347(−09) | 0.335(−09) | 0.316(−09) | 0.444(−09) | 0.797(−09) | 0.477(−09) |
| 12 | 0.793(−11) | 0.770(−11) | 0.730(−11) | 0.965(−11) | 0.188(−10) | 0.140(−10) |
| 13 | 0.157(−12) | 0.153(−12) | 0.145(−12) | 0.182(−12) | 0.368(−12) | 0.335(−12) |
| 14 | 0.270(−14) | 0.264(−14) | 0.235(−14) | 0.304(−14) | 0.613(−14) | 0.669(−14) |
| 15 | 0.410(−16) | 0.403(−16) | 0.386(−16) | 0.448(−16) | 0.888(−16) | 0.113(−15) |
| 16 | 0.553(−18) | 0.543(−18) | 0.523(−18) | 0.589(−18) | 0.113(−17) | 0.163(−17) |
| 17 | 0.665(−20) | 0.655(−20) | 0.633(−20) | 0.697(−20) | 0.128(−19) | 0.205(−19) |
| 18 | 0.720(−22) | 0.710(−22) | 0.687(−22) | 0.742(−22) | 0.130(−21) | 0.226(−21) |
| 19 | 0.703(−24) | 0.694(−24) | 0.674(−24) | 0.717(−24) | 0.120(−23) | 0.220(−23) |
| 20 | 0.624(−26) | 0.617(−26) | 0.600(−26) | 0.631(−26) | 0.101(−25) | 0.192(−25) |

$\left|\frac{U_m(v, z)}{\Gamma(v + 1)}\right|, \quad v = 5/2 + \omega i, \quad z = 8$

| $m/\omega$ | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| 6 | 0.868(−12) | 0.143(−09) | 0.196(−07) | 0.137(−05) | 0.600(−04) | 0.193(−02) |
| 7 | 0.988(−13) | 0.174(−10) | 0.279(−08) | 0.233(−06) | 0.119(−04) | 0.426(−03) |
| 8 | 0.130(−13) | 0.240(−11) | 0.432(−09) | 0.420(−07) | 0.248(−05) | 0.100(−03) |
| 9 | 0.193(−14) | 0.368(−12) | 0.730(−10) | 0.805(−08) | 0.542(−06) | 0.246(−04) |
| 10 | 0.317(−15) | 0.621(−13) | 0.133(−10) | 0.163(−08) | 0.124(−06) | 0.625(−05) |
| 11 | 0.567(−16) | 0.114(−13) | 0.259(−11) | 0.348(−09) | 0.292(−07) | 0.164(−05) |
| 12 | 0.109(−16) | 0.223(−14) | 0.536(−12) | 0.779(−10) | 0.718(−08) | 0.443(−06) |
| 13 | 0.225(−17) | 0.467(−15) | 0.117(−12) | 0.182(−10) | 0.183(−08) | 0.123(−06) |
| 14 | 0.492(−18) | 0.103(−15) | 0.270(−13) | 0.445(−11) | 0.479(−09) | 0.350(−07) |
| 15 | 0.113(−18) | 0.241(−16) | 0.649(−14) | 0.113(−11) | 0.130(−09) | 0.102(−07) |
| 16 | 0.273(−19) | 0.586(−17) | 0.163(−14) | 0.297(−12) | 0.363(−10) | 0.305(−08) |
| 17 | 0.687(−20) | 0.149(−17) | 0.425(−15) | 0.808(−13) | 0.104(−10) | 0.933(−09) |
| 18 | 0.180(−20) | 0.393(−18) | 0.115(−15) | 0.227(−13) | 0.307(−11) | 0.291(−09) |
| 19 | 0.489(−21) | 0.108(−18) | 0.321(−16) | 0.656(−14) | 0.929(−12) | 0.928(−10) |
| 20 | 0.137(−21) | 0.304(−19) | 0.926(−17) | 0.195(−14) | 0.288(−12) | 0.302(−10) |

$\left|\frac{L_n(\nu, z)}{\Gamma(\nu + 1)}\right|, \quad \nu = 9/2 + \omega i, \quad z = 8$

| $n/\omega$ | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| 6 | 0 136(−03) | 0.130(−03) | 0.125(−03) | 0.191(−03) | 0.126(−03) | 0.327(−04) |
| 7 | 0.673(−05) | 0.650(−05) | 0.627(−05) | 0.951(−05) | 0.883(−05) | 0.278(−05) |
| 8 | 0.269(−06) | 0.261(−06) | 0.252(−06) | 0.369(−06) | 0.447(−06) | 0.174(−06) |
| 9 | 0.884(−08) | 0.864(−08) | 0.836(−08) | 0.116(−07) | 0.171(−07) | 0.828(−08) |
| 10 | 0.244(−09) | 0.239(−09) | 0.232(−09) | 0.307(−09) | 0.511(−09) | 0.309(−09) |
| 11 | 0.572(−11) | 0.562(−11) | 0.547(−11) | 0.691(−11) | 0.124(−10) | 0.925(−11) |
| 12 | 0.116(−12) | 0.114(−12) | 0.111(−12) | 0.135(−12) | 0.250(−12) | 0.226(−12) |
| 13 | 0.203(−14) | 0.201(−14) | 0.196(−14) | 0.230(−14) | 0.429(−14) | 0.460(−14) |
| 14 | 0.314(−16) | 0.310(−16) | 0.303(−16) | 0.347(−16) | 0.636(−16) | 0.791(−16) |
| 15 | 0.429(−18) | 0.425(−18) | 0.416(−18) | 0.465(−18) | 0.828(−18) | 0.116(−17) |
| 16 | 0.524(−20) | 0.519(−20) | 0.509(−20) | 0.558(−20) | 0.958(−20) | 0.148(−19) |
| 17 | 0.573(−22) | 0.568(−22) | 0.558(−22) | 0.602(−22) | 0.994(−22) | 0.166(−21) |
| 18 | 0.566(−24) | 0.562(−24) | 0.553(−24) | 0.589(−24) | 0.932(−24) | 0.164(−23) |
| 19 | 0.507(−26) | 0.504(−26) | 0.496(−26) | 0.523(−26) | 0.795(−26) | 0.146(−25) |
| 20 | 0.414(−28) | 0.412(−28) | 0.406(−28) | 0.424(−28) | 0.620(−28) | 0.116(−27) |

$\left|\frac{U_m(\nu, z)}{\Gamma(\nu + 1)}\right|, \quad \nu = 9/2 + \omega i, \quad z = 8$

| $m/\omega$ | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| 6 | 0.241(−10) | 0.321(−08) | 0.259(−06) | 0.952(−05) | 0.230(−03) | 0.449(−02) |
| 7 | 0.204(−11) | 0.310(−09) | 0.336(−07) | 0.168(−05) | 0.508(−04) | 0.113(−02) |
| 8 | 0.214(−12) | 0.354(−10) | 0.473(−08) | 0.304(−06) | 0.114(−04) | 0.296(−03) |
| 9 | 0.265(−13) | 0.465(−11) | 0.728(−09) | 0.571(−07) | 0.260(−05) | 0.789(−04) |
| 10 | 0.375(−14) | 0.686(−12) | 0.121(−09) | 0.112(−07) | 0.605(−06) | 0.213(−04) |
| 11 | 0.590(−15) | 0.112(−12) | 0.218(−10) | 0.230(−08) | 0.144(−06) | 0.585(−05) |
| 12 | 0.102(−15) | 0.199(−13) | 0.418(−11) | 0.494(−09) | 0.352(−07) | 0.163(−05) |
| 13 | 0.191(−16) | 0.381(−14) | 0.853(−12) | 0.111(−09) | 0.885(−08) | 0.460(−06) |
| 14 | 0.384(−17) | 0.780(−15) | 0.184(−12) | 0.260(−10) | 0.229(−08) | 0.132(−06) |
| 15 | 0.819(−18) | 0.169(−15) | 0.418(−13) | 0.634(−11) | 0.609(−09) | 0.387(−07) |
| 16 | 0.185(−18) | 0.387(−16) | 0.995(−14) | 0.160(−11) | 0.167(−09) | 0.116(−07) |
| 17 | 0.438(−19) | 0.928(−17) | 0.247(−14) | 0.421(−12) | 0.468(−10) | 0.352(−08) |
| 18 | 0.109(−19) | 0.232(−17) | 0.638(−15) | 0.114(−12) | 0.135(−10) | 0.109(−08) |
| 19 | 0.280(−20) | 0.605(−18) | 0.171(−15) | 0.319(−13) | 0.400(−11) | 0.344(−09) |
| 20 | 0.751(−21) | 0.164(−18) | 0.473(−16) | 0.918(−14) | 0.121(−11) | 0.111(−09) |

For the evaluation of $|U_m(\nu, z)/\Gamma(\nu + 1)|$ we employed (13) with $O(k^{-3})$ neglected. For the computation of $|L_n(\nu, z)/\Gamma(\nu + 1)|$, we used (6) modified as follows so that gamma functions with positive integer arguments only need be computed. To this end, we replaced $\Gamma(n + \nu + 1 - a)$ by $\zeta\Gamma(n + \delta + 1 - a)$, $\zeta = \Gamma(n + \delta + 1 - a + \beta)/\Gamma(n + \delta + 1 - a)$, where $\delta$ is the largest positive integer or zero contained in $\mu - 1$ and $\beta = \nu - \delta$. Then $\zeta$ was approximated by use of the asymptotic expansion of the ratio of gamma functions for large $n + \delta + 1 - a$ with terms of $O(n^{-3})$ omitted. The other two gamma functions in (6) were treated in a similar fashion.

To manifest the remarkable efficiency of our asymptotic estimates of the error, Table 3 compares values of $|L_n(\nu, z)/\Gamma(\nu + 1)|$ and $U_n(\nu, z)/\Gamma(\nu + 1)$ with the corresponding values of $S(n)$ and $Y(n)$, respectively (see Table 1) for $v + 1 = \mu + \omega i$, $\mu = \frac{7}{2}$, $\omega = 0, 4, 8$ and $n = 6, 7, 8, 9$.

TABLE 3

*Comparison of asymptotic estimates of the error and the true error*

| $n$ | $\frac{L_n(\nu, z)}{\Gamma(\nu + 1)}$ | $S(n)$ | $\frac{U_n(\nu, z)}{\Gamma(\nu + 1)}$ | $Y(n)$ |
|---|---|---|---|---|
| | | $\nu = 5/2$ | | |
| 6 | 0.380(−02) | 0.381(−02) | 0.868(−12) | 0.868(−12) |
| 7 | 0.226(−03) | 0.226(−03) | 0.988(−13) | 0.987(−13) |
| 8 | 0.106(−04) | 0.106(−04) | 0.130(−13) | 0.130(−13) |
| 9 | 0.407(−06) | 0.408(−06) | 0.193(−14) | 0.187(−14) |
| | | $\nu = 5/2 + 4i$ | | |
| 6 | 0.323(−02) | 0.291(−02) | 0.197(−07) | 0.199(−07) |
| 7 | 0.195(−03) | 0.182(−03) | 0.279(−08) | 0.280(−08) |
| 8 | 0 933(−05) | 0.888(−05) | 0.432(−09) | 0.433(−09) |
| 9 | 0.363(−06) | 0.350(−06) | 0.730(−10) | 0.730(−10) |
| | | $\nu = 5/2 + 8i$ | | |
| 6 | 0.228(−02) | 0.171(−02) | 0.600(−04) | 0.594(−04) |
| 7 | 0.225(−03) | 0.116(−03) | 0.119(−04) | 0.118(−04) |
| 8 | 0.152(−04) | 0.605(−05) | 0.248(−05) | 0.247(−05) |
| 9 | 0.744(−06) | 0.251(−06) | 0.542(−06) | 0.539(−06) |

TABLE 4

*Comparison of approximations for $J_n(\nu, z)$*

| $n$ | $J_n^*(\nu, z)$ | $J_n^{**}(\nu, z)$ | $J_n(\nu, z)$ | $J_n^*(\nu, z)$ | $J_n^{**}(\nu, z)$ | $J_n(\nu, z)$ |
|---|---|---|---|---|---|---|
| | | $\nu = 1/2$ | | | $\nu = 5/2$ | |
| 6 | 0.433 | 0.433 | 0.434 | 0.602 | 0.601 | 0.597 |
| 7 | 0.477 | 0.476 | 0.481 | 0.635 | 0.633 | 0.631 |
| 8 | 0.512 | 0.514 | 0.515 | 0.661 | 0.660 | 0.657 |
| 9 | 0.546 | 0.546 | 0.549 | 0.685 | 0.683 | 0.678 |
| | | $\nu = 1/2 + 4i$ | | | $\nu = 5/2 + 4i$ | |
| 6 | 0.456 | 0.510 | 0.496 | 0.641 | 0.673 | 0.648 |
| 7 | 0.504 | 0.540 | 0.534 | 0.672 | 0.693 | 0.676 |
| 8 | 0.542 | 0.568 | 0.562 | 0.694 | 0.710 | 0.697 |
| 9 | 0.576 | 0.593 | 0.589 | 0.715 | 0.727 | 0.715 |
| | | $\nu = 1/2 + 8i$ | | | $\nu = 5/2 + 8i$ | |
| 6 | 0.406 | 0.738 | 0.665 | 0.560 | 0.885 | 0.816 |
| 7 | 0.434 | 0.729 | 0.678 | 0.581 | 0.868 | 0.818 |
| 8 | 0.461 | 0.727 | 0.693 | 0.601 | 0.858 | 0.821 |
| 9 | 0.486 | 0.729 | 0.705 | 0.622 | 0.853 | 0.827 |

For another set of interesting data, let

$$J_n(v, z) = \left| \frac{L_n(v, z)\Gamma(v + 1)}{L_{n+1}(v, z)\Gamma(v + 3)} \right|$$

as determined from the values of $S(n)$ as illustrated by the calculations in Table 1; $J_n^*(v, z)$ be $J_n(v, z)$ as determined from the modified procedure for the evaluation of $L_n(v, z)/\Gamma(v + 1)$ as described in this section, $N = n$; $J_n^{**}(v, z)$ be $J_n(v, z)$ as determined from (34) with $O(n^{-3})$ omitted.

Values of these three quantities are presented in Table 4 for the same set of parameters given in Table 3 and also for $v = \frac{1}{2} + i\omega$.

**5. Acknowledgment.** I am indebted to Miss Rosemary Moran for the calculations. Together, we will publish elsewhere a complete FORTRAN program for the evaluation of $\Gamma(v + 1)$ for complex $v$.

REFERENCES

[1] Y. L. LUKE, *The Special Functions and Their Approximations*, vols. 1 and 2, Academic Press, New York, 1969.

[2] H. E. SALZER, *Table of the Gamma Function for Complex Arguments*, Introduction, National Bureau of Standards, Applied Mathematics Series 34, U.S. Government Printing Office, Washington, D.C., 1954.

SIAM J. MATH. ANAL.
Vol. 1, No. 2, May 1970

# CONTINUOUS DEPENDENCE ON THE DATA FOR A STEFAN PROBLEM*

C. Y. CHAN†

**Abstract.** The continuous dependence on the data for the Stefan problem is proved by using comparison lemmas and properties of the fundamental singularity for the heat equation.

**1. Introduction.** Sherman [9], [10, pp. 4–5] gave a unified treatment of a problem considered by Friedman [4], Kyner [5] and Miranker [7], and of a problem considered by Boley and Weiner [1], Citron [3], Landau [6] and Sherman [8]. Sherman proved the existence and uniqueness of the set of solutions $\{u(x,t), s(t)\}$ of the following system:

$$\begin{aligned}
&cu_{xx}(x,t) = u_t(x,t) \quad \text{for } 0 < x < s(t), \quad t > 0,\\
&u(x,0) = \phi(x) \leqq 0, \qquad \phi(a) = 0,\\
&u_x(0,t) = f(t) \geqq 0 \quad \text{for } t > 0,\\
&u(s(t),t) = 0 \quad \text{for } t > 0,\\
&-s'(t) + ku_x(s(t),t) = q(t) \geqq 0 \quad \text{for } t > 0,\\
&s(0) = a > 0.
\end{aligned}$$

Here $c$ and $k$ are (positive) constants; $\phi(x)$ is continuously differentiable; $f(t)$ and $q(t)$ are continuous. (The symbol $'$ denotes "the derivative.")

Under the assumptions that $f(t)$ and $q(t)$ have continuous first derivatives and $\phi(x)$ is twice continuously differentiable, Sherman [10] further proved the continuous dependence of the set of solutions on the set of data, $\{f(t), q(t), \phi(x), a\}$. We are to prove, in this paper, the continuous dependence without this extra regularity assumption.

**2. Continuous dependence.** Without loss of generality in the final result, let us assume that $c = 1$ in the proofs. Let $\{u_1(x,t), s_1(t)\}$ and $\{u_2(x,t), s_2(t)\}$ be two sets of solutions (on $0 \leqq t \leqq T < \infty$) corresponding respectively to the two sets of Stefan data, $\{f_1(t), q_1(t), \phi_1(x), a_1\}$ and $\{f_2(t), q_2(t), \phi_2(x), a_2\}$. An argument similar to that used in establishing Theorem 6 in [2, p. 12] for the case $b_1 < b_2$ gives us the following comparison lemma.

LEMMA 1. *If $f_1(t) \leqq f_2(t)$ for $0 < t < \infty$, $\phi_1(x) \geqq \phi_2(x)$ over their common domain of definition, $q_1(t) \geqq q_2(t)$ for $0 < t < \infty$, and $a_1 < a_2$, then $s_1(t) < s_2(t)$ for $0 < t < \infty$.*

Using Lemma and the strong maximum principle for the function $u_2 - u_1$, we have the next comparison lemma.

LEMMA 2. *If the assumptions of Lemma 1 are valid with strict inequalities, then $u_1(x,t) > u_2(x,t)$ in $\{0 < x \leqq s_1(t), 0 < t < \infty\}$.*

* Received by the editors June 19, 1969, and in final revised form November 29, 1969.

† Department of Mathematics, Florida State University, Tallahassee, Florida 32306. This work forms a part of the author's doctoral dissertation at the University of Toronto, and was supported by a Postgraduate Scholarship from the National Research Council of Canada.

Let $\|g\|$ denote l.u.b. $|g|$ taken on the domain for which $g$ is defined. For $i = 1, 2$, let

$$n_i = [\|q_i\|^2 T/\pi]^* + 1$$

with $[x]^*$ denoting the largest integer $\leqq x$,

$$\frac{1}{\Gamma_i} = 1 - \|q_i\| \sqrt{\frac{T}{n_i \pi}} > 0,$$

and

$$v_i = \frac{2\|f_i\|\Gamma_i\{(4\Gamma_i)^{n_i} - 1\}}{4\Gamma_i - 1} + (4\Gamma_i)^{n_i} \max(\|\phi_i'\|, f_i(0)).$$

Then by Sherman [10, pp. 5–7], we have, for $0 \leqq t \leqq T$,

$$u_{i_x}(s_i(t), t) \leqq v_i. \tag{2.1}$$

The main result of this paper is the following theorem.

THEOREM. *The set of solutions $\{u(x, t), s(t)\}$ depends continuously on the data $\{f(t), q(t), \phi(x), a\}$ for $0 \leqq t \leqq T$ and for $x < s(t)$, where $f(t)$ and $q(t)$ are continuous, and $\phi(x)$ is continuously differentiable.*

*Proof.* Let $\|f_1 - f_2\| < \varepsilon$, $\|q_1 - q_2\| < \delta$, $\|\phi_1 - \phi_2\| < \eta$ and $|a_1 - a_2| < \alpha$. Also let $\{u^*(x, t), s^*(t)\}$ be the set of solutions corresponding to the set of Stefan data $\{f_1(t) + \varepsilon, q_1(t) - \delta, \phi_1(x) - \eta, a_1 + \alpha\}$. Here $\phi_1 - \eta$ is extended to $x = a_1 + \alpha$. Also $q_1 - \delta$ may be negative; in this case, Lemmas 1 and 2 remain valid while the existence and uniqueness of a set of solutions can be proved by using the technique of Sherman [9], [10, pp. 4–5]. Then by Lemmas 1 and 2,

$$s^*(t) > s_1(t) \quad \text{and} \quad s^*(t) > s_2(t) \quad \text{for } 0 \leqq t < \infty,$$

$$u^*(x, t) < u_1(x, t) \quad \text{in } \{0 < x \leqq s_1(t), 0 < t < \infty\},$$

and

$$u^*(x, t) < u_2(x, t) \quad \text{in } \{0 < x \leqq s_2(t), 0 < t < \infty\}.$$

Using the initial boundary conditions and evaluating

$$\int_0^t \int_0^{s(\tau)} (u_{\xi\xi} - u_\tau) \, d\xi \, d\tau = 0,$$

we get

$$s(t) = a + k \int_0^t f(\tau) \, d\tau - \int_0^t q(\tau) \, d\tau - k \int_0^a \phi(\xi) \, d\xi + k \int_0^{s(t)} u(\xi, t) \, d\xi.$$

Now,

$$k\int_0^t (f_1(\tau) + \varepsilon - f_i(\tau))\,d\tau < 2kt\varepsilon,$$

$$\int_0^t \{q_i(\tau) - (q_1(\tau) - \delta)\}\,d\tau < 2t\delta,$$

$$k\left\{\int_0^{a_i} \phi_i(\xi)\,d\xi - \int_0^{a_1+\alpha} (\phi_1(\xi) - \eta)\,d\xi\right\}$$
$$= k\left\{\int_0^{a_i} (\phi_i(\xi) - \phi_1(\xi) + \eta)\,d\xi - \int_{a_i}^{a_1+\alpha} (\phi_1(\xi) - \eta)\,d\xi\right\}$$
$$< 2ka_i\eta + 2\|\phi_1 - \eta\|k\alpha,$$

and

$$k\left(\int_0^{s^*(t)} u^*(\xi, t)\,d\xi - \int_0^{s_i(t)} u_i(\xi, t)\,d\xi\right)$$
$$= k\left\{\int_0^{s_i(t)} (u^*(\xi, t) - u_i(\xi, t))\,d\xi + \int_{s_i(t)}^{s^*(t)} u^*(\xi, t)\,d\xi\right\} < 0$$

since $u^*(x, t) \leqq 0$ [10, pp. 4–5]. From these we have, for $i = 1, 2$,

$$0 < s^*(t) - s_i(t) < 2\alpha + 2kt\varepsilon + 2t\delta + 2ka_i\eta + 2\|\phi_1 - \eta\|k\alpha. \tag{2.2}$$

We denote the right-hand side of the last inequality by $E$. Therefore, for $0 \leqq t < \infty$,

$$\begin{aligned} |s_1(t) - s_2(t)| &\leqq |s_1(t) - s^*(t)| + |s^*(t) - s_2(t)| \\ &< 4\alpha + 4kt\varepsilon + 4t\delta + 4kA^*\eta + 4\|\phi_1 - \eta\|k\alpha, \end{aligned} \tag{2.3}$$

where $A^* = \max\{a_1, a_2\}$.

The fundamental singularity for the heat equation is given by

$$K(x, t; \xi, \tau) = [2\sqrt{\pi(t - \tau)}]^{-1} \exp\left[-\frac{(x - \xi)^2}{4(t - \tau)}\right].$$

Hence, the Neumann function for the half-plane $x > 0$ is given by

$$N(x, t; \xi, \tau) = K(x, t; \xi, \tau) + K(-x, t; \xi, \tau).$$

Integrating Green's identity

$$\frac{\partial}{\partial\xi}(Nu_\xi - uN_\xi) - \frac{\partial}{\partial\tau}(Nu) = 0$$

over the domain $\{0 < \xi < s(\tau), \varepsilon < \tau < t - \varepsilon\}$, and letting $\varepsilon \to 0$, we get

$$u(x, t) = \int_0^a N(x, t; \xi, 0)\phi(\xi)\,d\xi + \int_0^t N(x, t; s(\tau), \tau)u_\xi(s(\tau), \tau)\,d\tau$$
$$- \int_0^t N(x, t; 0, \tau)f(\tau)\,d\tau.$$

Hence, over the common domain of definition,

$$u_i(x,t) - u^*(x,t) = \left\{\int_0^{a_i} N(x,t;\xi,0)\phi_i(\xi)\,d\xi - \int_0^{a_1+\alpha} N(x,t;\xi,0)(\phi_1(\xi) - \eta)\,d\xi\right\}$$

$$+ \int_0^t (N(x,t;s_i(\tau),\tau)u_{i_\xi}(s_i(\tau),\tau) - N(x,t;s^*(\tau),\tau)u_\xi^*(s^*(\tau),\tau))\,d\tau$$

$$+ \int_0^t N(x,t;0,\tau)(f_1(\tau) + \varepsilon - f_i(\tau))\,d\tau$$

$$\equiv F_1 + F_2 + F_3.$$

Then

$$|F_1| < 2\eta \int_0^{a_i} N(x,t;\xi,0)\,d\xi + \|\phi_1 - \eta\| \int_{a_i}^{a_1+\alpha} N(x,t;\xi,0)\,d\xi$$

$$< 4\eta + \frac{\|\phi_1 - \eta\|}{2\sqrt{\pi t}} \int_{a_i}^{a_1+\alpha} \left[\exp\left\{-\frac{(x-\xi)^2}{4t}\right\} + \exp\left\{-\frac{(x+\xi)^2}{4t}\right\}\right] d\xi.$$

Over the range of integration we have $0 < \exp\{-(x-\xi)^2/(4t)\} \leqq 1$ for $0 < t < \infty$, and $0 < \exp\{-(x+\xi)^2/(4t)\} < 1$ for $0 < t < \infty$. As a function of $w^2$, $\exp\{-w^2/(4t)\}$ is monotonically decreasing. Thus for each arbitrary fixed $x < s(t)$,

$$|F_1| < 4\eta + \|\phi_1 - \eta\| B^*(t)\alpha,$$

where $B^*(t) \to 0$ as $t \to 0$ or as $t \to \infty$. And

$$F_2 = \int_0^t u_{i_\xi}(s_i(\tau),\tau)(N(x,t;s_i(\tau),\tau) - N(x,t;s^*(\tau),\tau))\,d\tau$$

$$+ \int_0^t (u_{i_\xi}(s_i(\tau),\tau) - u_\xi^*(s^*(\tau),\tau))N(x,t;s^*(\tau),\tau)\,d\tau$$

$$\equiv H_1 + H_2.$$

By the mean value theorem and (2.1), we have

$$|H_1| \leqq v_i \max_{0 \leqq \tau \leqq t} |s_i(\tau) - s^*(\tau)| \left|\int_0^t N_\xi(x,t;\sigma(\tau),\tau)\,d\tau\right|,$$

where $\sigma(\tau)$ lies between $s_i(\tau)$ and $s^*(\tau)$. Hence for any arbitrary fixed $(x, t)$,

$$|H_1| < v_i EF,$$

where $F$ is a constant, since the integral exists. Also

$$|H_2| = \frac{1}{k}\left|\int_0^t (q_i(\tau) + s_i'(\tau) - q^*(\tau) - s^{*\prime}(\tau))N(x,t;s^*(\tau),\tau)\,d\tau\right|$$

$$< \frac{2\delta}{k}\int_0^t \sqrt{\{\pi(t-\tau)\}^{-1}}\,d\tau + \frac{1}{k}\left|\int_0^t N(x,t;s^*(\tau),\tau)(s_i'(\tau) - s^{*\prime}(\tau))\,d\tau\right|$$

$$< \frac{4\sqrt{t}}{k\sqrt{\pi}}\delta + \frac{1}{k}|s_i(0) - s^*(0)|N(x,t;a_1+\alpha,0)$$

$$+ \frac{1}{k}\max_{0\leqq\tau\leqq t}|s_i(\tau) - s^*(\tau)|N(x,t;a_1+\alpha,0)$$

$$< \frac{4\sqrt{t}}{k\sqrt{\pi}}\delta + \frac{1}{k}N(x,t;a_1+\alpha,0)(2\alpha+E),$$

and

$$|F_3| < 2\varepsilon\int_0^t N(x,t;0,\tau)\,d\tau \leqq \frac{4\sqrt{t}}{\sqrt{\pi}}\varepsilon.$$

Hence, for arbitrary fixed $(x, t)$, we have

$$|u_1(x,t) - u_2(x,t)| \leqq |u_1(x,t) - u^*(x,t)| + |u^*(x,t) - u_2(x,t)|$$

$$< 8\eta + 2\|\phi_1 - \eta\|B^*(t)\alpha + (v_1 + v_2)EF \tag{2.4}$$

$$+ \frac{8\sqrt{t}}{k\sqrt{\pi}}\delta + \frac{2}{k}N(x,t;a_1+\alpha,0)(2\alpha+E) + \frac{8\sqrt{t}}{\sqrt{\pi}}\varepsilon.$$

From (2.2), $E \to 0$ as $\{\varepsilon, \delta, \eta, \alpha\} \to \{0, 0, 0, 0\}$. Therefore, it follows from (2.3) and (2.4) that the set of solutions $\{u(x, t), s(t)\}$ depends continuously on the initial boundary data.

**Acknowledgment.** The author would like to express his gratitude to his supervisor, Professor G. F. D. Duff.

REFERENCES

[1] B. A. Boley and J. H. Weiner, *Theory of Thermal Stresses*, John Wiley, New York, 1950.
[2] J. R. Cannon and C. D. Hill, *Existence, uniqueness, stability, and monotone dependence in a Stefan problem for the heat equation*, J. Math. Mech., 17 (1967), pp. 1–19.
[3] S. J. Citron, *Heat conduction in a melting slab*, J. Aero/Space Sci., 27 (1960), pp. 219–228.
[4] A. Friedman, *Free boundary problems for parabolic equations. I: Melting of solids*,, J. Math. Mech., 8 (1959), pp. 499–518.
[5] W. T. Kyner, *On a free boundary value problem for the heat equation*, Quart. Appl. Math., 17 (1959), pp. 305–310.
[6] H. G. Landau, *Heat conduction in a melting solid*, Ibid., 8 (1950), pp. 81–94.
[7] W. L. Miranker, *A free boundary value problem for the heat equation*, Ibid., 16 (1958), pp. 121–130.

[8] B. SHERMAN, *A free boundary problem for the heat equation with heat input at a melting interface*, Ibid., 23 (1966), pp. 337–347.
[9] ———, *A free boundary problem for the heat equation with prescribed flux at both fixed face and melting interface*, Ibid., 25 (1967), pp. 53–63.
[10] ———, *Continuous dependence of the solution on the data in a free boundary problem for the heat equation*, Res. Rep. 67–8, Rocketdyne Corp., Canoga Park, California, 1967, pp. 1–37.

SIAM J. MATH. ANAL.
Vol. 1, No. 2, May 1970

# THE FRACTIONAL DERIVATIVE OF A COMPOSITE FUNCTION*

THOMAS J. OSLER†

**1. Introduction.** In the elementary calculus one considers the derivative of order $N$ of the composite function $f(z) = F(h(z))$ and obtains the formula [5, p. 19]

$$D_z^N f(z) = \sum_{n=0}^{N} \frac{U_n(z) D_{h(z)}^n f(z)}{n!}, \tag{1.1}$$

where

$$U_n(z) = \sum_{r=0}^{n} \binom{n}{r} (-h(z))^r D_z^N h(z)^{n-r}.$$

In this paper we consider the extension of (1.1) to fractional derivatives. We derive the fundamental result

$$D_{g(z)}^{\alpha} f(z) = D_{h(z)}^{\alpha} \left\{ \frac{f(z)g'(z)}{h'(z)} \left( \frac{h(z) - h(w)}{g(z) - g(w)} \right)^{\alpha+1} \right\} \Bigg|_{w=z}, \tag{1.2}$$

where the notation $D_{g(z)}^{\alpha} f(z)$ means the fractional derivative of order $\alpha$ of $f(z)$ with respect to $g(z)$. The Leibniz rule for fractional derivatives is then applied to (1.2) to obtain the new series expansion

$$\begin{aligned} D_{g(z)}^{\alpha} f(z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma+n} D_{h(z)}^{\gamma+n} \frac{f(z)}{F(z,w)} \\ \cdot D_{h(z)}^{\alpha-\gamma-n} \left\{ \frac{F(z,w)g'(z)}{h'(z)} \left( \frac{h(z) - h(w)}{g(z) - g(w)} \right)^{\alpha+1} \right\} \Bigg|_{w=z}, \end{aligned} \tag{1.3}$$

where

$$\binom{\alpha}{\gamma+n} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\gamma-n+1)\Gamma(\gamma+n+1)}.$$

The formula (1.1) from the elementary calculus is shown to be a special case of the "generalized chain rule" (1.3). A few specific examples of these general results are studied.

The concept of the fractional derivative with respect to an arbitrary function has been used in recent papers [3], [4]. However, to the best of the author's knowledge, the full definition and notation $D_{g(z)}^{\alpha} f(z)$, introduced in his paper [8], are new. Indeed, it is this new notation which suggests the possibility of investigating the fractional derivative of a composite function so as to generalize the calculus formula (1.1).

The Leibniz rule for fractional derivatives

$$D_{g(z)}^{\alpha} u(z)v(z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma+n} D_g^{\alpha-\gamma-n} u D_g^{\gamma+n} v$$

* Received by the editors July 10, 1969.

† Department of Mathematics, St. Joseph's College, Philadelphia, Pennsylvania 19131.

was first studied by Watanabe [10] in 1931. Recently the author [8] found a new proof for this Leibniz rule which revealed its precise region of convergence in the $z$-plane. Since the Leibniz rule plays a central role in the investigation of the series expansions in this paper, the reader should consult [8] before proceeding.

Finally, a few specific examples of (1.3) are examined. Novel derivations of the known result

$$F(\alpha, 1 - \alpha; p - \alpha + 1; 1/2) = \frac{2^\alpha \Gamma(p - \alpha + 1)\Gamma(p/2 + 1)}{\Gamma(p/2 - \alpha + 1)\Gamma(p + 1)}$$

and Kummer's formula

$$F(\alpha + 1, p; p - \alpha; -1) = \frac{\Gamma(p/2)\Gamma(p - \alpha)}{2\Gamma(p)\Gamma(p/2 - \alpha)}$$

are obtained as well as new results.

The generalized chain rule, the Leibniz rule, the relation $D^\alpha D^\beta = D^{\alpha+\beta}$, and other operations illustrate that the fractional calculus exists as a natural extension of the elementary calculus. Recent papers illustrating the application of the fractional calculus to problems in ordinary, partial and integral equations [3], [4], [6], [7], [9] demonstrate that it is a highly suggestive tool. Higgins [7] has observed that "although results using fractional integral operators can always be obtained by other methods, the succinct simplicity of the formulation may often suggest approaches which might not be evident in a classical approach." It is hoped that this paper will further reveal the uses of fractional derivatives.

**2. Fractional derivatives and Leibniz rule.** We now review briefly the definition of fractional derivative and the statement of the Leibniz rule. A full discussion of these ideas is found in [8].

DEFINITION 1. The *fractional derivative* or order $\alpha$ of $f(z)$ with respect to $h(z)$ is

$$D^\alpha_{h(z)} f(z) = \frac{\Gamma(\alpha + 1)}{2\pi i} \int_{h^{-1}(0)}^{(z+)} \frac{f(t)h'(t)\,dt}{(h(t) - h(z))^{\alpha+1}}. \tag{2.1}$$

The branch line for $(h(t) - h(z))^{\alpha+1}$ starts at $t = z$, passes through $t = h^{-1}(0)$, and continues to infinity. The contour of integration starts at $t = h^{-1}(0)$, encircles $t = z$ in the positive sense once, and returns to $t = h^{-1}(0)$ without crossing the branch line of $(h(t) - h(z))^{\alpha+1}$. $f(z)$ and $h(z)$ are assumed to possess sufficient regularity to give the integral (2.1) meaning.

The critical use of the Leibniz rule for fractional derivatives,

$$D^\alpha_{h(z)} u(z)v(z) = \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma + n} D^{\alpha-\gamma-n}_{h(z)} u(z) D^{\gamma+n}_{h(z)} v(z), \tag{2.2}$$

requires a description of the region in the $z$-plane over which the series (2.2) converges. To simplify the following discussion we describe this region of convergence as the "Leibniz region" and give its definition.

DEFINITION 2. Let $u(h^{-1}(z))$ and $v(h^{-1}(z))$ be defined and analytic on the simply connected region $\mathscr{R}$. Let $z = 0$ be an interior or boundary point of $\mathscr{R}$.

$\mathscr{S}(\mathscr{R})$ denotes the set of all $z$ such that the closed disk $|t - z| \leqq |z|$ contains only points $t$ in $\mathscr{R} \cup \{0\}$. We call the set of all $z$ in $h^{-1}(\mathscr{S}(\mathscr{R}))$ the *Leibniz region of u and v with respect to h* and denote it by $\mathscr{L}(u, v; h)$.

We state the precise version of the Leibniz rule from [8] as a theorem for future reference.

THEOREM 1 (Leibniz rule). *Let $u(h^{-1}(z))$ and $v(h^{-1}(z))$ be defined and analytic in the simply connected region $\mathscr{R}$. Let $\oint u(h^{-1}(z))\,dz$, $\oint v(h^{-1}(z))\,dz$ and $\oint u(h^{-1}(z)) \cdot v(h^{-1}(z))\,dz$ vanish over any simple closed contour in $\mathscr{R} \cup \{0\}$ passing through the origin. Then the Leibniz rule* (2.2) *is true for z in $\mathscr{L}(u, v; h)$ and arbitrary $\gamma$ for which $\binom{\alpha}{\gamma + n}$ is defined.*

Having reviewed the definition of fractional derivative and the Leibniz rule we proceed to study the fractional derivative of a composite function. We shall see that the results are easy applications of the Leibniz rule and Definition 1.

**3. The generalized chain rule.** We begin by deriving the fundamental relation

$$D^{\alpha}_{g(z)} f(z) = D^{\alpha}_{h(z)} \left\{ \frac{f(z)g'(z)}{h'(z)} \left( \frac{h(z) - h(w)}{g(z) - g(w)} \right)^{\alpha + 1} \right\} \Bigg|_{w = z}. \tag{3.1}$$

This relation combined with the Leibniz rule yields the generalized chain rule for fractional derivatives.

THEOREM 2. *Let $f(g^{-1}(z))$ and $f(h^{-1}(z))$ be defined and analytic on the simply connected region $\mathscr{R}$, and let the origin be an interior or boundary point of $\mathscr{R}$. Suppose also that $g^{-1}(z)$ and $h^{-1}(z)$ are regular univalent functions on $\mathscr{R}$ and that $h^{-1}(0) = g^{-1}(0)$. Let $\oint f(g^{-1}(z))\,dz$ vanish over every simple closed contour in $\mathscr{R} \cup \{0\}$ through the origin. Then the fundamental relation* (3.1) *is valid.*

*Proof.* The result follows immediately upon converting both sides of the fundamental relation (3.1) to contour integrals by means of the definition of fractional derivative (2.1).

The Leibniz rule can be applied to the right-hand side of (3.1) once we select $u(z)$ and $v(z)$ such that

$$u(z)v(z) = \frac{f(z)g'(z)}{h'(z)} \left( \frac{h(z) - h(w)}{g(z) - g(w)} \right)^{\alpha + 1}.$$

To indicate all possible methods of factoring, we introduce the arbitrary function $F(z, w)$ and set

$$u(z) = \frac{F(z, w)g'(w)}{h'(z)} \left( \frac{h(z) - h(w)}{g(z) - g(w)} \right)^{\alpha + 1} \tag{3.2}$$

and

$$v(z) = \frac{f(z)}{F(z, w)}.$$

The Leibniz rule (2.2) yields the desired generalized chain rule at once:

$$(3.3)\qquad \begin{aligned} D^{\alpha}_{g(z)}f(z) &= \sum_{n=-\infty}^{\infty} \binom{\alpha}{\gamma+n} D^{\gamma+n}_{h(z)} \frac{f(z)}{F(z,w)} \\ &\quad \cdot D^{\alpha-\gamma-n}_{h(z)} \left\{ \frac{F(z,w)g'(z)}{h'(z)} \left( \frac{h(z)-h(w)}{g(z)-g(w)} \right)^{\alpha+1} \right\} \bigg|_{w=z}. \end{aligned}$$

The precise conclusion is stated as a theorem.

THEOREM 3. *Let $f(z)$, $g(z)$ and $h(z)$ satisfy the conditions of Theorem 2. Let $F(h^{-1}(z), h^{-1}(w))$ be regular on $\mathscr{R} \times \mathscr{R}$. Let $u(z)$ and $v(z)$ be defined by* (3.2) *and satisfy the conditions of Theorem* 1. *Then the generalized chain rule* (3.3) *is valid for $z$ in the Leibniz region $\mathscr{L}(u, v; h)$ and arbitrary $\gamma$ for which $\binom{\alpha}{\gamma+n}$ is defined.*

We conclude the analytical investigation of the generalized chain rule by converting (3.3) into a form somewhat like the elementary calculus formula (1.1). The new form is

$$(3.4)\qquad D^{\alpha}_{g(z)}f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} \{D^{\alpha}_{g(z)}F(z,w)(h(z)-h(w))^n\} D^{n}_{h(z)} \frac{f(z)}{F(z,w)} \bigg|_{w=z},$$

where

$$(3.5)\qquad D^{\alpha}_{g(z)}F(z,w)(h(z)-h(w))^n|_{w=z} = \sum_{r=0}^{n} \binom{n}{r} (-h(z))^r D^{\alpha}_{g(z)}F(z,w)h(z)^{n-r} \bigg|_{w=z}.$$

The elementary calculus formula (1.1) is seen as the special case in which $\alpha$ is an integer, $g(z) \equiv z$ and $F(z, w) \equiv 1$.

THEOREM 4. *With the hypothesis of Theorem* 3, *the relations* (3.4) *and* (3.5) *are valid.*

*Proof.* Set $\gamma = 0$ in the generalized chain rule (3.3). The summation now extends from $n = 0$ to $\infty$ rather than from $-\infty$ to $\infty$. We see at once that

$$\begin{aligned} \binom{\alpha}{n} D^{\alpha-n}_{h(z)} \left\{ \frac{F(z,w)g'(z)}{h'(z)} \left( \frac{h(z)-h(w)}{g(z)-g(w)} \right)^{\alpha+1} \right\} \bigg|_{w=z} \\ = \frac{1}{n!} D^{\alpha}_{g(z)}F(z,w)(h(z)-h(w))^n \bigg|_{w=z} \end{aligned}$$

upon writing both sides as a contour integral by (2.1). The relation (3.5) is obtained at once upon expanding $(h(z) - h(w))^n$ by the binomial theorem.

It is useful to note that Theorem 4 is valid even when $h^{-1}(0)$ and $g^{-1}(0)$ are not equal. This is seen at once upon replacing $h(z)$ by $h(z) - h(g^{-1}(0))$ in (3.4) and observing that $h'(z)$ and $D^{n}_{h(z)}$ do not change.

We have demonstrated that the formulas from the elementary calculus for the derivatives of a composite function generalize to fractional derivatives in a natural way. We proceed to investigate some consequences of the generalized chain rule through the study of a few specific examples.

**4. Examples.** We conclude this paper with an examination of a few special cases of the generalized chain rule. These require the evaluation of the fractional derivatives of elementary functions. A table of fractional derivatives or integrals such as that found in [2, vol. 2, pp. 185–214] is useful for this purpose.

The notation for the special functions used is that of Erdélyi, Magnus, Oberhettinger and Tricomi [1], [2].

*Example* 1. Setting $f(z) = z^{p-2}$, $g(z) = z^2$ and $h(z) = z$ in the fundamental relation (3.1), we obtain

$$D_z^\alpha z^{p-2} = D_z^\alpha 2z^{p-1}(z + w)^{-\alpha-1}|_{w=z}.$$

The left-hand side is evaluated with the aid of the relation

$$D_z^\alpha z^q = \frac{\Gamma(q+1)z^{q-\alpha}}{\Gamma(q-\alpha+1)},$$

after replacing $z$ by $z^2$. Using [2, vol. 2, no. 9, p. 186] we obtain Kummer's formula

$$F(\alpha+1, p; p-\alpha; -1) = \frac{\Gamma(p/2)\Gamma(p-\alpha)}{2\Gamma(p)\Gamma(p/2-\alpha)}.$$

*Example* 2. Letting $f(z) = z^p$, $g(z) = z^2$, $h(z) = z$, $F(z, w) = z^p(z + w)^{\alpha+1}$ and $\gamma = 0$ in the generalized chain rule (3.3), we obtain

$$\frac{\Gamma(p/2+1)}{\Gamma(p/2-\alpha+1)} = \sum_{n=0}^{\infty} \frac{\Gamma(1-\alpha+n)\Gamma(p+1)(-1)^n(2)^{-\alpha-n}}{\Gamma(1-\alpha-n)\Gamma(p-\alpha+n+1)n!},$$

which reduces to

$$\frac{\Gamma(p-\alpha+1)\Gamma(p/2+1)2^\alpha}{\Gamma(p/2-\alpha+1)\Gamma(p+1)} = F(\alpha, 1-\alpha; p-\alpha+1; 1/2). \tag{4.1}$$

It may be noted that this novel method for determining the known relation (4.1) provides a direct evaluation of the hypergeometric series of argument $z = 1/2$.

*Example* 3. Setting $g(z) = z$, $h(z) = z^k$ and $F(z, w) = z^q$ in (3.4), we obtain

$$D_z^\alpha f(z) = \frac{\Gamma(q+1)z^{q-\alpha}}{\Gamma(q-\alpha+1)} \sum_{n=0}^{\infty} \frac{(-z^k)^n}{n!} D_{z^k}^n f(z) z^{-q}$$
$$\cdot {}_{k+1}F_k\left(-n, \frac{q+1}{k}, \frac{q+2}{k}, \ldots, \frac{q+k}{k}; \frac{q-\alpha+1}{k}, \frac{q-\alpha+2}{k}, \ldots, \frac{q-\alpha+k}{k}; 1\right)$$

with the aid of [2, vol. 2, no. 11, p. 186]. This is the generalized chain rule for the fractional derivative of the composite function $f(z) = F(z^k)$ in terms of derivatives with respect to $z^k$.

*Example* 4. Setting $g(z) = z^p$, $h(z) = z$ and $F(z, w) = z^{q-p+1}$ in (3.4), we obtain

$$D_{z^p}^\alpha f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} D_{z^p}^\alpha z^q (w-z)^n \Big|_{w=z} \cdot D_z^n f(z) z^{p-q-1}.$$

Using the Cauchy integral formula for fractional derivatives (2.1), we easily see that

$$\frac{(-1)^n}{n!} D_{z^p}^{\alpha} z^q (w-z)^n \bigg|_{w=z} = \frac{(-1)^{n-\alpha-1} p}{\Gamma(-\alpha)} D_z^{-n-1} (z^p - w^p)^{-\alpha-1} z^q \bigg|_{w=z}.$$

Finally, using [2, vol. 2, no. 11, p. 186] we obtain

$$D_{z^p}^{\alpha} f(z) = \frac{p\Gamma(q+1) z^{q-\alpha p-p+1}}{\Gamma(-\alpha)} \cdot \sum_{n=0}^{\infty} \frac{(-z)^n D_z^n f(z) z^{p-q-1}}{\Gamma(q+n+2)} {}_{p+1}F_p$$

$$\cdot \left(\alpha+1, \frac{q+1}{p}, \frac{q+2}{p}, \cdots, \frac{q+p}{p}; \frac{q+n+2}{p}, \frac{q+n+3}{p}, \cdots, \frac{q+n+p+1}{p}; 1\right).$$

Computation of the coefficient of $D_z^n f(z) z^{p-q-1}$ by means of (3.5) rather than by the procedure outlined above reveals that

$${}_{p+1}F_p\left(\alpha+1, \frac{q+1}{p}, \frac{q+2}{p}, \cdots, \frac{q+p}{p}; \frac{q+n+2}{p}, \frac{q+n+3}{p}, \cdots, \frac{q+n+p+1}{p}\ 1\right)$$

equals the finite sum of gamma functions

$$\frac{\Gamma(-\alpha)\Gamma(q+n+2)}{p\Gamma(q+1)} \sum_{r=0}^{n} \frac{(-1)^{n+r}\Gamma((q+n-r+1)/p)}{(n-r)!\,\Gamma((q+n-r+1-p\alpha)/p)}.$$

Replacement of $z$ by $z^{1/p}$ in (4.2) yields the generalized chain rule for the fractional derivative of $f(z^{1/p})$ in terms of derivatives with respect to $z^{1/p}$.

**5. Acknowledgment.** The author wishes to express his indebtedness to Professor S. N. Karp of the Courant Institute of Mathematical Sciences at New York University for his advice and encouragement.

REFERENCES

[1] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, 3 vols., McGraw-Hill, New York, 1953, 1955.

[2] ———, *Tables of Integral Transforms*, 2 vols., McGraw-Hill, New York, 1954.

[3] A. Erdélyi, *An integral equation involving Legendre functions*, J. Soc. Indust. Appl. Math., 12 (1964), pp. 15–30.

[4] ———, *Axially symmetric potentials and fractional integration*, Ibid., 13 (1965), pp. 216–228.

[5] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, 1965.

[6] T. P. Higgins, *A hypergeometric function transform*, Ibid., 12 (1964), pp. 601–612.

[7] ———, *The use of fractional integral operators for solving nonhomogeneous differential equations*, Document D1-82-0677, Boeing Scientific Research Laboratories, 1967.

[8] T. J. Osler, *Leibniz rule for fractional derivatives and an application to infinite series*, SIAM J. Appl. Math., 18 (1970), pp. 658–674.

[9] M. Riesz, *L'intégrale de Riemann–Liouville et le problème de Cauchy*, Acta Math., 81 (1949), pp. 1–233.

[10] Y. Watanabe, *Notes on the generalized derivative of Riemann–Liouville and its application to Leibnitz's formula. I and II*, Tôhoku Math. J., 34 (1931), pp. 8–41.

SIAM J. MATH ANAL.
Vol. 1, No. 3, August 1970

# EQUICONVERGENCE AND ALMOST EVERYWHERE CONVERGENCE OF HERMITE AND LAGUERRE SERIES*

BENJAMIN MUCKENHOUPT†

**1. Introduction.** The purpose of this paper is to prove equiconvergence theorems for both Laguerre and Hermite series with weaker hypotheses than those given by Szegö in [8]. As usual, these hypotheses are conditions on the absolute value of the function, and they are shown to be the best possible conditions of this type. These results along with Hunt's theorem for almost everywhere convergence of Fourier series give general almost everywhere convergence theorems for Hermite and Laguerre series.

The results proved here are stated in § 2. The principal tool used to obtain them is an estimate for Laguerre polynomials in [3] for $\alpha \geqq 0$ and in [5] for $-1 < \alpha < 0$. Except for the use of this estimate at one crucial point, the proofs in § 3 of the equiconvergence theorems, Theorems 1 and 2, are the same as those given by Szegö in [8]. Szegö gives two sets of hypotheses for Laguerre series and two for Hermite series; it is easy to show that these hypotheses imply the conditions required here. A simple example in § 12 for the Laguerre case shows that there is a function that satisfies the conditions given here but neither of Szegö's sets of hypotheses. Furthermore, this function cannot be written as the sum of two functions, each satisfying one set of Szegö's hypotheses. Similar functions exist in the Hermite case.

Many other sets of conditions that are sufficient to imply equiconvergence can be derived from Theorems 1 and 2. Corollaries 3 and 4 contain a collection that include those given by Szegö.

The almost everywhere convergence results, Corollaries 1 and 2, are proved in § 4. They are an obvious application of the theorems.

As discussed in [8], there has been a history of progressively weaker conditions on the absolute value of the function in equiconvergence theorems for Hermite and Laguerre series. The ones obtained here are the best possible of this type since if $|f|$ does not satisfy the conditions, then by adjusting the algebraic sign of $f$ properly the conclusion is false. This is the content of Theorems 3 and 4, and §§ 5–11 are devoted to proving them.

Theorems 3 and 4 are intrinsically much more difficult to prove than Theorems 1 and 2. For Theorem 1, estimates of integrals not containing Laguerre polynomials must be used to estimate integrals that do contain Laguerre polynomials. An accurate upper bound for the absolute value of the polynomial is sufficient for this. The proof of Theorem 3 is the opposite problem; here estimates of integrals containing Laguerre polynomials must be used to bound integrals not containing Laguerre polynomials. Since the Laguerre polynomials have zeros in the relevant

* Received by the editors September 8, 1969.

† Department of Mathematics, Rutgers University, The State University of New Jersey, New Brunswick, New Jersey 08903. This work was supported in part by the National Science Foundation under Grants GP 7539 and GP 11403.

ranges of integration, good asymptotic formulas and careful reasoning must be used to do this.

In § 12, in addition to the function described above, examples of other functions are given to show that the theorems proved here cannot be greatly improved. Examples are also given to shed some light on the meaning of the conditions in Theorems 1 and 2.

**2. Statements of the results.** As usual, for a fixed $\alpha > -1$ the Laguerre polynomials, $L_n^\alpha(x)$, will be defined by

$$\sum r^n L_n^\alpha(x) = (1-r)^{-\alpha-1} \exp\left(\frac{-rx}{1-r}\right);$$

they are orthogonal on $[0, \infty)$ with weight function $e^{-x}x^\alpha$. The Hermite polynomials will be defined by $\sum H_n(x) r^n/n! = \exp(2xr - r^2)$; they are orthogonal on $(-\infty, \infty)$ with weight function $\exp(-x^2)$.

The equiconvergence theorems to be proved are the following.

THEOREM 1. *Given $\alpha > -1$, assume that $f(x)$ has a Laguerre series with parameter $\alpha$, let $s_n^\alpha(f, x)$ denote the n-th partial sum of that series and assume that*

$$\int_{m/2}^{m} \frac{|f(x)|\, e^{-x/2} x^{\alpha/2 - 1/2}}{(m + m^{1/3} - x)^{1/4}}\, dx = o(1), \qquad m \to \infty, \tag{2.1}$$

$$\int_1^\infty |f(x)|\, e^{-x/2} x^{\alpha/2 - 3/4}\, dx < \infty \tag{2.2}$$

*and, in case $\alpha > -\frac{1}{2}$,*

$$\int_0^1 |f(x)| x^{\alpha/2 - 1/4}\, dx < \infty. \tag{2.3}$$

*Then if $y > 0$ and $0 < \delta < y^{1/2}$,*

$$\lim_{n\to\infty} \left[ s_n^\alpha(f, y) - \frac{1}{\pi} \int_{y^{1/2}-\delta}^{y^{1/2}+\delta} f(t^2) \frac{\sin 2n^{1/2}(y^{1/2} - t)}{y^{1/2} - t}\, dt \right] = 0. \tag{2.4}$$

*This holds uniformly for every fixed positive interval $[r, s]$ if $0 < \delta < r^{1/2}$.*

THEOREM 2. *Assume that $f(x)$ has a Hermite series, let $s_n(f, x)$ denote the n-th partial sum of that series and assume that*

$$\int_{m^{1/2}/2}^{m^{1/2}} \frac{|f(-x)| + |f(x)|}{(m + m^{1/3} - x^2)^{1/4}} \exp\left(-\frac{1}{2}x^2\right) x^{-1/2}\, dx = o(1), \qquad m \to \infty, \tag{2.5}$$

*and*

$$\int_1^\infty \frac{|f(-x)| + |f(x)|}{x} \exp\left(-\frac{1}{2}x^2\right) dx < \infty. \tag{2.6}$$

*Then if $\delta > 0$ and $y$ is real,*

$$\lim_{n\to\infty} \left[ s_n(f, y) - \frac{1}{\pi} \int_{y-\delta}^{y+\delta} f(t) \frac{\sin (2n)^{1/2}(y - t)}{y - t}\, dt \right] = 0. \tag{2.7}$$

*This holds uniformly for $y$ in every finite interval.*

The exact statements of the sense in which Theorems 1 and 2 are the best possible are the following.

THEOREM 3. *If $f(x)$ does not satisfy one or more of the conditions,* (2.1), (2.2) *and* (2.3), *then there exists a function, $f_1(x)$, such that $|f_1(x)| = |f(x)|$, and for almost every $y > 0$ and every $\delta$ satisfying $0 < \delta < y^{1/2}$, equation* (2.4), *with $f$ replaced by $f_1$, is false.*

THEOREM 4. *If $f(x)$ does not satisfy* (2.5), (2.6) *or both, then there exists a function, $f_1(x)$, such that $|f_1(x)| = |f(x)|$, and for almost every real $y$ and every $\delta > 0$, equation* (2.7), *with $f$ replaced by $f_1$, is false.*

In § 12 an example will be given to show that "almost every" in Theorem 3 cannot be replaced by "every" or "all but a countable number of." Examples will also be given in § 12 to show that a function may violate (2.1), (2.2) or (2.3) but still have the conclusion of Theorem 1 valid.

It will appear in the proofs and from other examples in § 12 that conditions (2.2) and (2.3) are much more subtle than (2.1). It is easy to see that if the conclusion of Theorem 1 holds, then the individual terms of $f$'s Laguerre series, $a_n^\alpha(f)L_n^\alpha(y)$, must satisfy $\lim_{n\to\infty} a_n^\alpha(f)L_n^\alpha(y) = 0$ for every $y > 0$. It will be shown in the proof of Theorem 3 that if $\lim_{n\to\infty} a_n^\alpha(f_1)L_n^\alpha(y) = 0$ for every $f_1$ satisfying $|f_1(x)| = |f(x)|$ and every $y > 0$, then (2.1) must hold. Examples in § 12 show, however, that this condition does not imply (2.2) or (2.3).

Similar comments apply to the Hermite case.

The almost everywhere convergence results are the following; in these $\log^+ x$ is the function equal to $\log x$ for $x \geqq 1$ and 0 otherwise.

COROLLARY 1. *Given $\alpha > -1$, assume that $f(x)$ has a Laguerre series with parameter, $\alpha$, let $s_n^\alpha(f, x)$ denote the n-th partial sum of that series and assume that $|f(x)|(\log^+|f(x)|)^2$ is integrable on $[a, b] \subset [0, \infty)$. If* (2.1), (2.2) *and* (2.3) *are true, then $s_n^\alpha(f, x)$ converges to $f(x)$ for almost every $x$ in $[a, b]$. If* (2.1), (2.2) *or* (2.3) *is false, there exists $f_1(x)$ such that $|f_1(x)| = |f(x)|$ and the set of all $x$ in $[a, b]$ for which $s_n^\alpha(f_1, x)$ converges to $f_1(x)$ has measure* 0.

COROLLARY 2. *Assume that $f(x)$ has a Hermite series, let $s_n(f, x)$ denote the n-th partial sum of that series and assume that $|f(x)|(\log^+|f(x)|)^2$ is integrable on $[a, b] \subset (-\infty, \infty)$. If* (2.5) *and* (2.6) *are true, then $s_n(f, x)$ converges to $f(x)$ for almost every $x$ in $[a, b]$. If* (2.5) *or* (2.6) *is false, there exists $f_1(x)$ such that $|f_1(x)| = |f(x)|$ and the set of all $x$ in $[a, b]$ for which $s_n(f_1, x)$ converges to $f_1(x)$ has measure* 0.

The relationship between Theorems 1 and 2 and the results in [8, Theorems 9.1.5 and 9.1.6, pp. 244–245] is illustrated by the following two corollaries.

COROLLARY 3. *Assume that $1 \leqq p \leqq \infty$ and the hypotheses of Theorem* 1 *are all satisfied except that* (2.1) *and, if $1 \leqq p < 4/3$,* (2.2) *are replaced by the condition*

$$(2.8) \qquad \left(\int_m^{2m} |f(x)\, e^{-x/2} x^{\alpha/2}|^p\, dx\right)^{1/p} = \begin{cases} o(m^{1/4 + 1/(3p)}), & 1 \leqq p < 4/3, \\ o(m^{1/2}(\log m)^{-1/4}), & p = 4/3, \\ o(m^{-1/4 + 1/p}), & p > 4/3. \end{cases}$$

*Then the conclusion of Theorem* 1 *is valid.*

COROLLARY 4. *Assume that* $1 \leqq p \leqq \infty$ *and the hypotheses of Theorem* 2 *are all satisfied except that* (2.5) *and, if* $1 \leqq p < 4/3$, (2.6) *are replaced by the condition*

$$\left(\int_m^{2m}\left[\frac{|f(x)| + |f(-x)|}{\exp(x^2/2)}\right]^p dx\right)^{1/p} = \begin{cases} o(n^{1-1/(3p)}), & 1 \leqq p < 4/3, \\ o(n^{3/4}(\log n)^{-1/4}), & p = 4/3, \\ o(n^{1/p}), & p > 4/3. \end{cases} \tag{2.9}$$

*Then the conclusion of Theorem* 2 *is valid.*

Corollary 3 with $p = 1$ has a hypothesis that appears slightly weaker than the first set of hypotheses in Theorem 9.1.5 of [8], but they are in fact equivalent. Similarly, Corollary 3 with $p = 2$ is equivalent to Theorem 9.1.5 with its second set of hypotheses. Corollary 4 with $p = 1$ is equivalent to Theorem 9.1.6 with the first set of hypotheses and with $p = 2$ is equivalent to Theorem 9.1.6 with the second set of hypotheses.

It is also interesting to note that for $p > 4/3$ the hypotheses of Corollaries 3 and 4 become stronger as $p$ increases.

**3. Proofs of Theorems 1 and 2.** Szegö's proof in [8, pp. 264–266] of his equiconvergence theorem, Theorem 9.1.5, for Laguerre series uses nothing more than the hypotheses of Theorem 1 up to the last line of the proof. At that point all that remains to be proved is that

$$n^{\alpha/2+3/4}\int_{3n}^{\infty} e^{-t}t^{\alpha-1}|f(t)L_n^{\alpha}(t)|\,dt = o(1) \tag{3.1}$$

and

$$n^{-\alpha/2+3/4}\int_{3n}^{\infty} e^{-t}t^{\alpha-1}|f(t)L_{n+1}^{\alpha}(t)|\,dt = o(1). \tag{3.2}$$

Consequently, it will be sufficient to prove (3.1) and (3.2) from the hypotheses of Theorem 1 to complete the proof of Theorem 1.

To prove (3.1) let $\nu = 4n + 2\alpha + 2$ and write the left side of (3.1) as the sum of

$$n^{-\alpha/2+3/4}\int_{3n}^{\nu} e^{-t}t^{\alpha-1}|f(t)L_n^{\alpha}(t)|\,dt, \tag{3.3}$$

$$n^{-\alpha/2+3/4}\int_{\nu}^{2\nu} e^{-t}t^{\alpha-1}|f(t)L_n^{\alpha}(t)|\,dt \tag{3.4}$$

and

$$n^{-\alpha/2+3/4}\int_{2\nu}^{\infty} e^{-t}t^{\alpha-1}|f(t)L_n^{\alpha}(t)|\,dt. \tag{3.5}$$

By [1, p. 699], given a fixed $\alpha \geqq 0$, there exist positive constants, $C$ and $D$, such that

$$(3.6)\qquad e^{-x/2}x^{\alpha/2}n^{-\alpha/2}|L_n^\alpha(x)| \leqq \begin{cases} \dfrac{Cv^{-1/4}}{(v+v^{1/3}-x)^{1/4}}, & 1/v \leqq x \leqq v, \\ \dfrac{C\exp[-D(x-v)^{3/2}v^{-1/2}]}{(x+v^{1/3}-v)^{1/4}v^{1/4}}, & v \leqq x \leqq 2v, \\ Ce^{-Dx}, & 2v \leqq x. \end{cases}$$

The restriction in [1] to $\alpha \geqq 0$ is caused by the fact that their proof was based on asymptotic estimates for $L_n^\alpha(x)$ that were proved in [3] only for $\alpha \geqq 0$. However, in [5] these estimates are proved for $-1 < \alpha < 0$. The reasoning in [1] then can be used to prove that (3.6) is true for all $\alpha > -1$.

Using (3.6) in (3.3) and the hypothesis (2.1) with $m$ the least integer greater than $v$ immediately shows that (3.3) is $o(1)$. Similarly, using (3.6) in (3.5) and the hypothesis (2.2) shows that (3.5) is $o(1)$.

To treat (3.4), observe that (2.1) implies that given $\varepsilon > 0$ there exists $u_0$ such that for all real $u > u_0$,

$$(3.7)\qquad \int_{u-u^{1/3}}^{u} |f(x)|\, e^{-\alpha/2}x^{\alpha/2-1/2}\,dx < \varepsilon u^{1/12}.$$

Using (3.6) in (3.4) shows that (3.4) is bounded by

$$(3.8)\qquad C\sum_{k=0}^{\infty}\int_{v+kv^{1/3}}^{v+(k+1)v^{1/3}} \frac{e^{-t/2}t^{\alpha/2-1/2}\exp(-Dk^{3/2})}{v^{1/12}(k+1)^{1/4}}|f(t)|\,dt.$$

If $v > u_0$, (3.7) can be used to show that (3.8) is less than $\varepsilon$ times a fixed constant. This shows that (3.4) is $o(1)$ and completes the proof of (3.1).

Showing that (3.2) holds is the same; this completes the proof of Theorem 1.

Theorem 2 follows from Theorem 1 in exactly the same way that Theorem 9.1.6 of [8] is derived from Theorem 9.1.5 of [8] as described in [8, pp. 268–269].

**4. Proofs of the corollaries.** To prove the first assertion in Corollary 1, choose $u$, $v$ and $\delta$ so that $0 < \delta < u$, $(u-\delta, v+\delta) \subset (a^{1/2}, b^{1/2})$ and $v - u + 2\delta < 2\pi$ and assume that (2.1), (2.2) and (2.3) are true. By Theorem 1,

$$(4.1)\qquad \frac{1}{\pi}\int_{y^{1/2}-\delta}^{y^{1/2}+\delta} f(t^2)\frac{\sin 2n^{1/2}(y^{1/2}-t)}{y^{1/2}-t}\,dt$$

has the same limit for $y^{1/2} > u$ as $s_n^\alpha(f, y)$ whenever (4.1) converges. Now let $m(n)$ denote the greatest integer less than or equal to $2n^{1/2}$ and let $D_n(x)$ be the ordinary Dirichlet kernel, $[\sin(n+\frac{1}{2})x]/[2\sin x/2]$. Then the fact that $f(t^2)$ is integrable on $(u-\delta, v+\delta)$ and the Riemann-Lebesgue theorem show that (4.1) and

$$(4.2)\qquad \frac{1}{\pi}\int_{u-\delta}^{v+\delta} f(t^2)D_{m(n)}(y^{1/2}-t)\,dt$$

have the same limit whenever (4.2) converges and $y^{1/2}$ is in $(u, v)$.

Since $|f(t^2)|(\log^+|f(t^2)|)^2$ is integrable on $(u-\delta, v+\delta)$, Hunt's convergence theorem [4, p. 235] implies that there is a set $E$ of measure 0 such that (4.2) converges to $f(y)$ for $y^{1/2}$ in $(u, v) - E$. If $E^*$ is the set of all $y$ such that $y^{1/2}$ is in $E$, then $E^*$ has measure 0 and (4.2) converges to $f(y)$ for $y$ in $(u^2, v^2) - E^*$. Therefore, $s_n^\alpha(f, y)$ converges to $f(y)$ for almost all $y$ in $(u^2, v^2)$. Since $(a, b)$ can be written as the countable union of intervals $(u^2, v^2)$ of this type, $s_n^\alpha(f, y)$ converges to $f(y)$ for almost every $y$ in $[a, b]$.

The second assertion in Corollary 1 is now easy to prove. Let $f$ not satisfy one of the conditions (2.1), (2.2) or (2.3) and let $f_1$ be the function whose existence is asserted by Theorem 3. Now the proof for the first part of Corollary 1 shows that (4.1), with $f$ replaced by $f_1$, converges to $f_1(y)$ for almost every $y$ in $[a, b]$. Since the difference between that expression and $s_n^\alpha(f_1, y)$ converges to 0 on at most a subset of measure 0 by Theorem 3, the set of $y$ in $[a, b]$ for which $s_n^\alpha(f_1, y)$ converges to $f_1(y)$ must have measure 0. This completes the proof of Corollary 1.

The proof of Corollary 2 is the same as the proof of Corollary 1.

To prove Corollary 3, use Hölder's inequality to show that the left side of (2.1) is bounded by

$$\left(\int_{m/2}^{m} |f(x)\, e^{-x/2} x^{1/2}|^p\, dx\right)^{1/p} \left(\int_{m/2}^{m} \left[\frac{2m^{-1/2}}{(m + m^{1/3} - x)^{1/4}}\right]^q dx\right)^{1/q}, \tag{4.3}$$

where $1/p + 1/q = 1$. It is now easy to compute the second integral in (4.3) and use (2.8) to show that (4.3) is $o(1)$. This implies that (2.1) is true.

If $1 \leqq p < 4/3$, (2.8) and Hölder's inequality show that

$$\int_{m/2}^{m} |f(x)\, e^{-x/2} x^{\alpha/2 - 3/4}|\, dx = o(m^{1/2 - 2/(3p)}). \tag{4.4}$$

This implies (2.2) in this case.

Since (2.8) implies the hypotheses of Theorem 1 that are not assumed in Corollary 3, Theorem 1 gives the conclusion of Corollary 3.

Corollary 4 is proved similarly.

**5. Known results.** The proofs of Theorems 3 and 4 are given in §§ 5–11. The proof of Theorem 3 requires considerable knowledge about Laguerre polynomials; the needed facts that appear elsewhere are gathered in this section. Section 6 contains two basic lemmas about sequences of Laguerre polynomials and preliminary results needed to prove them. Section 7 contains simple estimates of various integrals containing Laguerre polynomials. Section 8 contains the most involved part of the proof in four similar lemmas. In § 9 it is shown that Theorem 3 is a fairly easy consequence of §§ 5–8. In § 10 certain basic facts about Hermite polynomials are quoted that are needed to prove Theorem 4. In § 11, Theorem 4 is proved by replacing the Hermite polynomials with Laguerre polynomials and then using the lemmas in §§ 6–8.

To simplify expressions the notation

$$\mathscr{L}_n^\alpha(x) = \left[\frac{n!}{\Gamma(n + \alpha + 1)}\right]^{1/2} x^{\alpha/2}\, e^{-x/2} L_n^\alpha(x) \tag{5.1}$$

will be used. In all that follows $\alpha$ will be assumed to be greater than $-1$ and fixed so that the dependence of constants on $\alpha$ will not be given. The notation $\nu = 4n + 2\alpha + 2$ will be used throughout; to avoid some ambiguities when $n = 0$ and $\alpha < 0$, $\nu$ will be defined to be 2 in these cases.

From [1, p. 699] there are positive constants, $C$ and $D$, such that for all $n \geqq 0$,

$$(5.2)\qquad |\mathscr{L}_n^\alpha(x)| \leqq \begin{cases} C(x\nu)^{\alpha/2}, & 0 \leqq x \leqq 1/\nu, \\ C(x\nu)^{-1/4}, & 1/\nu \leqq x \leqq \frac{1}{2}\nu, \\ C\nu^{-1/4}x^{-1/12}, & 1 \leqq x < \infty, \\ \dfrac{C\nu^{-1/4}}{[|x-\nu| + \nu^{1/3}]^{1/4}}, & \frac{1}{2}\nu \leqq x \leqq 2\nu, \\ Ce^{-Dx}, & 2\nu \leqq x < \infty, \end{cases}$$

and for all $n \geqq 1$,

$$(5.3)\qquad |\mathscr{L}_{n+1}^\alpha(x) - \mathscr{L}_{n-1}^\alpha(x)| \leqq \begin{cases} Cx^{\alpha/2}\nu^{\alpha/2-1}, & 0 \leqq x \leqq 1/\nu, \\ Cx^{1/4}\nu^{-3/4}, & 1/\nu \leqq x. \end{cases}$$

The third estimate in (5.2) is a consequence of the second, fourth and fifth. It is a poor estimate, but it is easier to use than the others and is sufficient for most of the computations. As noted after (3.6), the estimates in [1] are stated only for $\alpha \geqq 0$, but the results of [5] show that (5.2) and (5.3) are valid for all $\alpha > -1$.

Two Laguerre polynomial identities will be needed:

$$(5.4)\qquad L_n^{\alpha+1}(x) = -(n+1)x^{-1}L_{n+1}^\alpha(x) + (n+\alpha+1)x^{-1}L_n^\alpha(x)$$

and

$$(5.5)\qquad L_{n-1}^\alpha(x) = -\frac{n+1}{n+\alpha}L_{n+1}^\alpha(x) + \frac{2n+\alpha+1-x}{n+\alpha}L_n^\alpha(x).$$

These are slight modifications respectively of (5.1.14) and (5.1.10) of [8, pp. 100–101].

The asymptotic expansion (5.4) of [3, p. 245] shows that for $1/\nu \leqq x \leqq \nu^{1/3}$,

$$(5.6)\qquad L_n^\alpha(x) = \frac{\Gamma(n+\alpha+1)}{n!}\left(\frac{4}{\nu x}\right)^{\alpha/2} e^{x/2}\left(J_\alpha([\nu x]^{1/2}) + O\left[\frac{x^{1/4}(\nu x+1)}{\nu^{7/4}}\right]\right).$$

Using the estimate for the Bessel function, (1.71.7) of [8, p. 15], we have

$$J_\alpha(x) = \left(\frac{2}{\pi x}\right)^{1/2}\cos\left(x - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O(x^{-3/2}), \qquad x \geqq 1.$$

Stirling's formula and (5.1) show that

$$(5.7)\qquad \mathscr{L}_n^\alpha(x) = \frac{\pi^{-1/2}}{n^{1/4}x^{1/4}}\left[\cos\left(2(nx)^{1/2} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1+x^2}{n^{1/2}x^{1/2}}\right)\right], \qquad n^{-1} \leqq x \leqq n^{1/3}.$$

Define $G = \frac{1}{4}\nu(2\theta - \sin 2\theta) - \frac{1}{4}\pi$ where $\theta = \cos^{-1}(x^{1/2}\nu^{-1/2})$. Expressions (7.6) and (7.7) of [7] are

$$(5.8)\qquad \mathscr{L}_n^\alpha(x) = \frac{(2/\pi)^{1/2}(-1)^n}{x^{1/4}(\nu-x)^{1/4}}\cos G + O\left[(\nu x)^{-3/4} + \frac{\nu^{1/4}}{(\nu-x)^{7/4}}\right], \qquad 1 \leqq x \leqq \nu - \nu^{1/3},$$

and

$$(5.9)\quad \mathscr{L}^{\alpha}_{n+1}(x) - \mathscr{L}^{\alpha}_{n-1}(x) = \frac{2^{5/2}(-1)^n x^{1/4}}{\pi^{1/2} v^{3/4}}\left(\sin G + O\left[\frac{x}{v} + \frac{1}{x}\right]\right), \qquad 1 \leqq x \leqq \tfrac{1}{2}v.$$

Next, define

$$(5.10)\qquad D_n(x, y) = \sum_{k=0}^{n} \mathscr{L}^{\alpha}_k(x)\mathscr{L}^{\alpha}_k(y),$$

$$(5.11)\qquad j_1(n, x, y) = \mathscr{L}^{\alpha}_n(x)\mathscr{L}^{\alpha}_n(y),$$

$$(5.12)\qquad j_2(n, x, y) = \frac{n\mathscr{L}^{\alpha}_n(y)[\mathscr{L}^{\alpha}_{n+1}(x) - \mathscr{L}^{\alpha}_{n-1}(x)]}{y - x}$$

and

$$(5.13)\qquad j_3(n, x, y) = j_2(n, y, x).$$

Then by (2.22) of [7] there are sequences $b_n$ and $c_n$ bounded above and below by positive constants depending only on $\alpha$ such that

$$(5.14)\qquad D_n = b_n j_1 + c_n(j_2 + j_3).$$

Three lemmas from [6] will be needed and are stated here as Lemmas 1–3. They are Lemmas 2, 4 and 5 respectively of [6]; for proofs see [6].

LEMMA 1. *Let $L$ be an integer greater than* 20 *and let $I$ be a set of $L$ consecutive integers. If for $n$ in $I$, $1/(3L) \leqq g(n+1) - g(n) \leqq \frac{1}{4}\pi$ and $g(n+1) - g(n)$ is monotone increasing in $n$, then for at least $\frac{2}{3}$ of the integers, $n$, in $I$, $|\cos g(n)| \geqq 1/200$.*

LEMMA 2. *If $y \geqq (90)^3$, $\alpha > -1$ and $x$ is a fixed number such that $5y/6 \leqq x \leqq y - y^{1/3}$, then for at least two thirds of the integers, $n$, such that $y \leqq 4n + 2\alpha + 2 \leqq y + y^{1/3}$, $|\cos G| \geqq 1/200$.*

LEMMA 3. *Let $w(x)$ be a nonnegative function and $t$ a positive real number. Let $f(n, x)$ be a function such that for every $x$ in a set $E$ and every integer in a finite set of integers, $I$, $0 \leqq f(n, x) \leqq 1$, and for each $x$ in $E$, $f(n, x) \geqq t$ for at least $\frac{2}{3}$ of the $n$'s in $I$. Then $\int_E f(n, x)w(x)\,dx \geqq (t/10)\int_E w(x)\,dx$ for at least $\frac{3}{5}$ of the $n$'s in $I$.*

**6. Sequences of Laguerre polynomials.** The purpose of this section is to prove Lemmas 5 and 6. Various preliminary results are given for these proofs.

First, it will be shown that for $v^{-1} \leqq x \leqq v^{1/3}$,

$$(6.1)\quad \mathscr{L}^{\alpha}_{n+1}(x) - \mathscr{L}^{\alpha}_{n-1}(x) = \frac{-2x^{1/4}}{\pi^{1/2} n^{3/4}}\left(\sin\left[2(nx)^{1/2} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right] + O\left[\frac{1 + x^2}{n^{1/2} x^{1/2}}\right]\right).$$

To prove this, assume that $x$ is in the indicated range. Equations (5.4) and (5.5) can be combined to show that

$$(6.2)\qquad \begin{aligned} L^{\alpha}_{n+1}(x) - L^{\alpha}_{n-1}(x) &= \frac{-vx}{2(n+1)(n+\alpha)} L^{\alpha+1}_n(x) \\ &\quad + \frac{2(n+1)x + \alpha v}{2(n+1)(n+\alpha)} L^{\alpha}_n(x). \end{aligned}$$

Next, replace all the Laguerre polynomials in (6.2) by using (5.1), and multiply the resulting identity by

$$e^{-\alpha/2}x^{\alpha/2}\left[\frac{(n-1)!}{\Gamma(n+d)}\right]^{1/2}.$$

This shows that $\mathscr{L}_{n+1}^{\alpha}(x) - \mathscr{L}_{n-1}^{\alpha}(x)$ equals

$$(6.3)\qquad \frac{-2x^{1/2}}{n^{1/2}}\mathscr{L}_n^{\alpha+1}(x)\left[1+O\left(\frac{1}{n}\right)\right] + O\left(\frac{x+1}{n}\right)\mathscr{L}_n^{\alpha}(x) + O\left(\frac{1}{n}\right)\mathscr{L}_{n+1}^{\alpha}(x).$$

Now using the fact obtained from (5.2) that $|\mathscr{L}_n^{\alpha}(x)| \leqq C(nx)^{-1/4}$ on the error terms in (6.3) and using (5.7) on the principal term completes the proof of (6.1).

The following will be needed in the proofs of Lemmas 5 and 6.

LEMMA 4. *If $r_k$ and $s_k$ are sequences, $\lim_{k\to\infty}|r_k| = \infty$ and $E$ is the set of all nonnegative $x$ for which* $\limsup_{k\to\infty}|\cos(r_k x^{1/2} + s_k)| < \frac{1}{2}$, *then $E$ has measure* 0.

Let $E^*$ be the set of all $x$ such that $x^2$ is in $E$. Let $E_k$ be the set of $x$ for which $|\cos(r_k x + s_k)| < \frac{1}{2}$. Then

$$(6.4)\qquad E^* = \bigcup_{j=1}^{\infty}\bigcap_{k=j}^{\infty} E_k.$$

If $b - a > 2\pi(r_k)^{-1}$, then by the definition of $E_k$, $m([a,b] \cap E_k) \leqq \frac{1}{2}(b-a)$ where $m(D)$ denotes the Lebesgue measure of $D$. Then for any $j$ and any interval $[a,b]$, $m([a,b] \cap \bigcap_{k=j}^{\infty} E_k) \leqq \frac{1}{2}(b-a)$. But then $\bigcap_{k=j}^{\infty} E_k$ has no points of density so its measure is 0 for each $j$. Equation (6.4) then shows that $m(E^*) = 0$. It follows immediately that $m(E) = 0$.

LEMMA 5. *If $\lim_{n\to\infty} a_n\mathscr{L}_n^{\alpha}(x) = 0$ for $x$ in a subset, $E$, of $[0,\infty)$ of positive measure, then $a_n = o(n^{1/4})$.*

Suppose that the conclusion is false so that there is an $\varepsilon > 0$ and a strictly monotone sequence $n_k$ such that

$$(6.5)\qquad a_{n_k}(n_k)^{-1/4} \geqq \varepsilon.$$

Using (5.7) shows that for each positive $x$ in $E$,

$$(6.6)\qquad \lim_{k\to\infty} a_{n_k}(n_k)^{-1/4}\left(\cos\left[2(n_k x)^{1/2} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right] + O[(n_k)^{-1/2}]\right) = 0.$$

But then (6.5) implies that

$$(6.7)\qquad \lim_{k\to\infty}\cos\left[2(n_k x)^{1/2} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right] = 0$$

for each positive $x$ in $E$. Lemma 4 then implies that $m(E) = 0$. This contradiction proves Lemma 5.

LEMMA 6. *If $t_n(x) = (a_n + x^r b_n)\mathscr{L}_n^{\alpha}(x) + c_n[\mathscr{L}_{n+1}^{\alpha}(x) - \mathscr{L}_{n-1}^{\alpha}(x)]$ is bounded for each $x$ in a subset of $[0,\infty)$ of positive measure, then $a_n = O(n^{1/4})$, $b_n = O(n^{1/4})$ and $c_n = O(n^{3/4})$.*

Suppose that the conclusion of the lemma is false. Then there is a strictly monotone sequence, $n_k$, such that

$$(6.8)\qquad (n_k)^{-1/4}(|a_{n_k}| + |b_{n_k}|) + (n_k)^{-3/4}|c_{n_k}| \geqq k$$

and either (i) $-a_{n_k}/b_{n_k}$ converges to a finite limit, $u$, or (ii) $|a_{n_k}/b_{n_k}|$ converges to $\infty$, or (iii) $a_{n_k} = b_{n_k} = 0$ for all $k$. Using the hypothesis, it is easy to see that there is a set $E \subset (0, \infty)$ of positive measure such that $t_n(x)$ is bounded for each $x$ in $E$, $x^r + u$ in case (i) has the same algebraic sign for all $x$ in $E$, and $x^r + u$ in case (i) and $x$ in all cases vary by no more than a factor of 11/10 for $x$ in $E$. In each case it follows readily that there exist constants, $C > 0$ and $k_0$, such that

$$|a_{n_k} + x^r b_{n_k}| \geqq C(|a_{n_k}| + |b_{n_k}|), \qquad x \in E, \quad k > k_0, \tag{6.9}$$

$$|a_{n_k} + x^r b_{n_k}| \leqq \tfrac{6}{5}|a_{n_k} + y^r b_{n_k}|, \qquad x \in E, \quad y \in E, \quad k > k_0, \tag{6.10}$$

and $a_{n_k} + x^r b_{n_k}$ has the same algebraic sign for all $x \in E$ provided that $k$ is fixed and greater than $k_0$.

Now define $R(k, x)$ and $\theta(k, x)$ to satisfy $R(k, x) \geqq 0$, $-\pi < \theta(k, x) \leqq \pi$,

$$R(k, x) \cos \theta(k, x) = \pi^{-1/2}(n_k x)^{-1/4}(a_{n_k} + x^r b_{n_k}) \tag{6.11}$$

and

$$R(k, x) \sin \theta(k, x) = 2\pi^{-1/2} x^{1/4}(n_k)^{-3/4} c_{n_k}. \tag{6.12}$$

Then using (5.7) and (6.1) shows that for $x$ in $E$,

$$t_{n_k}(x) = R(k, x)\left(\cos\left[2(n_k x)^{1/2} + \theta(k, x) - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right] + O([n_k]^{-1/2})\right). \tag{6.13}$$

Now let $x_0$ be a fixed point in $E$ and $k$ an integer greater than $k_0$. Using (6.10) and the facts that for $x$ in $E$, $a_{n_k} + x^r b_{n_k}$ has constant algebraic sign and $x$ varies by no more than a factor of 11/10, shows that for $x$ in $E$, $\theta(k, x)$ stays in one quadrant and $\tan \theta(k, x)$ varies by no more than a factor of 4/3. Consequently, if $k > k_0$ and $x \in E$,

$$|\theta(k, x) - \theta(k, x_0)| \leqq \sup_u \left|\frac{u/3}{1 + u^2}\right| \leqq \frac{1}{6} \tag{6.14}$$

by use of the law of the mean on the function $\tan^{-1} u$. By (6.8) and (6.9) $\lim_{k \to \infty} R(k, x) = \infty$ for each $x$ in $E$. Then (6.13) shows that if $x$ is in $E$,

$$\lim_{k \to \infty} \cos\left[2(n_k x)^{1/2} + \theta(k, x) - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right] = 0.$$

This and (6.14) show that for $x$ in $E$,

$$\limsup_{k \to \infty} \left|\cos\left[2(n_k x)^{1/2} + \theta(k, x_0) - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right]\right| \leqq \frac{1}{6}.$$

Lemma 4 then shows that $E$ has measure 0. This contradiction proves Lemma 6.

## 7. Simple consequences of Lemmas 1–3.

LEMMA 7. *Lemma 2 remains valid if, in the conclusion,* "cos" *is replaced by* "sin."

This follows from the fact that Lemma 2 is proved directly from Lemma 1 and the fact that the proof of Lemma 1 is equally valid if "cos" is replaced by "sin" in its conclusion.

LEMMA 8. *If* $0 < y \leqq 1/30, \alpha > -1$, *and* $x$ *is a fixed number such that* $y \leqq x \leqq 1$, *then for at least two thirds of the integers*, $n$, *such that* $y^{-1} \leqq n \leqq 2y^{-1}$,

$$\left|\cos\left[2(nx)^{1/2} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right]\right| \geqq \frac{1}{200}.$$

This is a simple application of Lemma 1. Let $g(n) = -2(-nx)^{1/2} + \alpha\pi/2 + \pi/4$, let $I$ be the set of integers such that $y^{-1} \leqq -n \leqq 2y^{-1}$ and let $L$ be the number of integers in $I$. Then $20 < y^{-1} - 1 \leqq L \leqq y^{-1} + 1$ and

$$g(n+1) - g(n) = \frac{2x^{1/2}}{(-n)^{1/2} + (-n-1)^{1/2}}.$$

It is easy to verify that for $n$ in $I$, $g(n+1) - g(n)$ is monotone increasing and $1/(3L) \leqq g(n+1) - g(n) \leqq \frac{1}{4}\pi$. By Lemma 1, $|\cos g(n)| \geqq 1/200$ for at least $\frac{2}{3}$ of the $n$'s in $I$. This immediately gives the conclusion of this lemma.

The integral estimates that will be needed in the proof of Theorems 3 and 4 will now be given as Lemmas 9–11.

LEMMA 9. *Given* $\alpha > -1$, *there exist constants* $C_1 \geqq 1$, $C_2 > 0$ *and* $y_0 > 0$ *such that if* $y > y_0$, $E_y = [5y/6, y - C_1 y^{1/3}]$ *and* $w(x)$ *is a measurable function, then*

$$\int_{E_y} |\mathscr{L}_n^\alpha(x) w(x)|\, dx \geqq C_2 \int_{E_y} \frac{|w(x)|\, dx}{y^{1/4}(y-x)^{1/4}} \tag{7.1}$$

*for at least* $\frac{3}{5}$ *of the* $n$*'s satisfying* $y \leqq 4n + 2\alpha + 2 \leqq y + y^{1/3}$.

Assume that $y \leqq 4n + 2\alpha + 2 \leqq y + y^{1/3}$. The estimate (5.8) shows that there is a constant, $C_3$, such that for any $C_1 \geqq 1$, the left side of (7.1) is bounded below by

$$(2/\pi)^{1/2} \int_{E_y} \frac{|w(x)\cos G|}{x^{1/4}(v-x)^{1/4}}\, dx - C_3 \int_{E_y} \frac{|w(x)| v^{1/4}\, dx}{(v-x)^{7/4}}. \tag{7.2}$$

Using Lemmas 2 and 3 on the first integral in (7.2) and the definition of $E_y$ on the second one shows that if $y \geqq (90)^3$ there is a positive constant, $C_4$, such that for $\frac{3}{5}$ of the indicated $n$'s, (7.2) is bounded below by

$$C_4 \int_{E_y} \frac{|w(x)|\, dx}{x^{1/4}(v-x)^{1/4}} - C_3 (C_1)^{-3/2} \int_{E_y} \frac{|w(x)|\, dx}{x^{1/4}(v-x)^{1/4}}. \tag{7.3}$$

Taking $C_1$ as the larger of 1 and $(2C_3/C_4)^{2/3}$ proves (7.1) with $C_2 = \frac{1}{2}C_4$ and $y_0 = (90)^3$.

LEMMA 10. *Given* $\alpha > -1$, *there exist constants* $C_1 \geqq 2$, $C_2 > 0$ *and* $y_0 > 0$ *such that if* $y > y_0$, $E_y = [C_1, y/C_1]$ *and* $w(x)$ *is a measurable function, then*

$$\int_{E_y} \left|\mathscr{L}_{n+1}^\alpha(x) - \mathscr{L}_{n-1}^\alpha(x)\right| |w(x)|\, dx \geqq C_2 \int_{E_y} |w(x)| x^{1/4} y^{-3/4}\, dx \tag{7.4}$$

*for at least* $\frac{3}{5}$ *of the* $n$*'s satisfying* $y \leqq 4n + 2\alpha + 2 \leqq y + y^{1/3}$.

The estimate (5.9) shows that there is a constant, $C_3$, such that for any $C_1 \geqq 2$ the left side of (7.4) is bounded below by

$$\frac{2^{5/2}}{\pi^{1/2} v^{3/4}} \int_{E_y} x^{1/4} |\sin G|\, |w(x)|\, dx - \frac{C_3}{C_1} \int_{E_y} \frac{x^{1/4}}{v^{3/4}} |w(x)|\, dx. \tag{7.5}$$

The proof is then like that of Lemma 9; Lemmas 3 and 7 are used and $C_1$ is made large enough to give the result.

LEMMA 11. *Given $\alpha > -1$, there exist constants $C_1 \geqq 1$, $C_2 > 0$ and $y_0 > 0$ such that if $0 < y \leqq y_0$, $E_y = [C_1 y, 1]$ and $w(x)$ is a measurable function, then*

$$\int_{E_y} |\mathscr{L}_n^\alpha(x) w(x)|\, dx \geqq C_2 \int_{E_y} |w(x)| x^{-1/4} y^{1/4}\, dx \tag{7.6}$$

*for at least $\frac{3}{5}$ of the n's satisfying $y^{-1} \leqq n \leqq 2y^{-1}$.*

The proof is like that of Lemmas 9 and 10; it uses (5.7) and Lemmas 8 and 3.

**8. More lemmas.** The four lemmas of this section are all similar and are specifically oriented toward proving Theorems 3 and 4.

LEMMA 12. *Let $g(x)$ be a nonnegative function and assume that $\alpha > -1$. If for every function, $g_1(x)$, satisfying $|g_1(x)| = g(x)$,*

$$\int_1^\infty g_1(x) \mathscr{L}_n^\alpha(x)\, dx = o(n^{1/4}), \qquad n \to \infty, \tag{8.1}$$

*then*

$$\int_{n/2}^n \frac{g(x)\, dx}{(n + n^{1/3} - x)^{1/4}} = o(n^{1/2}), \qquad n \to \infty. \tag{8.2}$$

Assume that (8.2) is false. Then there is an $\varepsilon > 0$ and arbitrarily large $n$'s such that

$$\int_{n/2}^n \frac{g(x)\, dx}{(n + n^{1/3} - x)^{1/4}} > \varepsilon n^{1/2}. \tag{8.3}$$

Let $E_y = [5y/6, y - C_1 y^{1/3}]$ where $C_1$ is as in Lemma 9. It is easy to see that there is a positive constant, $C$, such that if $x$ is in $E_y$ and $E_y \subset [n/2, n]$, then $(y - x)^{-1/4} \geqq C(n + n^{1/3} - x)^{-1/4}$. Furthermore, since $(5/6)^4 < \frac{1}{2}$, for large $n$, $[n/2, n]$ can be written as the union of $4E_y$'s that are subsets of $[n/2, n]$. These facts and (8.3) show that there are arbitrarily large $y$'s for which

$$\int_{E_y} \frac{g(x)\, dx}{(y - x)^{1/4}} > \frac{C\varepsilon y^{1/2}}{5}. \tag{8.4}$$

Three sequences, $n_k$, $y_k$ and $z_k$, will be defined inductively. Let $z_1 = 1$. Given $z_k$, the numbers $y_k$, $n_k$ and $z_{k+1}$ will be chosen as described below.

Choose $y_k$ so that $y_k > 2z_k$, $y_k$ is greater than the $y_0$ of Lemma 9,

$$\int_1^{z_k} x^{-1/4} g(x)\, dx < (y_k)^{1/4} \tag{8.5}$$

and (8.4) is true with $y = y_k$. The integral in (8.5) exists since (8.1) implies that $g(x)$ is locally integrable. With $C_2$ as in Lemma 9, Lemma 9 shows that

$$\int_{E_{y_k}} |\mathscr{L}_n^\alpha(x)| g(x)\, dx \geqq C_2 \int_{E_{y_k}} \frac{g(x)\, dx}{y_k^{1/4}(y_k - x)^{1/4}} \tag{8.6}$$

for at least $\frac{3}{5}$ of the $n$'s satisfying $y_k \leqq 4n + 2\alpha + 2 \leqq y_k + y_k^{1/3}$. Let $n_k$ be one of these $n$'s. Combining (8.6) with (8.4) shows that

$$\int_{E_{y_k}} |\mathscr{L}_{n_k}^{\alpha}(x)| g(x)\, dx \geqq \frac{\varepsilon}{5} C C_2 y_k^{1/4}. \tag{8.7}$$

By hypothesis $\int_1^\infty g(x)\mathscr{L}_{n_k}^{\alpha}(x)\, dx$ exists. Consequently a number, $z_{k+1}$, can be chosen so that $z_{k+1} > y_k$ and

$$\int_{z_{k+1}}^\infty |\mathscr{L}_{n_k}^{\alpha}(x)| g(x)\, dx < 1. \tag{8.8}$$

Now define $g_1(x) = g(x) \operatorname{sgn} \mathscr{L}_{n_k}^{\alpha}(x)$ for $z_k \leqq x < z_{k+1}$, $k = 1, 2, \cdots$. Given a $k$,

$$\int_1^\infty \mathscr{L}_{n_k}^{\alpha}(x) g_1(x)\, dx \tag{8.9}$$

is bounded below by the difference of

$$\int_{z_k}^{z_{k+1}} |\mathscr{L}_{n_k}^{\alpha}(x)| g(x)\, dx \tag{8.10}$$

and

$$\int_1^{z_k} |\mathscr{L}_{n_k}^{\alpha}(x)| g(x)\, dx + \int_{z_{k+1}}^\infty |\mathscr{L}_{n_k}^{\alpha}(x)| g(x)\, dx. \tag{8.11}$$

The first integral in (8.11) is bounded by use of (5.2), (8.5) and the fact that $2z_k < y_k \leqq 4n_k + 2\alpha + 2$. The second integral in (8.11) is bounded by 1 because of (8.8). The integral (8.10) is greater than a constant times $(n_k)^{1/4}$ by (8.7) and the facts that $E_{y_k} \subset [z_k, z_{k+1}]$ and $4n_k + 2\alpha + 2 \leqq 2y_k$. Therefore, for sufficiently large $k$, (8.9) is bounded below by a constant times $n_k^{1/4}$. This contradiction of (8.1) completes the proof of Lemma 12.

LEMMA 13. *Let $g(x)$ be a nonnegative function and assume that $\alpha > -1$. If for every function, $g_1(x)$, satisfying $|g_1(x)| = g(x)$,*

$$\int_0^1 g_1(x)\mathscr{L}_n^{\alpha}(x)\, dx = o(n^{1/4}), \qquad n \to \infty, \tag{8.12}$$

*then*

$$\int_{1/n}^{2/n} g(x)\, dx = o(n^{1/4}), \qquad n \to \infty. \tag{8.13}$$

The proof is similar to that of Lemma 12. Assume that (8.13) is false, let $C_1$, $C_2$ and $y_0$ be as in Lemma 11 and let $E_y = [C_1 y, 2C_1 y]$. Then there are an $\varepsilon > 0$ and positive $y$'s arbitrarily close to zero such that

$$\int_{E_y} g(x) x^{-1/4}\, dx > \varepsilon y^{-1/2}. \tag{8.14}$$

Sequences $z_k$, $y_k$ and $n_k$ will be defined inductively. Let $z_1 = 1$. Given $z_k$, the numbers $y_k$, $n_k$ and $z_{k+1}$ will be defined as described below.

Choose $y_k$ so that $2C_1 y_k \leqq z_k$, $0 < y_k \leqq y_0$, (8.14) holds with $y = y_k$ and

$$\int_{z_k}^{1} g(x)x^{-1/4}\,dx \leqq y_k^{-1/4}. \tag{8.15}$$

By Lemma 11,

$$\int_{E_{y_k}} |\mathscr{L}_n^\alpha(x)| g(x)\,dx \geqq C_2 \int_{E_{y_k}} g(x)x^{-1/4} y_k^{1/4}\,dx \tag{8.16}$$

for at least $\frac{3}{5}$ of the $n$'s satisfying $y_k^{-1} \leqq n \leqq 2y_k^{-1}$. Let $n_k$ be one such $n$. Combining (8.14) and (8.16) shows that

$$\int_{E_{y_k}} |\mathscr{L}_{n_k}^\alpha(x)| g(x)\,dx > \varepsilon C_2 y_k^{-1/4}. \tag{8.17}$$

Define $z_{k+1}$ so that $0 < z_{k+1} \leqq C_1 y_k$ and

$$\int_0^{z_{k+1}} |\mathscr{L}_{n_k}^\alpha(x)| g(x)\,dx \leqq 1. \tag{8.18}$$

Now let $g_1(x) = g(x) \operatorname{sgn} \mathscr{L}_{n_k}^\alpha(x)$ for $z_{k+1} < x \leqq z_k$, $k = 1, 2, \cdots$. Given a $k$, write

$$\int_0^1 g_1(x)\mathscr{L}_{n_k}^\alpha(x)\,dx \tag{8.19}$$

as the sum of integrals over $[0, z_{k+1}]$, $[z_{k+1}, z_k]$ and $[z_k, 1]$. Inequalities (5.2) and (8.15) show that the third of these is bounded by a constant independent of $k$ while (8.18) shows the same for the first part. Inequality (8.17) shows that the second part is greater than a constant times $n_k^{1/4}$, and, therefore, the same is true for (8.19). This contradicts (8.12) and completes the proof of Lemma 13.

LEMMA 14. *Let $g(x)$ be a nonnegative function and assume that $\alpha > -1$. If for every function, $g_1(x)$, satisfying $|g_1(x)| = g(x)$,*

$$\int_1^\infty g_1(x)[\mathscr{L}_{n+1}^\alpha(x) - \mathscr{L}_{n-1}^\alpha(x)]\,dx = O(n^{-3/4}), \tag{8.20}$$

*then*

$$\int_1^\infty g(x)x^{1/4}\,dx < \infty. \tag{8.21}$$

Assume that (8.21) is not true. Let $C_1$, $C_2$ and $y_0$ be as in Lemma 10. Let $z_1 = C_1$; given $z_k$, numbers $y_k$, $n_k$ and $z_{k+1}$ will be chosen as described below.

Choose $y_k$ so that $y_k \geqq C_1 z_k$, $y_k > y_0$ and

$$\int_{z_k}^{y_k/C_1} g(x)x^{1/4}\,dx \geqq \frac{8C}{C_2}\int_1^{z_k} g(x)x^{1/4}\,dx, \tag{8.22}$$

where $C$ is the constant in (5.3). This is possible since (8.21) is assumed false and, as shown by (8.20), $g(x)$ is locally integrable.

Next choose $n_k$ so that $y_k \leqq 4n_k + 2\alpha + 2 \leqq 2y_k$ and

$$\int_{z_k}^{y_k/C_1} |\mathscr{L}^{\alpha}_{1+n_k}(x) - \mathscr{L}^{\alpha}_{-1+n_k}(x)| g(x)\,dx \geqq C_2 \int_{z_k}^{y_k/C_1} \frac{g(x)x^{1/4}\,dx}{y_k^{3/4}}; \tag{8.23}$$

this is possible by Lemma 10 since $z_k \leqq y_k/C_1$ and, as will be shown by the definition of the $z_k$'s, $C_1 \leqq z_k$.

Finally, choose $z_{k+1}$ so that $z_{k+1} \geqq 2y_k/C_1$ and

$$\int_{z_{k+1}}^{\infty} |\mathscr{L}^{\alpha}_{1+n_k}(x) - \mathscr{L}^{\alpha}_{-1+n_k}(x)| g(x)\,dx < n_k^{-3/4}; \tag{8.24}$$

this is possible because of (8.20).

Now define $g_1(x) = g(x)\operatorname{sgn}[\mathscr{L}^{\alpha}_{1+n_k}(x) - \mathscr{L}^{\alpha}_{-1+n_k}(x)]$ for $z_k \leqq x < z_{k+1}$, $k = 1, 2, \cdots$, and $g_1(x) = g(x)$ for $1 \leqq x < z_1$. Then for $n = n_k$ the left side of (8.20) can be written as the sum of integrals over $[1, z_k]$, $[z_k, z_{k+1}]$ and $[z_{k+1}, \infty]$. Using (5.3) on the first of these, (8.23) and (8.22) on the second and (8.24) on the third shows that the left side of (8.20) with $n = n_k$ and $k$ sufficiently large is bounded below by

$$n_k^{-3/4}\left[-1 + C\int_1^{z_k} g(x)x^{1/4}\,dx\right]. \tag{8.25}$$

Since (8.21) was assumed to be false, this contradicts (8.20). This completes the proof of Lemma 14.

LEMMA 15. *Let $g(x)$ be a nonnegative function and assume that $\alpha > -1$. If for every function, $g_1(x)$, satisfying $|g_1(x)| = g(x)$,*

$$\int_0^1 g_1(x)\mathscr{L}^{\alpha}_n(x)\,dx = O(n^{-1/4}), \tag{8.26}$$

*then*

$$\int_0^1 g(x)x^{-1/4}\,dx < \infty. \tag{8.27}$$

Assume that (8.27) is false, let $C_1$, $C_2$ and $y_0$ be as in Lemma 11 and let $C$ be as in (5.2). Let $z_1 = 1$; given $z_k$, numbers $y_k$, $n_k$ and $z_{k+1}$ will be chosen as described below.

Choose $y_k$ so that $C_1 y_k < z_k$, $0 < y_k \leqq y_0$, and

$$\int_{C_1 y_k}^{z_k} g(x)x^{-1/4}\,dx \geqq \frac{3C}{C_2}\int_{z_k}^{1} g(x)x^{-1/4}\,dx. \tag{8.28}$$

Next, choose $n_k$ to be an integer satisfying $y_k^{-1} \leqq n_k \leqq 2y_k^{-1}$ and

$$\int_{C_1 y_k}^{z_k} g(x)|\mathscr{L}^{\alpha}_{n_k}(x)|\,dx \geqq C_2 \int_{C_1 y_k}^{z_k} g(x)x^{-1/4}y_k^{1/4}\,dx; \tag{8.29}$$

this is possible by Lemma 11 since $C_1 y_k \leqq z_k$ and, as will be shown by the definition of the $z_k$'s, $z_k \leqq 1$.

Finally, choose $z_{k+1}$ so that $0 < z_{k+1} \leqq \frac{1}{2}C_1 y_k$ and

$$\int_0^{z_{k+1}} g(x)\mathscr{L}^{\alpha}_{n_k}(x)\,dx \leqq n_k^{-1/4}; \tag{8.30}$$

this is possible because of (8.26).

Define $g_1(x) = g(x)\operatorname{sgn}\mathscr{L}^{\alpha}_{n_k}(x)$ for $z_{k+1} < x \leqq z_k$, $k = 1, 2, \cdots$. As in the last proof, it is now easy to prove that the left side of (8.26) with $n = n_k$ is bounded below by

$$n_k^{-1/4}\left(-1 + C\int_{z_k}^{1} g(x)x^{-1/4}\,dx\right). \tag{8.31}$$

The assumption that (8.27) is false then produces a contradiction of (8.26). This completes the proof of Lemma 15.

**9. Proof of Theorem 3.** Theorem 3 will now be proved by showing that the negation of its conclusion implies the negation of its hypothesis. Accordingly, assume that $f(x)$ is a function such that for each $f_1(x)$ satisfying $|f_1(x)| = f(x)$ there exists a subset, $E$, of $[0, \infty)$ of positive measure and for each $y$ in $E$ a $\delta$ satisfying $0 < \delta < y^{1/2}$ for which (2.4) is true with $f$ replaced by $f_1$. It will be proved that (2.1), (2.2) and (2.3) are true.

Given an $f_1$ and the corresponding set $E$, choose a $y \in E$ and its $\delta$ and subtract the $(n-1)$th term in (2.4), with $f$ replaced by $f_1$, from the $n$th term. Since (2.4) is true and

$$\frac{\sin(2n)^{1/2}(y^{1/2}-x)}{y^{1/2}-x} - \frac{\sin(2n-2)^{1/2}(y^{1/2}-x)}{y^{1/2}-x} = O(n^{-1/2}),$$

uniformly for all $x$, it follows that

$$s^{\alpha}_n(f_1, y) - s^{\alpha}_{n-1}(f_1, y) = o(1) + O(n^{-1/2})\int_{y^{1/2}-\delta}^{y^{1/2}+\delta} f_1(x^2)\,dx. \tag{9.1}$$

Since $f_1$ must have a Laguerre series to make (2.4) meaningful, $f_1$ is integrable on $[y^{1/2} - \delta, y^{1/2} + \delta]$ so the right side of (9.1) is $o(1)$. Therefore,

$$\lim_{n\to\infty} L^{\alpha}_n(y)\frac{n!}{\Gamma(n+\alpha+1)}\int_0^{\infty} f_1(x)L^{\alpha}_n(x)\,e^{-x}x^{\alpha}\,dx = 0. \tag{9.2}$$

Since (9.2) is true for all $y$ in $E$ and $E$ has positive measure, Lemma 5 and (5.1) show that

$$\int_0^{\infty} f_1(x)\,e^{-x/2}x^{\alpha/2}\mathscr{L}^{\alpha}_n(x)\,dx = o(n^{1/4}). \tag{9.3}$$

Since (9.3) is true for all $f_1$ with $|f_1(x)| = f(x)$, it is true for the sum of an $f_1$ and the function that equals $f_1$ for $x$ in $I$ and is $-f_1$ for $x$ not in $I$. This shows that for any $I \subset [0, \infty)$,

$$\int_I f_1(x)\,e^{-x/2}x^{\alpha/2}\mathscr{L}^{\alpha}_n(x)\,dx = o(n^{1/4}). \tag{9.4}$$

Taking $I = [1, \infty)$, we can now apply Lemma 12 with $g(x) = |f(x)|\,e^{-x/2}x^{\alpha/2}$; this immediately proves (2.1).

Several other simple consequences of (9.4) will be needed at a later stage in this proof. First, (2.1) implies that

$$\int_{n/2}^{n} |f(x)|\, e^{-x/2}x^{\alpha/2}\, dx = o(n^{3/4}). \tag{9.5}$$

Now writing the integral as the sum of integrals over $[2^k, 2^{k+1}]$, $k = 0, 1, 2, \cdots$, and using (9.5) shows that

$$\int_{1}^{\infty} |f(x)|\, e^{-x/2}x^{\alpha/2-1}\, dx < \infty. \tag{9.6}$$

Taking $I = [0, 1]$ in (9.4) and applying Lemma 13 proves that

$$\int_{1/n}^{2/n} |f(x)| x^{\alpha/2}\, dx = o(n^{1/4}). \tag{9.7}$$

Again, splitting up the integral and applying (9.7) proves

$$\int_{0}^{1} f(x) x^{\alpha/2+1/2}\, dx < \infty. \tag{9.8}$$

A simple consequence of (9.8) is the fact that

$$n^{\alpha/2+1/2} \int_{0}^{1/n} f(x) x^{\alpha+1}\, dx = o(1). \tag{9.9}$$

To prove (2.2) and (2.3) the whole expression for the partial sums will be used. Again, given $f_1$, let $E$ denote the corresponding set. Choose $a$ and $b$ so that $0 < a \leqq 1 \leqq b < \infty$ and $E_1 = [2a, \frac{1}{2}b] \cap E$ has positive measure. As mentioned before, the fact that $f_1$ has a Laguerre series implies that $f_1$ is integrable on any compact subset of $(0, \infty)$. This fact, the hypothesis and the Riemann-Lebesgue theorem show that for $y \in E_1$,

$$\lim_{n\to\infty} \left[ s_n^\alpha(f_1, y) - \int_{a^{1/2}}^{b^{1/2}} f_1(x^2) \frac{\sin 2n^{1/2}(y^{1/2} - x)}{y^{1/2} - x}\, dx \right] = 0. \tag{9.10}$$

Now let $F(x) = f_1(x)$ for $a \leqq x \leqq b$ and $F(x) = 0$ elsewhere. Since $F(x)$ satisfies the hypotheses of Theorem 1 it satisfies the conclusion. The reasoning above then shows that (9.10) is satisfied with $f_1$ replaced by $F$ for every $y$ in $E_1$. Subtracting this result from (9.10) and using the definition of $F$ then shows that for every $y$ in $E_1$,

$$\lim_{n\to\infty} [s_n^\alpha(f_1, y) - s_n^\alpha(F, y)] = 0. \tag{9.11}$$

Using (5.1), (5.10) and (5.14) and writing $R = [0, a) \cup (b, \infty)$ shows that $s_n^\alpha(f_1, y) - s_n^\alpha(F, y)$ equals the sum of

$$b_n\, e^{y/2} y^{-\alpha/2} \int_R f_1(x)\, e^{-x/2} x^{\alpha/2} j_1(n, x, y)\, dx \tag{9.12}$$

and

$$c_n\, e^{y/2} y^{-\alpha/2} \int_R f_1(x)\, e^{-x/2} x^{\alpha/2} (j_2(n, x, y) + j_3(n, x, y))\, dx. \tag{9.13}$$

Using (9.4), (5.11) and (5.2) shows that the integral in (9.12) is $o(1)$. Since $b_n$ and $c_n$ are bounded above and below by positive constants, (9.12) is $o(1)$ and (9.11) implies that the integral in (9.13) is $o(1)$ for $y$ in $E_1$.

Now, if $y \in E_1$ and $x \in [0, a]$, then $(y - x)^{-1} = y^{-1}[1 + O(x)]$ while if $y \in E_1$ and $x \in [b, \infty)$, then $(y - x)^{-1} = x^{-1}[-1 + O(x^{-1})]$. Now use (5.12) and (5.13) to replace $j_2$ and $j_3$ in (9.13) and replace the resulting $(y - x)^{-1}$ by the estimates just given. Using (5.2), (5.3), (9.6), (9.8) and (9.9), it is easy to show that for each $y \in E_1$ the error terms are bounded as $n \to \infty$. The principal term resulting from $j_3$ and integrating over $[b, \infty)$ is also bounded for each $y$ in $E_1$. Since the integral in (9.13) was a bounded function of $n$ for each $y$ in $E_1$, the sum of the remaining parts,

$$\frac{n}{y}\mathscr{L}_n^\alpha(y)\int_0^a f_1(x)\, e^{-x/2}x^{\alpha/2}[\mathscr{L}_{n+1}^\alpha(x) - \mathscr{L}_{n-1}^\alpha(x)]\, dx, \tag{9.14}$$

$$-n\mathscr{L}_n^\alpha(y)\int_b^\infty f_1(x)\, e^{-x/2}x^{\alpha/2-1}[\mathscr{L}_{n+1}^\alpha(x) - \mathscr{L}_{n-1}^\alpha(x)]\, dx \tag{9.15}$$

and

$$-\frac{n}{y}[\mathscr{L}_{n+1}^\alpha(y) - \mathscr{L}_{n-1}^\alpha(y)]\int_0^a f_1(x)\, e^{-x/2}x^{\alpha/2}\mathscr{L}_n^\alpha(x)\, dx, \tag{9.16}$$

must be a bounded function of $n$ for each $y$ in $E_1$.

Now apply Lemma 6 to $y$ times the sum of (9.14), (9.15) and (9.16). This shows that

$$n\int_b^\infty f_1(x)\, e^{-x/2}x^{\alpha/2-1}[\mathscr{L}_{n+1}^\alpha(x) - \mathscr{L}_{n-1}^\alpha(x)]\, dx = O(n^{1/4}) \tag{9.17}$$

and

$$n\int_0^a f_1(x)\, e^{-x/2}x^{\alpha/2}\mathscr{L}_n^\alpha(x)\, dx = O(n^{3/4}). \tag{9.18}$$

Since (9.17) is true for every $f_1$ with $|f_1(x)| = f(x)$, Lemma 14 can be applied to show that

$$\int_b^\infty |f(x)|\, e^{-x/2}x^{\alpha/2-3/4}\, dx < \infty. \tag{9.19}$$

Since $f$ must be integrable on $[1, b]$ to have a Laguerre series, (2.2) is proved. Similarly, (9.18) and Lemma 15 imply (2.3). This completes the proof of Theorem 3.

**10. Hermite polynomial expressions.** Theorem 4 can be proved either by proving analogues of the lemmas in §§ 5–8 for Hermite polynomials or by reducing the Hermite polynomial expressions that occur to Laguerre polynomial expressions and using the lemmas in §§ 5–8. The second method will be used. Various needed facts about Hermite polynomials will be quoted in this section. The proof of Theorem 4 is given in § 11.

To simplify expressions, the function

$$\mathscr{H}_n(x) = \exp(-\tfrac{1}{2}x^2)\pi^{-1/4}(2^n n!)^{-1/2}H_n(x) \tag{10.1}$$

will be used. Define $d_n(x, y)$ by

$$d_n(x, y) = \sum_{k=0}^{n} \mathscr{H}_k(x)\mathscr{H}_k(y); \tag{10.2}$$

using (5.5.9) of [8, p. 105] and (10.1) shows that

$$d_n(x, y) = \left(\frac{n+1}{2}\right)^{1/2} \cdot \frac{\mathscr{H}_{n+1}(x)\mathscr{H}_n(y) - \mathscr{H}_n(x)\mathscr{H}_{n+1}(y)}{x - y}. \tag{10.3}$$

Now, starting with the identity,

$$H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x), \tag{10.4}$$

(5.5.8) of [8, p. 105], it is easy to verify that

$$H_{n+1}(x) = \frac{nH_{n+2}(x) + 4(n+1)x^2H_n(x) - 4(n^3 - n)H_{n-2}(x)}{(4n+2)x}. \tag{10.5}$$

Now use (10.5) to eliminate the $\mathscr{H}_{n+1}(x)$ and $\mathscr{H}_{n+1}(y)$ in (10.3). The result is that there are sequences of constants, $a_n$, $b_n$, $c_n$, bounded above and below by positive constants such that for $n \geqq 2$, $d_n(x, y)$ equals the sum of

$$a_n\mathscr{H}_n(x)\mathscr{H}_n(y), \tag{10.6}$$

$$\frac{-nb_n}{y(x-y)}\mathscr{H}_n(x)[\mathscr{H}_{n+2}(y) - \mathscr{H}_{n-2}(y)], \tag{10.7}$$

$$\frac{nb_n}{x(x-y)}\mathscr{H}_n(y)[\mathscr{H}_{n+2}(x) - \mathscr{H}_{n-2}(x)] \tag{10.8}$$

and

$$\frac{-c_n\mathscr{H}_n(x)\mathscr{H}_{n-2}(y)}{y(x-y)} + \frac{c_n\mathscr{H}_n(y)\mathscr{H}_{n-2}(x)}{x(x-y)}. \tag{10.9}$$

Expression (5.6.1) of [8, p. 105] states that

$$H_{2n}(x) = (-1)^n 2^{2n} n! L_n^{-1/2}(x^2). \tag{10.10}$$

Combining this with (5.1) and (10.1) shows that

$$\mathscr{H}_{2n}(x) = (-1)^n |x|^{1/2} \mathscr{L}_n^{-1/2}(x^2). \tag{10.11}$$

This, (5.2) and (5.3) show that there are positive constants, $C$ and $D$, such that for $n \geqq 1$,

$$|\mathscr{H}_{2n}(x)| \leqq Cn^{-1/4}(1 + |x|^{1/3}) \quad \text{for all } x, \tag{10.12}$$

$$|\mathscr{H}_{2n}(x)| \leqq C\exp(-Dx^2) \qquad \text{for } x^2 \geqq 9n, \tag{10.13}$$

and

$$|\mathscr{H}_{2n+2}(x) - \mathscr{H}_{2n-2}(x)| \leqq Cn^{-3/4}(1 + |x|) \quad \text{for all } x. \tag{10.14}$$

**11. Proof of Theorem 4.** As in the proof of Theorem 3 it will be shown that the negation of the conclusion of Theorem 4 implies the negation of its hypothesis. Therefore, assume that $f(x)$ is a function such that for each $f_1$ satisfying $|f_1(x)| = |f(x)|$ there exists a subset, $E$, of $(-\infty, \infty)$ of positive measure and for each $y$ in $E$ a $\delta > 0$ for which (2.7) is true with $f$ replaced by $f_1$. It will be proved that (2.5) and (2.6) are true.

Given an $f_1$ and the corresponding set $E$, it follows from (2.7) that for $y$ in $E$,

$$\lim_{n\to\infty} \frac{H_n(y)}{2^n n!} \int_{-\infty}^{\infty} f_1(x) H_n(x) \exp(-x^2)\, dx = 0 \tag{11.1}$$

in the same way that (9.2) was derived from (2.4). For $n = 2m$, (10.1) and (10.11) can be used in (11.1) to show that for $y$ in $E$ and $y \neq 0$,

$$\lim_{m\to\infty} \mathscr{L}_m^{-1/2}(y^2) \int_{-\infty}^{\infty} f_1(x)|x|^{1/2} \mathscr{L}_m^{-1/2}(x^2) \exp(-\tfrac{1}{2}x^2)\, dx = 0. \tag{11.2}$$

Lemma 5 then implies that the integral in (11.2) is $o(m^{1/4})$; with a change of variable this becomes

$$\int_0^{\infty} [f_1(x^{1/2}) + f_1(-x^{1/2})] x^{-1/4} e^{-x/2} \mathscr{L}_m^{-1/2}(x)\, dx = o(m^{1/4}) \tag{11.3}$$

for every $f_1(x)$ satisfying $|f_1(x)| = |f(x)|$. By the reasoning used to obtain (9.4) from (9.3) it follows that for any subset, $I$, of $[0, \infty)$,

$$\int_I [f_1(x^{1/2}) + f_1(-x^{1/2})] x^{-1/4} e^{-x/2} \mathscr{L}_m^{-1/2}(x)\, dx = o(m^{1/4}). \tag{11.4}$$

Now since all functions $g(x)$ with $|g(x)| = |f(x^{1/2})| + |f(-x^{1/2})|$ can be written in the form $f_1(x^{1/2}) + f_1(-x^{1/2})$, Lemma 12 can be applied to (11.4) with $I = [1, \infty)$ to prove that

$$\int_{n/2}^{n} \frac{|f(x^{1/2})| + |f(-x^{1/2})|}{(n + n^{1/3} - x)^{1/4}} x^{-1/4} e^{-x/2}\, dx = o(n^{1/2}). \tag{11.5}$$

This immediately implies (2.5).

Other consequences of (11.4) will be needed in the proof of (2.6). First, from (2.5) it follows that

$$\int_{n^{1/2}/2}^{n^{1/2}} [|f(x)| + |f(-x)|] n^{-1/4} x^{-1/2} \exp(-\tfrac{1}{2}x^2)\, dx = o(1). \tag{11.6}$$

From this it follows easily that

$$\int_1^{\infty} [|f(x)| + |f(-x)|] x^{-3/2} \exp(-\tfrac{1}{2}x^2)\, dx < \infty. \tag{11.7}$$

From (11.7) it is easy to show that for $b \geqq 1$,

$$\int_{|x|\geqq b} |f(x)\mathscr{H}_{2m}(x)| x^{-1} \exp(-\tfrac{1}{2}x^2)\, dx = o(m^{1/4}) \tag{11.8}$$

by splitting the integral into integrals over $b \leqq |x| \leqq 3m^{1/2}$ and $3m^{1/2} \leqq |x|$ and using (10.12) on the first and (10.13) on the second. From (11.4) and (10.11) it follows that for any $b$,

$$\int_{|x| \geqq b} f_1(x)\mathscr{H}_{2m}(x) \exp\left(-\tfrac{1}{2}x^2\right) dx = o(m^{1/4}). \tag{11.9}$$

Now, given $f_1(x)$ and its corresponding set, $E$, choose $b \geqq 1$ so that $E_1 = [-\frac{1}{2}b, \frac{1}{2}b] \cap E$ has positive measure. Define $F(x) = f_1(x)$ for $|x| \leqq b$ and $F(x) = 0$ elsewhere. Equation (2.7) and the reasoning that produced (9.11) from (2.4) show that for $y \in E_1$,

$$\lim_{n \to \infty} [s_n(f_1, y) - s_n(F, y)] = 0. \tag{11.10}$$

Using the notation (10.2), we see that for $y$ in $E_1$,

$$\lim_{n \to \infty} \int_{|x| \geqq b} d_{2n}(x, y) f_1(x) \exp\left(-\tfrac{1}{2}x^2\right) dx = 0. \tag{11.11}$$

Now replace $d_{2n}(x, y)$ in (11.11) by the sum of (10.6)–(10.9) and write $(x - y)^{-1} = x^{-1} + O(x^{-2})$. Using (11.9) on the term resulting from (10.6), (11.8) on the term resulting from (10.9), and (10.12), (10.14) and (11.7) on the error terms resulting from (10.7) and (10.8) shows that all these integrals are bounded functions of $n$ for each $y \neq 0$ in $E_1$. Since $a_n$, $b_n$ and $c_n$ are bounded above and below by positive constants, the sum of the principal terms resulting from (10.7) and (10.8),

$$\frac{-n}{y}[\mathscr{H}_{2n+2}(y) - \mathscr{H}_{2n-2}(y)] \int_{|x| \geqq b} \frac{f_1(x)\mathscr{H}_{2n}(x)}{x \exp\left(\frac{1}{2}x^2\right)} dx \tag{11.12}$$

and

$$n\mathscr{H}_{2n}(y) \int_{|x| \geqq b} \frac{f_1(x)[\mathscr{H}_{2n+2}(x) - \mathscr{H}_{2n-2}(x)]}{x^2 \exp(x^2/2)} dx, \tag{11.13}$$

is a bounded function of $n$ for each $y \neq 0$ in $E_1$.

Next use (10.11) to replace the $\mathscr{H}$'s in (11.12) and (11.13) and apply Lemma 6. This implies that the integral in (11.13) is $O(n^{-3/4})$. With a change of variable this becomes

$$\int_{b^2}^{\infty} \frac{f_1(x^{1/2}) + f_1(-x^{1/2})}{x^{5/4}\, e^{x/2}} [\mathscr{L}_{n+1}^{-1/2}(x) - \mathscr{L}_{n-1}^{-1/2}(x)]\, dx = O(n^{-3/4}). \tag{11.14}$$

Now with the remark used to derive (11.5), Lemma 14 implies that

$$\int_{b^2}^{\infty} [|f(x^{1/2})| + |f(-x^{1/2})|] x^{-1} e^{-x/2}\, dx < \infty. \tag{11.15}$$

Changing the variable and using the fact that a function with a Hermite series must be integrable on a compact set implies (2.6). This completes the proof of Theorem 4.

**12. Examples of functions.** To show some of the advantages and limitations of the theorems proved, various examples of functions will be given. First a list of the interesting characteristics of these functions will be given, then a discussion of what each function illustrates, and finally the definition of the functions and proofs that they have the asserted characteristics. These examples will only concern the Laguerre results, Theorems 1 and 3 and Corollary 3; similar examples could be given for the Hermite results.

As before, let $a_n^\alpha(f)$ denote the coefficient of $L_n^\alpha(x)$ in $f$'s Laguerre expansion with parameter $\alpha$. Functions, $f(x)$, will be produced with each of the following characteristics:

(a) Satisfies hypotheses of Theorem 1 but cannot be written as the sum of two functions, each satisfying one set of hypotheses of Theorem 9.1.5 of [8, pp. 244–245].

(b) Satisfies hypotheses of Theorem 1 but does not satisfy hypotheses of Corollary 3 for any $p$.

(c) Satisfies hypotheses of Theorem 1 but does not satisfy

$$\int_0^\infty \frac{|f(x)|\, e^{-x/2}x^{\alpha/2-1/4}(1+x)^{-1/4}}{[|n-x|+n^{1/3}]^{1/4}}\,dx = o(1). \tag{12.1}$$

(d) Violates (2.1) but satisfies the conclusion of Theorem 1.

(e) Violates (2.2) but satisfies the conclusion of Theorem 1.

(f) Violates (2.3) when $\alpha > -\frac{1}{2}$ but satisfies conclusion of Theorem 1.

(g) Violates (2.2) but for every $f_1$ satisfying $|f_1(x)| = |f(x)|$, $\lim_{n\to\infty} a_n^\alpha(f_1)L_n^\alpha(y) = 0$ for every $y$ in $(0, \infty)$.

(h) Violates (2.3) when $\alpha > -\frac{1}{2}$ but for every $f_1$ satisfying $|f_1(x)| = |f(x)|$, $\lim a_n^\alpha(f_1)L_n^\alpha(y) = 0$ for every $y$ in $(0, \infty)$.

(i) Violates (2.1) but for every $f_1$ satisfying $|f_1(x)| = |f(x)|$, equation (2.4), with $f$ replaced by $f_1$, holds for an uncountable set of $y$'s.

Functions (a) and (b) show that Theorem 1 is actually stronger than its principal competitors. Function (c) shows that an obvious amalgamation of (2.1)–(2.3) is not a substitute for them so that the statement of Theorem 1 cannot be simplified by using (12.1) in place of (2.1)–(2.3). It is easy to see that if the integral in (12.1) is taken over $[0, n] \cup [an, \infty)$ for any fixed $a > 1$, then the condition would be equivalent to (2.1)–(2.3). Requiring that the integral over $(n, an)$ also be $o(1)$, however, is an additional unneeded condition.

Examples (d)–(f) show that none of the conditions (2.1)–(2.3) is necessary for the conclusion of Theorem 1 to hold. In fact, the sum of these three functions would violate all three of those conditions but still satisfy the conclusion of Theorem 1. Consequently, there must be less restrictive conditions that will give the conclusion of Theorem 1; because of Theorem 3, however, they cannot be conditions on the absolute value of the function.

Examples (g) and (h) illustrate an essential difference between (2.1) and the other two hypotheses. The first part of the proof of Theorem 3 in § 9 showed that if for every $f_1$ satisfying $|f_1(x)| = |f(x)|$, $\lim a_n^\alpha(f_1)L_n^\alpha(y) = 0$ for a set of $y$'s of positive measure, then (2.1) holds. Examples (g) and (h) show that this condition on the terms of the series does not imply (2.2) or (2.3).

Example 9 shows that "almost every" in Theorem 3 cannot be replaced by anything stronger such as "every" or "all but a countable number of."

Now the definitions of the functions will be given. For example (a) take

$$f(x) = \begin{cases} e^{x/2}x^{-\alpha/2+1/4}(\log x)^{-2}, & 2^k - 2^{k/2} \leqq x \leqq 2^k, \quad k = 1, 2, \cdots, \\ 0, & \text{elsewhere}. \end{cases}$$

It is easy to verify (2.1)–(2.3). If $f(x)$ could be written as the sum of $g(x) + h(x)$ with $g$ satisfying the first set of Szegö's conditions in Theorem 9.1.5 and $h$ satisfying the second set, then they would satisfy the following:

$$\int_n^{2n} e^{-x/2}x^{\alpha/2}|g(x)|\, dx = o(n^{7/12}) \tag{12.2}$$

and

$$\int_n^{2n} |e^{-x/2}x^{\alpha/2}h(x)|^2\, dx = o(n^{1/2}). \tag{12.3}$$

Now if $x$ is in $[2^k - 2^{k/2}, 2^k]$, $e^{-x/2}x^{\alpha/2}|f(x)| \geqq 1$ and it is easy to show that

$$|e^{-x/2}x^{\alpha/2}f(x)| \leqq |2\, e^{-x/2}x^{\alpha/2}g(x)| + |\, e^{-x/2}x^{\alpha/2}h(x)|^2. \tag{12.4}$$

If (12.2) and (12.3) were true, then (12.4) could be used to show that

$$\int_{2^k - 2^{k/2}}^{2^k} e^{-x/2}x^{\alpha/2}|f(x)|\, dx = o(2^{7k/12}). \tag{12.5}$$

On the other hand a direct computation shows that the left side of (12.5) is greater than a positive constant times $k^{-2}2^{3k/4}$. This shows that such a decomposition is impossible.

It is easy to verify that example (a) also has the characteristics of example (b).

Example (c) is the function, $f(x)$, defined by

$$f(x) = \begin{cases} \dfrac{e^{x/2}x^{-\alpha/2+1/2}k^{-1/2}}{(x - 2^k + 2^{k/3})^{3/4}}, & 2^k \leqq x \leqq 2^k + 2^{k/2}, \quad k = 1, 2, \cdots, \\ 0, & \text{elsewhere}. \end{cases}$$

Taking $n = 2^k$, it is easy to see that $f(x)$ does not satisfy (12.1) by integrating over $[2^k, 2^k + 2^{k/2}]$. Verifying (2.1)–(2.3) is routine.

For example (d) let $I_k$ be the set of all $n$ such that $2^{k-1} \leqq 4n + 2\alpha + 2 \leqq 2^{k+1}$. Define

$$f(x) = \begin{cases} e^{x/2}x^{-\alpha/2+1/4}\operatorname{sgn}(\sin a_k x), & 2^k - 2^{k/3} \leqq x \leqq 2^k, \quad k = 1, 2, \cdots, \\ 0, & \text{elsewhere}, \end{cases}$$

where $a_k$ is chosen so large that

$$\left| \int_{2^k - 2^{k/3}}^{2^k} f(x)\, e^{-x/2}x^{\alpha/2}\mathscr{L}_n^{\alpha}(x)\, dx \right| \leqq 1 \tag{12.6}$$

for $n$ in $I_k$; this is possible by the Riemann–Lebesgue theorem for Rademacher functions. It is easy to verify for $m = 2^k$ that (2.1) is violated by $f(x)$.

To prove that this function and the functions in the next two examples satisfy the conclusion of Theorem 1, the following lemma will be used.

LEMMA 16. *Let $f(x)$ be a function that has a Laguerre series with parameter $\alpha$, let $f^b(x)$ denote the function that equals $f(x)$ for $x \geqq b$ and is 0 elsewhere and let $f_b(x) = f(x) - f^b(x)$. If*

$$\lim_{b\to\infty} \limsup_{n\to\infty} |s_n^\alpha(f^b, y)| = 0 \tag{12.7}$$

*and*

$$\lim_{b\to 0} \limsup_{n\to\infty} |s_n^\alpha(f_b, y)| = 0 \tag{12.8}$$

*and these limits are uniform for $y$ in any closed subinterval of $(0, \infty)$, then $f$ satisfies the conclusion of Theorem 1.*

First, consider a function, $f(x)$, that is 0 on $[0, 1]$. Given $\varepsilon > 0$ and a closed subinterval, $[r, s]$, of $(0, \infty)$, choose $b$ so that $b > 4s$ and

$$\limsup_{n\to\infty} |s_n^\alpha(f^b, y)| < \tfrac{1}{4}\varepsilon \tag{12.9}$$

for all $y$ in $[r, s]$; this is possible by (12.7). Using (12.9) and the fact that $f_b$ satisfies the hypotheses of Theorem 1 shows that for a fixed $\delta$ satisfying $0 < \delta < r^{1/2}$, there is an $N$ such that

$$|s_n^\alpha(f^b, y)| < \tfrac{1}{2}\varepsilon \tag{12.10}$$

and

$$\left| s_n^\alpha(f_b, y) - \frac{1}{\pi}\int_{y^{1/2}-\delta}^{y^{1/2}+\delta} f(t^2)\frac{\sin 2n^{1/2}(y^{1/2} - t)}{y^{1/2} - t}\,dt \right| < \frac{1}{2}\varepsilon \tag{12.11}$$

provided that $n > N$ and $y$ is in $[r, s]$. The fact that $s_n^\alpha(f, y) = s_n^\alpha(f_b, y) + s_n^\alpha(f^b, y)$ then shows that

$$\left| s_n^\alpha(f, y) - \frac{1}{\pi}\int_{y^{1/2}-\delta}^{y^{1/2}+\delta} f(t^2)\frac{\sin 2n^{1/2}(y^{1/2} - t)}{y^{1/2} - t}\,dt \right| < \varepsilon \tag{12.12}$$

provided that $n > N$ and $y$ is in $[r, s]$. This shows that $f$ satisfies the conclusion of Theorem 1.

If $f(x)$ is 0 on $(1, \infty)$, a similar proof can be given using (12.8) instead of (12.7). Since any function defined on $[0, \infty)$ can be written as the sum of a function that is 0 on $[0, 1]$ and one that is 0 on $(1, \infty)$, this completes the proof of the lemma.

Returning to example (d), observe that (12.8) is trivially satisfied. Now

$$s_n^\alpha(f^b, y) = e^{y/2}y^{-\alpha/2}\int_b^\infty f(x)\, e^{-x/2}x^{\alpha/2}D_n(x, y)\,dx, \tag{12.13}$$

where $D_n$ is the function defined in (5.10). If $y$ is in an interval $[r, s] \subset (0, \infty)$ and $b \geqq 2s$, it is easy to show that the part of $s_n(f^b, y)$ contributed by $j_2$ and $j_3$ is less than a constant times $b^{-1/6}$ by use of (5.2), (5.3) and absolute value inequalities. The contribution of $j_1$ is handled similarly except for the integration over intervals $[2^k - 2^{k/3}, 2^k]$ for which $n$ lies in $I_k$; these are estimated by using (12.6). The contribution of $j_1$ is found to be less than a constant times $n^{-1/6}$. Since all these parts satisfy (12.7), $f(x)$ satisfies the conclusion of Theorem 1.

For example (e) use the function

$$f(x) = \begin{cases} \dfrac{x^{-\alpha/2-1/4}\, e^{x/2}}{(\log x)(\log\log x)}, & x \geqq 3, \\ 0, & x < 3. \end{cases}$$

This clearly violates (2.2). On the other hand, (6.1) shows easily that if $y$ is in an interval $[r, s] \subset (0, \infty)$ and $b \geqq 2s$, then there is a constant, $C$, such that

$$(12.14) \qquad \left| e^{y/2} y^{-\alpha/2} \int_b^{n^{1/8}} f(x)\, e^{-x/2} x^{\alpha/2} j_2(n, x, y)\, dx \right| \leqq C n^{-1/8}.$$

To show that $f(x)$ satisfies the conclusion of Theorem 1 it is sufficient to prove (12.7) by use of (12.13). Inequalities (5.2) and (5.3) can be used to show that the contribution to $s_n^\alpha$ from $j_1$ is less than a constant times $(\log n)^{-1}$ and the contribution from $j_3$ is less than a constant times $b^{-1/3}$. For the $j_2$ part write the integral as the sum of integrals over $[b, n^{1/8}]$, $[n^{1/8}, 2v]$ and $[2v, \infty)$. On the first of these use (12.14), on the second use (5.2) and (5.3) and on the third use the fact obtained from (5.2) that $|\mathscr{L}_{n+1}^\alpha(x) - \mathscr{L}_{n-1}^\alpha(x)|$ is bounded by a constant times $e^{-Dx}$. The contribution from the $j_2$ part is then seen to be less than a constant times $(\log \log n)^{-1}$. It follows from these estimates that (12.7) is true and, thereby, that $f(x)$ satisfies the conclusion of Theorem 1.

For example (f) choose an $\alpha > -\frac{1}{2}$ and use the function

$$f(x) = \begin{cases} x^{-\alpha/2-3/4} (\log x)^{-1}, & 0 \leqq x < \frac{1}{2}, \\ 0, & \frac{1}{2} \leqq x. \end{cases}$$

It is easy to verify that (2.3) fails. If $y$ is in an interval $[r, s] \subset (0, \infty)$ and $0 < b \leqq \frac{1}{2}r$, then (5.7) shows that there is a constant, $C$, such that

$$(12.15) \qquad e^{y/2} y^{-\alpha/2} \left| \int_{n^{-1}}^{b} f(x)\, e^{-x/2} x^{\alpha/2} j_3(n, x, y)\, dx \right| \leqq \frac{C}{\log n}.$$

To show that $f(x)$ satisfies the conclusion of Theorem 1 it is sufficient to prove (12.8) by use of the analogue of (12.13) for $s_n^\alpha(f_b, y)$. The contributions from $j_1$ and $j_2$ cause no difficulty and the contribution from $j_3$ is estimated using (12.15).

For examples (g) and (h) it is easy to verify by use of (5.2) that examples (e) and (f) respectively have the desired properties.

Two lemmas will be needed to obtain the properties of example (i).

LEMMA 17. *Given a sequence,* $a_k$, *such that* $a_1 \geqq 9$ *and* $a_{k+1} \geqq (2\pi a_k)^2$, *let* $S_1 = [1, 2]$, *and for* $k > 1$, *let* $S_k$ *be the set of all* $x$ *in* $S_{k-1}$ *for which* $|\cos(a_k^{1/2}x - \alpha\pi/2 - \pi/4)| \leqq a_k^{-1/2}$. *Then* $S = \bigcap_{k=1}^\infty S_k$ *is an uncountable set.*

It will be shown that each closed interval subset of $S_{k-1}$ of length greater than or equal to $2/a_{k-1}$ contains at least two disjoint closed interval subsets of $S_k$ of length greater than or equal to $2/a_k$. This is sufficient to prove the lemma since this gives an uncountable collection of sequences of nested closed intervals with the $k$th interval of each sequence in $S_k$.

Given a closed interval subset of $S_{k-1}$ with length greater than or equal to $2/a_{k-1}$, $f_k(x) = \cos(a_k^{1/2}x - \alpha\pi/2 - \pi/4)$ has at least two full periods in this interval and, therefore, equals 1 at least twice. Between these two points where $f_k(x) = 1$ there are two points where $f_k(x) = 0$; call them $x_1$ and $x_2$. Now if $x$ is in $[x_1 - a_k^{-1}, x_1 + a_k^{-1}]$ or $[x_2 - a_k^{-1}, x_2 + a_k^{-1}]$, the law of the mean shows easily that $|f_k(x)| \leqq a_k^{-1/2}$. Since these intervals are subsets of $S_{k-1}$ they are also subsets of $S_k$. It is also clear that they are disjoint since $|x_1 - x_2| \geqq \pi a_k^{-1/2}$. This completes the proof of Lemma 17.

LEMMA 18. *Let $a_k$ satisfy the hypotheses of Lemma 17, let $S$ be the corresponding set, and let $S^*$ be the set of all $x$ such that $x^{1/2}$ is in $S$. Then there exists a constant, $C$, such that for all $x$ in $S^*$ and all $k$,*

$$|\mathscr{L}_n^\alpha(x)| \leqq Cn^{-1/4} \min\left(1, \frac{1 + |4n - a_k|}{n^{1/2}}\right).$$

Given a $k$, this follows easily from the fact that $x^{1/2}$ is in $S_k$ and (5.7).

Now for example (i) let $a_k$ be a sequence satisfying the hypotheses of Lemma 17 and define

$$f(x) = \begin{cases} e^{x/2}x^{-\alpha/2+1/4}, & a_k - a_k^{1/3} \leqq x \leqq a_k, \\ 0, & \text{elsewhere}. \end{cases}$$

It will be shown that

$$\lim_{b\to\infty} \limsup_{n\to\infty} s_n^\alpha(f_1^b, y) = 0 \tag{12.16}$$

uniformly for $y$ in the set $S^*$ described in Lemma 18 and $f_1$ satisfying $|f_1(x)| = f(x)$. That (2.4) is true uniformly for $y$ in $S^*$ and all such functions, $f_1$, can then be proved in the same way that Lemma 16 was proved.

Using (12.13) and the estimates (5.2) and (5.3) shows that the parts of $s_n^\alpha(f_1^b, y)$ resulting from $j_2$ and $j_3$ are less than a constant times $b^{-1/6}$. The term resulting from $j_1$ is bounded above by

$$e^{y/2}y^{-\alpha/2}|\mathscr{L}_n^\alpha(y)| \sum_{k=1}^{\infty} \int_{a^k - a_k^{1/3}}^{a^k} x^{1/4}|\mathscr{L}_n^\alpha(x)|\, dx. \tag{12.17}$$

To estimate (12.17) split the summation in (12.17) into the terms for which $a_k \leqq \frac{1}{2}v$, $\frac{1}{2}v \leqq a_k \leqq 2v$, and $a_k \geqq 2v$ where $v = 4n + 2 + 2$. The first and third parts are easily seen to be less than a constant times $n^{-1/6}$ by use of (5.2). The middle part consists of at most one term; using Lemma 18 and (5.2) shows that that term is bounded by a constant times

$$v^{-1/4} \min\left[1, \frac{1 + |v - a_k|}{v^{1/2}}\right] \int_{a_k - a_k^{1/3}}^{a_k} \frac{a_k^{1/4}v^{-1/4}\, dx}{[|v - x| + v^{1/3}]^{1/4}}. \tag{12.18}$$

If $|v - a_k| \geqq v^{1/2}$, (12.18) is bounded by a constant times $v^{-1/24}$. If $|v - a_k| \leqq v^{1/2}$, (12.18) is bounded by a constant times $[|a_k - v| + v^{1/3}]^{3/4}v^{-5/12}$, and this is also bounded by a constant times $v^{-1/24}$. Therefore, this part of $s_n^\alpha(f_1^b, y)$ also satisfies (12.16). This completes the proof concerning example (i).

## REFERENCES

[1] R. Askey and S. Wainger, *Mean convergences of expansions in Laguerre and Hermite series*, Amer. J. Math., 87 (1965), pp. 695–708.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi, *Higher Transcendental Functions*, vol. 2, McGraw-Hill, New York, 1953.

[3] A. Erdélyi, *Asymptotic forms for Laguerre polynomials*, J. Indian Math. Soc., 24 (1960), pp. 235–250.

[4] R. Hunt, *On the convergence of Fourier series*, Orthogonal Expansions and their Continuous Analogues, D. T. Haimo, ed., Southern Illinois University Press, Carbondale and Edwardsville, 1968, pp. 235–255.

[5] B. Muckenhoupt, *Asymptotic forms for Laguerre polynomials*, Proc. Amer. Math. Soc., 147 (1970), pp. 419–432.

[6] ———, *Mean convergence of Hermite and Laguerre series. I*, Trans. Amer. Math. Soc., 147 (1970), pp. 433–460.

[7] ———, *Mean convergence of Hermite and Laguerre series. II*, Ibid., 147 (1970), pp. 433–460.

[8] G. Szegö, *Orthogonal Polynomials*, revised ed., Colloquium Publications, vol. 23, American Mathematical Society, New York, 1959.

SIAM J. MATH. ANAL.
Vol. 1, No. 3, August 1970

# THE HANKEL TRANSFORMATION OF NEGATIVE ORDER FOR DISTRIBUTIONS OF RAPID GROWTH*

E. L. KOH†

**1. Introduction.** The Hankel transformation has been extended to orders $\mu < -\frac{1}{2}$ in such a way that an inverse transformation is defined [1]. This extension was made for certain distributions of slow growth such that for $\mu \geqq -\frac{1}{2}$, the transform coincides with that discussed in [2]. In this paper, we remove the restriction on the growth of the distributions. Specifically, we extend the Hankel transform of arbitrary order to distributions of rapid growth [3].

**2. Notation.** We shall use the notation and terminology in [1]–[3]. $I$ denotes the interval $(0, \infty)$; $x$, $y$ and $w$ are real one-dimensional variables with $x$ restricted to $I$. $\eta$ denotes a complex variable, $y + iw$, and a function of $\eta$ will be restricted to its principal branch. By a smooth function we mean one that has derivatives of all orders at every point of $I$. $\phi$ and $\Phi$ will denote testing functions while $f$ and $F$ are generalized functions. The number assigned by $f$ to some testing function $\phi$ will be denoted by $\langle f, \phi \rangle$. We use the following operators:

$$D^k = D_z^k = \frac{d^k}{dz^k}, \qquad k = 0, 1, 2, \cdots,$$

$$N_\mu = z^{\mu+1/2} D z^{-\mu-1/2},$$

$$N_\mu^{-1} = z^{\mu+1/2} \int_\infty^z t^{-\mu-1/2} \cdots dt,$$

$$M_\mu = z^{-\mu-1/2} D z^{\mu+1/2}.$$

Here, the variable $z$ may at times be complex. Other symbols will be defined as they are used.

**3. The spaces $\mathscr{B}_{\mu,b}$, $\mathscr{B}_\mu$, $\mathscr{Y}_{\mu,b}$, $\mathscr{Y}_\mu$ and their duals.** We now summarize some of the results obtained in [3].

Let $\mu$ be a fixed number in $(-\infty, \infty)$ and $b$ be some positive number. $\mathscr{B}_{\mu,b}$ is the topological vector space of all smooth, complex-valued functions $\phi(x)$ on $0 < x < \infty$ such that $\phi(x) \equiv 0$ on $b < x < \infty$ and for every nonnegative integer, $k$,

$$\gamma_k^\mu(\phi) \triangleq \sup_{0 < x < \infty} |(x^{-1}D)^k x^{-\mu-1/2} \phi(x)| < \infty.$$

The countable set of seminorms $\{\gamma_k^\mu\}_{k=0}^\infty$ generates the topology for $\mathscr{B}_{\mu,b}$. It was shown in [3] that $\mathscr{B}_{\mu,b}$ is a Hausdorff, locally convex, first countable, complete, countably normed space (see [4, p. 6]). Also for $b < c$, $\mathscr{B}_{\mu,b} \subset \mathscr{B}_{\mu,c}$ and the topology induced on $\mathscr{B}_{\mu,b}$ by $\mathscr{B}_{\mu,c}$ is identical to the topology of $\mathscr{B}_{\mu,b}$.

* Received by the editors July 10, 1969, and in revised form December 24, 1969. This work was supported in part by the National Research Council of Canada under grant number A7184.

† Department of Mathematics, University of Saskatchewan, Regina Campus, Regina, Saskatchewan.

$\mathscr{B}_\mu$ is then defined to be the strict inductive limit (see [4, p. 21]) of the $\mathscr{B}_{\mu,b}$, where $b$ traverses a monotonically increasing sequence of positive numbers tending to $\infty$. Its dual $\mathscr{B}'_\mu$ is the space of generalized functions (distributions) on which the Hankel transform of arbitrary order is to be extended. These generalized functions are of arbitrary growth as $x \to \infty$. Both $\mathscr{B}_\mu$ and $\mathscr{B}'_\mu$ are complete.

Similarly, for the given $\mu$ and $b$, $\mathscr{Y}_{\mu,b}$ is defined to be the space of functions $\Phi(\eta)$ such that $\eta^{-\mu-1/2}\Phi(\eta)$ is an even entire function of $\eta$ and for $k = 0, 1, 2, \cdots$,

$$\alpha_{b,k}^{\mu}(\Phi) = \sup_{\eta} |e^{-b|w|}\eta^{2k-\mu-1/2}\Phi(\eta)| < \infty,$$

where $\eta = y + iw$ and the supremum is taken over the entire $\eta$-plane. Again, $\mathscr{Y}_{\mu,b}$ holds the same properties listed above for $\mathscr{B}_{\mu,b}$.

$\mathscr{Y}_\mu$ is the strict inductive limit of the $\mathscr{Y}_{\mu,b}$, and $\mathscr{Y}'_\mu$ is the dual space of $\mathscr{Y}_\mu$. These spaces are again complete.

**4. The Hankel transformation on $\mathscr{B}'_\mu$ for $\mu \geqq -\frac{1}{2}$.** By an application of Griffith's theorem [5], it was shown that for $\mu \geqq -\frac{1}{2}$, the Hankel transformation $h_\mu$ is an isomorphism from $\mathscr{B}_{\mu,b}$ onto $\mathscr{Y}_{\mu,b}$ (see [3, Theorem 1]) and from $\mathscr{B}_\mu$ onto $\mathscr{Y}_\mu$ (see [3, Theorem 2]). Consequently, the generalized Hankel transformation for $\mu \geqq -\frac{1}{2}$ was defined on $\mathscr{B}'_\mu$ as the adjoint of $h_\mu$ on $\mathscr{Y}_\mu$. Specifically, if $\mu \geqq -\frac{1}{2}$, $f \in \mathscr{B}'_\mu$, $\phi \in \mathscr{B}_\mu$ and $\Phi = h_\mu\phi \in \mathscr{Y}_\mu$, then $h'_\mu f$ is defined as a functional on $\mathscr{Y}_\mu$ by

$$\langle h'_\mu f, \Phi\rangle \triangleq \langle f, \phi\rangle. \tag{1}$$

Under this definition, $h'_\mu$ is an isomorphism from $\mathscr{B}'_\mu$ onto $\mathscr{Y}'_\mu$, and its inverse is itself, i.e., $h'_\mu = (h'_\mu)^{-1}$.

**5. The Hankel transformation on $\mathscr{B}'_\mu$ for arbitrary $\mu$.** We now develop the Hankel transformation of arbitrary order for generalized functions in $\mathscr{B}'_\mu$. But first we prove certain operations on $\mathscr{B}_{\mu,b}$ and $\mathscr{Y}_{\mu,b}$.

LEMMA 1. *For any real value of $\mu$, the mapping $\phi \to N_\mu\phi$ is an isomorphism from $\mathscr{B}_{\mu,b}$ onto $\mathscr{B}_{\mu+1,b}$, the inverse mapping being $\phi \to N_\mu^{-1}\phi$.*

*Proof.* It has been shown [3, Lemma 3] that $\phi \to N_\mu\phi$ is a continuous linear mapping of $\mathscr{B}_{\mu,b}$ into $\mathscr{B}_{\mu+1,b}$. Since $N_\mu^{-1}$ is clearly linear and is the inverse of $N_\mu$, we only need to prove that $N_\mu^{-1}$ maps $\mathscr{B}_{\mu+1,b}$ continuously into $\mathscr{B}_{\mu,b}$. By a proof analogous to that of Lemma 2 of [1], we have, for $\phi(x) \in \mathscr{B}_{\mu+1,b}$,

$$\gamma_k^{\mu}(N_\mu^{-1}\phi) = \gamma_{k-1}^{\mu+1}(\phi), \qquad k = 1, 2, 3, \cdots. \tag{2}$$

For the case $k = 0$, we have

$$\gamma_0^{\mu}(N_\mu^{-1}\phi(x)) \leqq \frac{b^2}{2}\gamma_0^{\mu+1}(\phi). \tag{3}$$

The results (2) and (3) prove that $\phi \to N_\mu^{-1}\phi$ is a continuous linear mapping of $\mathscr{B}_{\mu+1,b}$ into $\mathscr{B}_{\mu,b}$. It follows that $\phi \to N_\mu\phi$ is one-to-one and onto. This completes the proof.

LEMMA 2. *For any integer $m$ and for any real value of $\mu$, the mapping $\Phi \to \eta^m\Phi$ is an isomorphism from $\mathscr{Y}_{\mu,b}$ onto $\mathscr{Y}_{\mu+m,b}$.*

*Proof.* This follows from the fact that $\alpha_{b,k}^{\mu+m}(\eta^m\Phi) = \alpha_{b,k}^{\mu}(\Phi)$ for every integer $m$.

In the sequel, we shall use the following operators. Let $\mu$ be a fixed real number and let $k$ be any positive integer such that $k \geqq -\mu - \frac{1}{2}$. Then for any $\phi(x) \in \mathscr{B}_{\mu,b}$ we set

(4) $$\Phi(\eta) = \hbar_{\mu,k}[\phi(x)] \triangleq (-1)^k \eta^{-k} \hbar_{\mu+k} N_{\mu+k-1} \cdots N_{\mu+1} N_\mu \phi(x),$$

and for any $\Phi(\eta) \in \mathscr{Y}_{\mu,b}$ we set

(5) $$\phi(x) = \hbar_{\mu,k}^{-1}[\Phi(\eta)] \triangleq (-1)^k N_\mu^{-1} N_{\mu+1}^{-1} \cdots N_{\mu+k-1}^{-1} \hbar_{\mu+k} \eta^k \Phi(\eta),$$

where $\hbar_{\mu+k}$ is the classical Hankel transformation given by

(6) $$(\hbar_\mu \psi)(y) = \int_0^\infty \psi(z) \sqrt{zy} J_\mu(zy)\, dz.$$

LEMMA 3. *For any real* $\mu$, *the transformation* $\hbar_{\mu,k}$ *as defined by* (4) *is an isomorphism from* $\mathscr{B}_{\mu,b}$ *onto* $\mathscr{Y}_{\mu,b}$. *Its inverse is defined by* (5). *Whenever* $\mu \geqq -\frac{1}{2}$, $\hbar_{\mu,k}$ *coincides with* $\hbar_\mu$ *as an isomorphism defined by* (6) *from* $\mathscr{B}_{\mu,b}$ *onto* $\mathscr{Y}_{\mu,b}$.

*Proof.* The first assertion follows from the facts that $\phi \to N_{\mu+k-1} \cdots N_{\mu+1} N_\mu \phi$ is an isomorphism from $\mathscr{B}_{\mu,b}$ onto $\mathscr{B}_{\mu+k,b}$ (Lemma 1), $\phi \to \hbar_{\mu+k}\phi$ is an isomorphism from $\mathscr{B}_{\mu+k,b}$ onto $\mathscr{Y}_{\mu+k,b}$ for $\mu + k \geqq -\frac{1}{2}$ (Theorem 1 of [3]), and $\Phi \to \eta^{-k}\Phi$ is an isomorphism from $\mathscr{Y}_{\mu+k,b}$ onto $\mathscr{Y}_{\mu,b}$ (Lemma 2).

The second assertion follows from Lemmas 1 and 2 and the fact that $\hbar_{\mu+k}^{-1} = \hbar_{\mu+k}$ on $\mathscr{Y}_{\mu+k,b}$. Thus, for $\mu + k \geqq -\frac{1}{2}$, $\hbar_{\mu+k}\eta^k\Phi(\eta) = \hbar_{\mu+k}^{-1}\eta^k\Phi(\eta) \in \mathscr{B}_{\mu+k,b}$ by Theorem 1 of [3].

To prove the last assertion, we let $\phi(x) \in \mathscr{B}_{\mu,b}$. Consider

$$-\eta^{-1}(\hbar_{\mu+1}N_\mu\phi)(\eta) = -\eta^{-1}\int_0^\infty x^{\mu+1/2}[Dx^{-\mu-1/2}\phi(x)]\sqrt{\eta x}J_{\mu+1}(\eta x)\, dx.$$

By an integration by parts we obtain

$$-\eta^{-1}(\hbar_{\mu+1}N_\mu\phi)(\eta) = -\eta^{-1}\left\{\phi(x)\sqrt{\eta x}J_{\mu+1}(\eta x)\Big|_0^\infty - \eta\int_0^\infty \sqrt{\eta x}J_\mu(\eta x)\phi(x)\, dx\right\}.$$

Since $\phi(x) = 0$ for $b \leqq x < \infty$ and $\phi(x)\sqrt{\eta x}J_{\mu+1}(\eta x) = O(x^{2\mu+2})$ as $x \to 0^+$, the limit terms vanish. Therefore $(\hbar_\mu\phi)(\eta) = -\eta^{-1}(\hbar_{\mu+1}N_\mu\phi)(\eta)$. By induction on $k$, we have

(7) $$(\hbar_\mu\phi)(\eta) = (-1)^k\eta^{-k}(\hbar_{\mu+k}N_{\mu+k-1}\cdots N_{\mu+1}N_\mu\phi)(\eta).$$

The right-hand side of (7) is $(\hbar_{\mu,k}\phi)(\eta)$. This completes the proof.

Equation (7) also implies the following corollary.

COROLLARY 1. *For any positive integers* $k$ *and* $p$ *both greater than* $-\mu - \frac{1}{2}$, $\hbar_{\mu,k} = \hbar_{\mu,p}$.

Since $\mathscr{B}_\mu$ and $\mathscr{Y}_\mu$ are strict inductive limits of the $\mathscr{B}_{\mu,b}$ and $\mathscr{Y}_{\mu,b}$ respectively, we deduce the following theorem from Lemma 3 and the properties of continuous linear operators on inductive limits [4, pp. 20–24].

THEOREM 1. *For any real value of* $\mu$, $\hbar_{\mu,k}$ *is an isomorphism from* $\mathscr{B}_\mu$ *onto* $\mathscr{Y}_\mu$. *Whenever* $\mu \geqq -\frac{1}{2}$, $\hbar_{\mu,k}$ *coincides with* $\hbar_\mu$ *as an isomorphism from* $\mathscr{B}_\mu$ *onto* $\mathscr{Y}_\mu$.

We now define the Hankel transformation of arbitrary order on $\mathscr{B}'_\mu$ as the adjoint to $\hbar_{\mu,k}$ on $\mathscr{B}_\mu$. Let $\mu$ be any real number. Let $k$ be any positive integer $\geqq -\mu - \frac{1}{2}$. For $f \in \mathscr{B}'_\mu$ and $\phi \in \mathscr{B}_\mu$ we define the Hankel transform $\hbar'_\mu f$ by

(8) $$\langle \hbar'_\mu f, \hbar_{\mu,k}\phi \rangle = \langle f, \phi \rangle.$$

In view of Theorem 1, we can state the following theorem.

THEOREM 2. *For any real value of* $\mu$, $h'_\mu$ *is an isomorphism from* $\mathscr{B}'_\mu$ *onto* $\mathscr{Y}'_\mu$.

When $\mu \geqq -\frac{1}{2}$, the definition (8) coincides with that in [3]. Note that our definition differs from that in [1] where the Hankel transformation on $H'_\mu$ is defined by

$$\langle h'_\mu f, \Phi\rangle = \langle f, h_{\mu,k}\Phi\rangle \tag{9}$$

for $f \in H'_\mu$ and $\Phi \in H_\mu$ (see [1, (13)]). This definition is possible there because $h_{\mu,k}$ is an automorphism on $H_\mu$. In the present work, we have seen that $h_{\mu,k}$ carries an element of $\mathscr{B}_\mu$ onto the space $\mathscr{Y}_\mu$. Thus (9) will not be valid for $f \in \mathscr{B}'_\mu$ whereas (8) is; the right-hand side of (8) defines the Hankel transform $h'_\mu f$ of any $f \in \mathscr{B}'_\mu$ as a functional on $\mathscr{Y}_\mu$.

**6. Some operational formulas.** We now establish certain transformation formulas relating to the Bessel-type differential operator $M_\mu N_\mu$. Since $\mathscr{B}_\mu$ is a subspace of $H_\mu$ (see [3, Lemma 6]), we have the following immediate result from [1, Lemma 5].

LEMMA 4. *Let* $\mu$ *be any fixed real number and k a positive integer* $\geqq -\mu - \frac{1}{2}$. *Then, for every* $\phi \in \mathscr{B}_\mu$,

$$M_\mu N_\mu h_{\mu,k}\phi = h_{\mu,k}(-x^2\phi). \tag{10}$$

We shall also need the following lemmas.

LEMMA 5. *Under the hypothesis of Lemma* 4,

$$h_{\mu+1,k}(N_\mu\phi) = -\eta h_{\mu,k}(\phi). \tag{11}$$

*Proof*. For $\phi \in \mathscr{B}_\mu$, it is clear from Lemmas 1 to 3 and Theorem 1 that both sides of (11) have a sense and belong to $\mathscr{Y}_{\mu+1}$. Equality follows from the definition of $h_{\mu,k}$ and Corollary 1 for

$$\begin{aligned} h_{\mu+1,k}(N_\mu\phi) &= (-1)^k\eta^{-k}h_{\mu+1+k}N_{\mu+k}\cdots N_{\mu+1}(N_\mu\phi) \\ &= -\eta h_{\mu,k+1}(\phi) = -\eta h_{\mu,k}(\phi). \end{aligned}$$

LEMMA 6. *Let* $\mu$ *and k be as in Lemma* 4. *Then for every* $\phi \in \mathscr{B}_{\mu+1}$,

$$h_{\mu,k}(M_\mu\phi) = \eta h_{\mu+1,k}(\phi). \tag{12}$$

*Proof*. By Lemmas 4 and 14 of [3], it follows that both sides of (12) are in $\mathscr{Y}_\mu$. Using the relation

$$N_{\mu+k-1}\cdots N_{\mu+1}N_\mu\phi(x) = x^{\mu+k+1/2}(x^{-1}D)^k x^{-\mu-1/2}\phi(x) \tag{13}$$

we have

$$\begin{aligned} h_{\mu,k}(M_\mu\phi) &= (-1)^k\eta^{-k}\int_0^\infty \sqrt{\eta x}J_{\mu+k}(\eta x)x^{\mu+k+1/2}x^2(x^{-1}D)^{k+1}x^{-\mu-1-1/2}\phi\, dx \\ &\quad +(-1)^k\eta^{-k}(2\mu + 2k + 2)\int_0^\infty \sqrt{\eta x}J_{\mu+k}(\eta x) \\ &\quad \cdot x^{\mu+k+1/2}(x^{-1}D)^k x^{-\mu-1-1/2}\phi\, dx. \end{aligned} \tag{14}$$

We now show that $\eta h_{\mu+1,k}(\phi)$ reduces to (14). Indeed,

$$\eta h_{\mu+1,k}(\phi) = (-1)^k \eta^{-k+1} \int_0^\infty \sqrt{\eta x} J_{\mu+k+1}(\eta x) x^{\mu+k+1+1/2} (x^{-1}D)^k x^{-\mu-1-1/2} \phi \, dy.$$

From the formula [6, (51), p. 11]

$$J_{\mu+k+1}(\eta x) = -\eta^{-1} x^{\mu+k} D x^{-\mu-k} J_{\mu+k}(\eta x)$$

and an integration by parts, we obtain

$$\begin{aligned}\eta h_{\mu+1,k}(\phi) &= (-1)^{k+1} \eta^{-k+1/2} \int_0^\infty x^{2\mu+2k+2} (x^{-1}D)^k x^{-\mu-1-1/2} \phi \\ &\qquad \cdot D[x^{-\mu-k} J_{\mu+k}(\eta x)] \, dx \\ &= (-1)^{k+1} \eta^{-k+1/2} \Big\{ x^{\mu+k+2} J_{\mu+k}(\eta x) (x^{-1}D)^k x^{\mu-1-1/2} \phi \Big|_0^\infty \\ &\qquad - \int_0^\infty x^{-\mu-k} J_{\mu+k}(\eta x) D[x^{2\mu+2k+2} (x^{-1}D)^k \\ &\qquad \cdot x^{-\mu-1-1/2} \phi] \, dx \Big\}.\end{aligned}$$

The limit terms vanish because $\phi \equiv 0$ on $b \leqq x < \infty$ for some $b$, $\gamma_k^{\mu+1}(\phi) < \infty$, and as $x \to 0^+$, $x^{\mu+k+2} J_{\mu+k}(\eta x) = O(x)$ for $\mu + k \geqq -\frac{1}{2}$. Since $D[x^{2\mu+2k+2}(x^{-1}D)^k \cdot x^{-\mu-1-1/2}\phi] = x^{2\mu+2k+3}(x^{-1}D)^{k+1} x^{-\mu-1-1/2}\phi + (2\mu + 2k + 2) x^{2\mu+2k+1} \cdot (x^{-1}D)^k x^{-\mu-1-1/2}\phi$, we see that $\eta h_{\mu+1,k}(\phi)$ equals the right-hand side of (14). This completes the proof.

Lemma 5 and 6 can be combined to yield the following lemma.

LEMMA 7. *Let $\mu$ be any fixed real number and $k$ a positive integer $\geqq -\mu - \frac{1}{2}$. Then, for every $\phi \in \mathscr{B}_\mu$,*

$$(15) \qquad h_{\mu,k}(M_\mu N_\mu \phi) = -\eta^2 h_{\mu,k}(\phi).$$

As usual, we define the operators $N_\mu, M_\mu$ and multiplication-by-$x$ on $\mathscr{B}'_\mu$ (or $\mathscr{Y}'_\mu$) as the respective adjoints of the operators $-N_\mu$, $-M_\mu$ and multiplication-by-$x$ on $\mathscr{B}_\mu$ (or $\mathscr{Y}_\mu$). As continuous linear maps on $\mathscr{B}'_\mu$ and $\mathscr{Y}'_\mu$, we have obvious analogues to Lemmas 8 to 10 and 21 to 23 of [3]. Since the proofs are identical we shall omit them. But we shall use these lemmas with the tacit understanding that the order $\mu$ can be any real number. Thus, in the next theorem, the operation $M_\mu N_\mu$ is a continuous linear mapping of $\mathscr{B}'_\mu$ (or $\mathscr{Y}'_\mu$) into itself. Similarly, the operation $F \to x^2 F$ is a continuous linear mapping from $\mathscr{B}'_\mu$ (or $\mathscr{Y}'_\mu$) into $\mathscr{B}'_{\mu-2}$ (or $\mathscr{Y}'_{\mu-2}$) and therefore into itself. (See [3, Lemmas 8 and 20].)

THEOREM 3. *For any real $\mu$ and $f \in \mathscr{B}'_\mu$,*

$$(16) \qquad M_\mu N_\mu h'_\mu f = h'_\mu[-x^2 f]$$

*and*

$$(17) \qquad h'_\mu M_\mu N_\mu f = -\eta^2 h'_\mu f.$$

*Proof*. In view of Theorem 2 and our comments in the previous paragraph, all terms in (16) and (17) can be identified as functionals belonging to $\mathscr{Y}'_\mu$. Let $\mu$

be any real number and $k$ a positive integer $\geqq -\mu - \frac{1}{2}$. Let $\phi \in \mathscr{B}_\mu$. Then, by the definitions of $h'_\mu$, multiplication-by-$x$ and $M_\mu N_\mu$ together with Lemma 4, we have

$$\begin{aligned}\langle h'_\mu(-x^2 f), h_{\mu,k}(\phi)\rangle &= \langle -x^2 f, \phi\rangle = \langle f, -x^2\phi\rangle \\ &= \langle h'_\mu f, h_{\mu,k}(-x^2\phi)\rangle = \langle h'_\mu f, M_\mu N_\mu h_{\mu,k}(\phi)\rangle \\ &= \langle M_\mu N_\mu h'_\mu f, h_{\mu,k}(\phi)\rangle.\end{aligned}$$

This proves (16). Analogously, we prove (17) by invoking Lemma 7. Thus

$$\begin{aligned}\langle h'_\mu M_\mu N_\mu f, h_{\mu,k}(\phi)\rangle &= \langle M_\mu N_\mu f, \phi\rangle = \langle f, M_\mu N_\mu \phi\rangle \\ &= \langle h'_\mu f, h_{\mu,k} M_\mu N_\mu \phi\rangle = \langle h'_\mu f, -\eta^2 h_{\mu,k}(\phi)\rangle \\ &= \langle -\eta^2 h'_\mu f, h_{\mu,k}(\phi)\rangle.\end{aligned}$$

This completes the proof.

Equation (17) can be applied to the differential equation

$$P(M_\mu N_\mu) f = 0 \tag{18}$$

which was solved for $f$ in [3] provided $\mu \geqq -\frac{1}{2}$. Here, the roots of the polynomial $P(-\eta^2)$ are assumed to be $\eta = \gamma_n \neq 0$ with the multiplicities $k_n (n = \pm 1, \cdots, \pm q$, $\gamma_n = \gamma_{-n}, k_n = k_{-n})$. If $\mu$ is allowed to be any real number, a similar computation as in [3] shows that a solution to (18) is $f \in \mathscr{B}'_\mu$ given by

$$\langle f, \phi\rangle = \sum_{n=1}^{q} \sum_{\nu=0}^{k_n - 1} b_{n\nu} \frac{\partial^\nu}{\partial \gamma_n^\nu} \Phi \bigg|_{\eta = \gamma_n}, \tag{19}$$

where $\Phi$ is the image of $\phi$ in $\mathscr{Y}_\mu$ under the $h_{\mu,k}$ transformation. Equation (19) reduces to the solution obtained in [3] whenever $\mu \geqq -\frac{1}{2}$.

**Acknowledgment.** The author wishes to thank one of the referees for his suggested improvements in the proofs of Lemmas 3 and 5.

REFERENCES

[1] A. H. ZEMANIAN, *Hankel transforms of arbitrary order*, Duke Math. J., 34 (1967), pp. 761–770.

[2] ———, *A distributional Hankel transformation*, SIAM J. Appl. Math., 14 (1966), pp. 561–576.

[3] ———, *The Hankel transformation of certain distributions of rapid growth*, Ibid., 14 (1966), pp. 678–690.

[4] A. FRIEDMAN, *Generalized Functions and Partial Differential Equations*, Prentice-Hall, Englewood Cliffs, N.J., 1963.

[5] J. L. GRIFFITH, *Hankel transforms of functions zero outside a finite interval*, J. Proc. Roy. Soc. New South Wales, 89 (1955), pp. 109–115.

[6] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Higher Transcendental Functions*, vol. II, McGraw-Hill, New York, 1953.

SIAM J. MATH. ANAL.
Vol. 1, No. 3, August 1970

# NOTE ON CERTAIN TRIANGULAR ARRAYS*

L. CARLITZ†

**1.** H. W. Gould [1] has introduced two triangular arrays $A_k^r$, $G_k^r$ in the following way. The array $A_k^r$ may be defined by

$$2^{[(r+3)/2]} \sum_{k=0}^{n} k^{2r+1} = \sum_{j=0}^{r} A_j^r n^{2r-2j+1} \sum_{k=0}^{n} k^{2j}. \tag{1.1}$$

He showed that

$$A_r^{r+k} = \begin{cases} \binom{2r+2k+1}{2k+1} 2^{[r/2]} A_0^k, & k \text{ odd}, \\ \binom{2r+2k+1}{2k+1} 2^{[(r+1)/2]} A_0^k, & k \text{ even}. \end{cases} \tag{1.2}$$

Thus it remains to determine $A_0^k$. The array $G_k^r$ may be defined by

$$\sum_{j=i}^{r} G_j^r A_i^j 2^{-[(j+3)/2]} = \begin{cases} 0, & i \neq r, \\ \dfrac{(2r+1)!}{r!2^r}, & i = r, \end{cases} \tag{1.3}$$

or alternatively

$$\sum_{j=i}^{r} A_j^r G_i^j \frac{j!2^j}{(2j+1)!} = \begin{cases} 0, & i \neq r, \\ 2^{[(r+3)/2]}, & i = r. \end{cases} \tag{1.4}$$

The object of the present note is to show that

$$A_0^r = -2^{[(r+1)/2]-2r} C_{2r+1} = 2^{[(r+3)/2]}(2^{2r+2} - 1)\frac{B_{2r+2}}{r+1}, \tag{1.5}$$

where the $B_n$ are the Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!},$$

while $C_n$ is defined by either of (see [2, p. 28])

$$\frac{2}{e^x + 1} = \sum_{n=0}^{\infty} C_n \frac{2^{-n}x^n}{n!}, \qquad C_n = 2^{n+1}(1 - 2^{n+1})\frac{B_{n+1}}{n+1}.$$

As for $G_k^r$ we show that

$$G_k^r = 2 \cdot 1 \cdot 3 \cdot 5 \cdots (2r-1)\binom{2r+1}{2k+1} B_{2r-2k}, \qquad 0 \leqq k \leqq r, \tag{1.6}$$

---

* Received by the editors October 28, 1969.

† Department of Mathematics, Duke University, Durham, North Carolina 27706. This work was supported in part by the National Science Foundation under Grant GP-7855.

and in particular

$$G_0^r = 2 \cdot 1 \cdot 3 \cdot 5 \cdots (2r+1) B_{2r}. \tag{1.7}$$

**2.** Since

$$\sum_{k=0}^{n} k^r = \frac{1}{r+1}[B_{r+1}(n+1) - B_{r+1}],$$

where

$$B_r(x) = \sum_{s=0}^{r} \binom{r}{s} B_{r-s} x^s, \tag{2.1}$$

we may replace (1.1) by

$$\frac{2^{[(r+3)/2]}}{2r+2}[B_{2r+2}(n+1) - B_{2r+2}] = \sum_{j=0}^{r} A_j^r n^{2r-2j+1} \frac{B_{2j+1}(n+1) - B_{2j+1}}{2j+1}. \tag{2.2}$$

Since (2.2) holds for $n = 0, 1, 2, \cdots$, it is an identity in $n$. We substitute from (2.1) in (2.2), divide by $n+1$ and then put $n = -1$. The result is

$$2^{[(r+3)/2]} B_{2r+1} = -\sum_{j=0}^{r} A_j^r B_{2j}. \tag{2.3}$$

Since

$$B_{2r+1} = \begin{cases} -\frac{1}{2}, & r = 0, \\ 0, & r > 0, \end{cases}$$

(2.3) reduces to

$$\sum_{j=0}^{r} A_j^r B_{2j} = \begin{cases} 1, & r = 0, \\ 0, & r > 0. \end{cases} \tag{2.4}$$

We now make use of (1.2). It will be convenient to put

$$A_0^r = 2^{[(r+1)/2]} \bar{A}_0^r. \tag{2.5}$$

Thus (1.2) becomes

$$A_r^{r+k} = \begin{cases} \binom{2r+2k+1}{2r} 2^{[r/2]+(k+1)/2} \bar{A}_0^k, & k \text{ odd}, \\ \binom{2r+2k+1}{2r} 2^{[(r+1)/2]+k/2} \bar{A}_0^k, & k \text{ even}. \end{cases}$$

Since

$$[r/2] + (k+1)/2 = [(r+k+1)/2], \qquad k \text{ odd},$$

$$[(r+1)/2] + k/2 = [(r+k+1)/2], \qquad k \text{ even},$$

it is clear that

$$A_r^{r+k} = \binom{2r+2k+1}{2r} 2^{[(r+k+1)/2]} \bar{A}_0^k. \tag{2.6}$$

Combining (2.4) and (2.6), we have

$$\sum_{j=0}^{r}\binom{2r+1}{2j}\bar{A}_0^{r-j}B_{2j}=\begin{cases}1, & r=0,\\ 0, & r>0.\end{cases} \tag{2.7}$$

Now multiply both sides of (2.7) by $x^{2r+1}/(2r+1)!$ and sum over $r$. We find that

$$\sum_{j=0}^{\infty}B_{2j}\frac{x^{2j}}{(2j)!}\sum_{r=0}^{\infty}\bar{A}_0^r\frac{x^{2r+1}}{(2r+1)!}=x.$$

Since $\sum_{n=0}^{\infty}B_n(x^n/n!)=x/(e^x-1)$, it follows that

$$\sum_{j=0}^{\infty}B_{2j}\frac{x^{2j}}{(2j)!}=\frac{x}{e^x-1}+\frac{x}{2}=\frac{x}{2}\frac{e^x+1}{e^x-1}.$$

Therefore

$$\sum_{r=0}^{\infty}\bar{A}_0^r\frac{x^{2r+1}}{(2r+1)!}=\frac{2(e^x-1)}{e^x+1}. \tag{2.8}$$

Comparing (2.8) with

$$\sum_{n=0}^{\infty}C_n\frac{2^{-n}x^n}{n!}=\frac{2}{e^x+1}, \tag{2.9}$$

it follows at once that $\bar{A}_0^r=-2^{-2r}C_{2r+1}$. Thus

$$A_0^r=-2^{[(r+1)/2]-2r}C_{2r+1} \tag{2.10}$$

and, by (2.6),

$$A_j^r=-\binom{2r+1}{2j}2^{[(r+1)/2]-2r+2j}C_{2r-2j+1}. \tag{2.11}$$

For example, (2.11) gives

$$A_r^r=(2r+1)2^{[(r+1)/2]}. \tag{2.12}$$

**3.** Returning to (1.1), we evidently have

$$2^{[(r+3)/2]}\left\{n^{2r+1}+\frac{1}{2r+2}[B_{2r+2}(n)-B_{2r+2}]\right\}$$
$$=\sum_{j=0}^{r}A_j^r n^{2r-2j+1}\left\{n^{2j}+\frac{1}{2j+1}[B_{2j+1}(n)-B_{2j+1}]\right\}.$$

Equating coefficients of $n^{2r-s+2}$, we get, for $0\leqq s\leqq 2r+1$,

$$\frac{2^{[(r+3)/2]}}{2r+2}\binom{2r+2}{s}B_s=\sum_{j=0}^{r}\frac{1}{2j+1}\binom{2j+1}{s}A_j^rB_s.$$

Replacing $s$ by $2s+2$, we have

$$\sum_{j=0}^{r}\frac{1}{2j+1}\binom{2j+1}{2s+2}A_j^r=\frac{2^{[(r+3)/2]}}{2r+2}\binom{2r+2}{2s+2},\qquad 0\leqq s<r. \tag{3.1}$$

Now put

$$A_j^r=\binom{2r+1}{2j}2^{[(r+1)/2]}\bar{A}_j^r. \tag{3.2}$$

Then (3.1) becomes

$$\sum_{j=s+1}^{r} \binom{2r-2s}{2j-2s-1} \bar{A}_j^r = 2$$

so that

$$\sum_{j=0}^{s-1} \binom{2s}{2j+1} \bar{A}_{r-j}^r = 2, \qquad 0 < s \leqq r. \tag{3.3}$$

We may rewrite (3.3) in the form

$$\sum_{j=0}^{s-1} \binom{2s}{2j+1} \bar{A}_{r+s-j}^{r+s} = 2, \qquad s = 1, 2, 3, \cdots. \tag{3.4}$$

It follows from (3.4) by an easy induction that $\bar{A}_{r-j}^r = \bar{A}_0^j$, so that (3.2) becomes

$$A_j^r = \binom{2r+1}{2j} 2^{[(r+1)/2]} \bar{A}_0^{r-j} \tag{3.5}$$

in agreement with (2.6). We have therefore proved (1.2).

**4.** Turning now to $G_i^r$, it is clear from (1.4) and (2.11) that

$$- \sum_{j=i}^{r} \binom{2r+1}{2j} 2^{[(r+1)/2]-2r+2j} C_{2r-2j+1} G_i^j \frac{j!2^j}{(2j+1)!} = \begin{cases} 0, & i \neq r, \\ 2^{[(r+3)/2]}, & i = r. \end{cases} \tag{4.1}$$

It is convenient to put

$$\bar{G}_i^j = G_i^j \frac{j!2^j}{(2j+1)!}. \tag{4.2}$$

Thus (4.1) becomes

$$- \sum_{j=i}^{r} \binom{2r+1}{2j} 2^{-(2r-2j+1)} C_{2r-2j+1} \bar{G}_i^j = \begin{cases} 0, & i \neq r, \\ 1, & i = r. \end{cases} \tag{4.3}$$

Now multiply by $x^{2r+1}/(2r+1)!$ and sum over $r$. We get

$$\begin{aligned} \frac{x^{2i+1}}{(2i+1)!} &= - \sum_{r=i}^{\infty} \frac{x^{2r+1}}{(2r+1)!} \sum_{j=1}^{r} \binom{2r+1}{2j} 2^{-(2r-2j+1)} C_{2r-2j+1} \bar{G}_i^j \\ &= - \sum_{j=i}^{\infty} \frac{x^{2j}}{(2j)!} \bar{G}_i^j \sum_{r=j}^{\infty} 2^{-(2r-2j+1)} C_{2r-2j+1} \frac{x^{2r-2j+1}}{(2r-2j+1)!} \\ &= - \sum_{j=i}^{\infty} \frac{x^{2j}}{(2j)!} \bar{G}_i^j \sum_{r=0}^{\infty} 2^{-(2r+1)} C_{2r+1} \frac{x^{2r+1}}{(2r+1)!}. \end{aligned}$$

By (2.9) we have $2/(e^x + 1) = 1 + \sum_{r=0}^{\infty} 2^{-(2r+1)} C_{2r+1} [x^{2r+1}/(2r+1)!]$, so that $- \sum_{r=0}^{\infty} 2^{-2r+1} C_{2r+1} [x^{2r+1}/(2r+1)!] = (e^x - 1)/(e^x + 1)$. Therefore

$$\sum_{j=i}^{\infty} \bar{G}_i^j \frac{x^{2j}}{(2j)!} = \frac{x^{2i+1}}{(2i+1)!} \frac{e^x + 1}{e^x - 1}. \tag{4.4}$$

Since

$$\begin{aligned}
\frac{x^{2i+1}}{(2i+1)!}\frac{e^x+1}{e^x-1} &= \frac{x^{2i+1}}{(2i+1)!} + \frac{2x^{2i}}{(2i+1)!}\frac{x}{e^x-1} \\
&= \frac{x^{2i+1}}{(2i+1)!} + \frac{2x^{2i}}{(2i+1)!}\sum_{n=0}^{\infty} B_n\frac{x^n}{n!} \\
&= \frac{x^{2i+1}}{(2i+1)!} + \frac{2}{(2i+1)!}\sum_{n=2i}^{\infty} B_{n-2i}\frac{x^n}{(n-2i)!} \\
&= \frac{x^{2i+1}}{(2i+1)!} + \frac{2}{2i+1}\sum_{n=2i}^{\infty}\binom{n}{2i}B_{n-2i}\frac{x^n}{n!} \\
&= \frac{2}{2i+1}\sum_{j=1}^{\infty}\binom{2j}{2i}B_{2j-2i}\frac{x^{2j}}{(2j)!}.
\end{aligned}$$

Thus (4.4) yields

$$\bar{G}_i^j = \frac{2}{2i+1}\binom{2j}{2i}B_{2j-2i}. \tag{4.5}$$

Finally, by (4.2),

$$G_i^j = 2\cdot 1\cdot 3\cdot 5\cdots(2j-1)\binom{2j+1}{2i+1}B_{2j-2i}. \tag{4.6}$$

In particular, we have

$$G_j^j = 2\cdot 1\cdot 3\cdot 5\cdots(2j-1) \tag{4.7}$$

and

$$G_0^j = 2\cdot 1\cdot 3\cdot 5\cdots(2j+1)B_{2j}. \tag{4.8}$$

## REFERENCES

[1] H. W. GOULD, *Power sum identities for arbitrary symmetric arrays*, SIAM J. Appl. Math., 17 (1969), pp. 307–316.

[2] N. E. NÖRLUND, *Vorlesungen über Differenzenrechnung*, Springer, Berlin, 1924.

SIAM J. MATH. ANAL.
Vol. 1, No. 3, August 1970

# THE LOCATION OF SINGULARITIES OF TWO-DIMENSIONAL HARMONIC FUNCTIONS. I: THEORY*

R. F. MILLAR†

**Abstract.** A procedure is developed for locating singularities of two-dimensional, exterior harmonic functions $\Phi$. The method is capable of extension to more general differential equations and, in principle at least, to higher dimensions. It utilizes the fact that $\Phi$ may be expanded in Fourier series about a point $P_0$ external to the boundary $C$. On the circle of convergence lies at least one singularity of $\Phi$. The envelope formed by the circles of convergence as $P_0$ describes a closed curve about $C$ will bound the singularities. The radius of the circle of convergence is obtained by determining the asymptotic behavior of the Fourier coefficients. For the Dirichlet problem, this is made possible by expressing $\Phi$ as the potential of a double layer, of density $\mu$, on $C$. The asymptotic behavior of the coefficients is governed by the singularities of $\mu$ in the plane of the complex arclength parameter $s$. Properties of $\mu$ are found by using the Fredholm integral equation of the second kind which it satisfies; the explicit solution is not required.

**1. Introduction.** In an increasing number of problems, one is interested in determining the location and nature of the singularities of the solution to an elliptic partial differential equation. By way of illustration, we need only look to the fields of fluid dynamics (see, for example, the later chapters of [1]) and geophysics; in both, the Laplace equation is involved.

In geophysics, such problems (which fall into the class of so-called inverse problems) are encountered with the interpretation of magnetic and gravitational anomalies [2]–[5]. In its simplest (two-dimensional) form, the following situation arises: on a simple closed curve $C$, one is given the values of a function $\Phi$ which is known to be harmonic on one side (the "exterior") of $C$. It is required to continue $\Phi$ analytically across $C$ and, thereby, to locate the singularities interior to $C$. These are equivalent to the real sources of $\Phi$ in the sense that their potential is equal to $\Phi$ on and outside $C$.

In a completely different context, we were recently led to examine a problem [6] quite similar to that just described. This arose through an attempt to justify an exterior electromagnetic field expansion and involved a search for the singularities of an exterior Green's function for the two-dimensional Laplace equation. A technique adequate for the purpose was developed. The intent of the present paper is to refine this method sufficiently to determine the convex hull of the possible singularities on and within $C$. In some cases, we may be able to deduce whether the singularities are isolated or form a continuum but, in many applications, knowledge of the convex hull alone is sufficient.

It is well known that two-dimensional problems for Laplace's equation may be discussed most elegantly by introducing a complex potential. Furthermore, the method of reflection (described, for example, in [1, Chap. 16, § 4]) provides a powerful and direct means of analytic continuation, even for nonlinear elliptic equations in two independent variables. Doubtless, the final results to be described in the following work could be obtained as readily by these methods. But while they are confined to two dimensions, the present notions are capable of

* Received by the editors October 23, 1968, and in revised form November 23, 1969.
† Antenna Engineering Section, National Research Council of Canada, Ottawa 7, Ontario.

extension to more general linear elliptic equations and to higher dimensions. It is the generality of our procedure which is its asset. We wish to describe the method; its principles are most readily demonstrated by application to Laplace's equation in two dimensions.

The basic idea is quite elementary, namely, that a function $\Phi$, harmonic outside a circle with center $P_0$, may be expanded in a Fourier series in $\theta$, the coefficients being inverse powers of $\rho$; here $(p, \theta)$ are polar coordinates with $\rho = 0$ at $P_0$. This series converges for all $\rho$ greater than some minimum value $\rho_m$. On $\rho = \rho_m$ lies at least one singularity of $\Phi$, for convergence of the series in $\rho > \rho_m$ implies analyticity of the solution in the same domain, and conversely. If $P_0$ is permitted to describe a closed path $C_0$ about $C$ (or, more precisely, about all the singularities of $\Phi$), the envelope interior to $C$ of the family of circles $\rho = \rho_m$ defines a closed convex curve bounding all singularities of $\Phi$. By expanding $C_0$ to infinity, this convex curve shrinks onto the convex hull of singularities of $\Phi$.

Evidently the most serious difficulty in this program lies in the determination of $\rho_m$. This is contingent upon knowledge of the asymptotic behavior of the Fourier coefficients. But if $\Phi$ is expressed as the potential of a double layer on $C$ of density $\mu$, it turns out that the asymptotic form of the Fourier coefficients depends in part on the singularities of $\mu \equiv \mu(s)$ in the complex plane of arclength $s$. The density $\mu$ satisfies a Fredholm integral equation of the second kind, from which necessary analytic properties of $\mu$ may be deduced without the explicit inversion of the equation.

Clearly, the proposed envelope method[1] cannot in general locate all singularities with precision. This uncertainty is the price that we must pay for not solving the integral equation.

An alternative approach is to expand $\Phi$ in a Fourier series with increasing powers of $\rho$. This will converge for all $\rho$ less than some maximum value, again denoted by $\rho_m$; it will be seen later that the pole $P_0$ must now lie outside $C$. Then the interior envelope of the family of circles $\rho = \rho_m$, obtained when $P_0$ describes a closed path $C_0$ about $C$, will bound the singularities of $\Phi$. It is clear that the closer $C_0$ is to $C$ itself, the more deeply can we enter the domain containing the singularities.

For a closed curve $C$ of finite length, these two procedures are largely equivalent. However, if $C$ is permitted to be an open contour, infinite in length, which divides the plane into two semi-infinite regions, then the first procedure cannot be effected and the second yields only that part of the convex hull of singularities which faces $C$.

In many respects, the present theory resembles the integral operator methods developed by S. Bergman [8] and extended by R. P. Gilbert (see, for example, [7]). In connection with two-dimensional problems, Bergman and Gilbert apply a suitable integral operator to an analytic function of a single complex variable to generate a solution of the differential equation; in our work, the analogue of this analytic function is $\mu$.

[1] This term has been used previously by R. P. Gilbert [7, p. 23] in a rather different sense. In his work, the envelope determines where the integral representation of a function of several complex variables may have pinch-type singularities, while here it bounds the singularity domain of $\Phi$.

In this paper, attention is confined to Laplace's equation in two dimensions; generalizations will be considered later. Only the formal aspects of the theory are presented here; its application to two examples is deferred until Part II.

The remainder of Part I is organized as follows. In § 2, the problem is formulated and the Fourier coefficients of $\Phi$ are expressed as line integrals on $C$. The analytic continuation of $\mu(s)$ is described in § 3, and, in as much generality as possible, the asymptotic determination of the Fourier coefficients is discussed. The convex hull of singularities is determined in § 4, and a few remarks of a general nature follow in § 5.

**2. Formulation.**

**2.1. Notational preliminaries.** We consider real two-dimensional potentials in the $w\,(\equiv u + iv)$-plane. Let $\Phi$ be harmonic outside a simple closed curve $C$ of length $l$; a point on $C$ is specified in terms of arclength $s$ by $w = W(s) \equiv u(s) + iv(s)$, $0 \leqq s \leqq l$, with $W(0) = W(l)$, $W'(s) \neq 0$. Suppose $\Phi$ is given on $C$, and denote by $D$ the set of singularities of $\Phi$; see Fig. 1. Then $D$ consists of points inside, and possibly on, $C$. By continuing $\Phi$ analytically across $C$, we intend to find $H(D)$, the convex hull of $D$. The origin $w = 0$ may be chosen arbitrarily; here it is assumed to lie within $C$ and the closed curve $C_0$ containing $C$ is taken to be a circle of radius $R_0$ centered at $w = 0$.

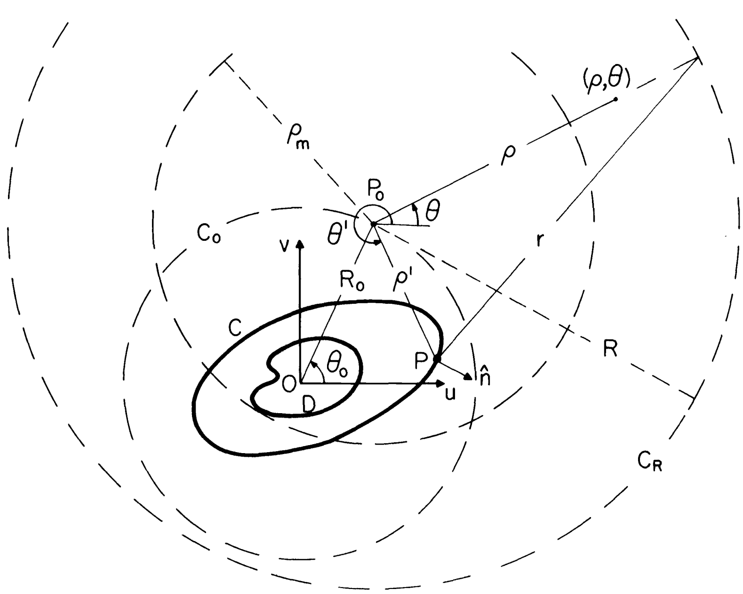


FIG. 1. *Geometric configuration and symbols for exterior expansion of* $\Phi$

**2.2. Exterior expansion.** We shall expand $\Phi$ as a series in $\theta$ and powers of $1/\rho$, $\rho = 0$ being the point $P_0$ on $C_0$. Then

$$\Phi(\rho, \theta) = \Phi_\infty + \sum_{n=1}^{\infty} (a_n e^{-in\theta} + b_n e^{in\theta})\rho^{-n}, \tag{2.1}$$

wherein $\Phi_\infty$ is a real constant.

Next, let $C_R$ be a circle of radius $R$, center $P_0$, which contains $C$. Because $b_n = \bar{a}_n$, we need only consider $a_n$:

$$a_n = \frac{R^n}{2\pi}\int_0^{2\pi} e^{in\theta}\Phi(R,\theta)\,d\theta. \tag{2.2}$$

To determine properties of $a_n$, we must relate $\Phi(R, \theta)$ to the prescribed values on $C$. This is accomplished by writing $\Phi(R, \theta) - \Phi_\infty$ as the potential of a double layer on $C$:

$$\Phi(R,\theta) = \int_0^l \mu(s)\left(\frac{1}{r}\frac{\partial r}{\partial n}\right) ds + \Phi_\infty. \tag{2.3}$$

Here $r$ denotes distance between points on $C$ and $C_R$, differentiation is in the direction of the unit normal $\hat{n}$ out of $C$, and $\mu$ is the double layer density; for future reference, we note that $s$ is assumed to increase in the clockwise direction.

We insert (2.3) into (2.2) and invert orders of integration. The $\theta$-integration may be performed with the aid of the expansion

$$\log r = \log R - \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\rho'}{R}\right)^m \cos m(\theta - \theta'), \qquad \rho' < R, \tag{2.4}$$

wherein $(\rho', \theta')$ are the polar coordinates of $P$ on $C$ with respect to the pole $P_0$. If we define

$$p \equiv \rho' e^{i\theta'}, \tag{2.5}$$

it is an easy calculation to show that

$$\hat{n}\cdot\nabla p^n = -inp^{n-1}(dp/ds)$$

and

$$a_n = \frac{1}{2}i\int_0^l \mu(s)p^{n-1}\frac{dp}{ds}\,ds, \qquad n = 1, 2, 3, \cdots. \tag{2.6}$$

Suppose that the $\varphi(s)$ ($\varphi(s)$ real) are the prescribed values of $\Phi$ on $C$. Upon letting $(R, \theta)$ in (2.3) approach a point specified by arclength $\tau$ on $C$, we see that $\mu$ satisfies

$$\pi\mu(\tau) = \int_0^l \mu(s)\left(\frac{1}{r}\frac{\partial r}{\partial n}\right) ds - \varphi(\tau) + \Phi_\infty; \tag{2.7}$$

the integral is a Cauchy principal value at $s = \tau$. With an appropriate choice of $\Phi_\infty$ and sufficiently smooth $\varphi(\tau)$ and $C$, this equation will possess a solution which is unique to within an unimportant additive constant; see, for example, [9, p. 214 ff.].

It is not necessary to solve (2.7). Of importance is its form which permits us to continue $\mu$ into the plane of complex $s$ and thereby assists us to estimate the magnitude of $a_n$ as $n \to \infty$.

To put the integral into a more familiar form, we write

$$\begin{aligned} p &\equiv W(s) - R_0 e^{i\theta_0} \\ &\equiv e^{\zeta(s)}, \end{aligned} \tag{2.8}$$

with $W(s)\,(\equiv u(s) + iv(s))$ specifying $P$ on $C$ and $\zeta(s)$ defined to within an arbitrary additive integral multiple of $2\pi i$. Consequently (2.6) becomes

$$a_n = \frac{1}{2}i \int_0^l \mu(s)\zeta'(s)\, e^{n\zeta(s)}\, ds, \qquad n = 1, 2, 3, \cdots, \tag{2.9}$$

a form which suggests estimation of $a_n$ for large $n$ by well-established asymptotic methods. Thus saddle points of $\zeta$ and singularities of $\mu$ and $\zeta$ in the complex $s$-plane play a decisive role.

Before continuing with the evaluation of $a_n$, we shall outline the procedure for expanding $\Phi$ in increasing powers of $\rho$.

**2.3. Interior expansion.** It is a simple matter to derive the corresponding interior expansion. For convenience, we use the same notation to specify analogues of symbols occurring in § 2.2.

We expand $\Phi$ as

$$\Phi(\rho, \theta) = \Phi_0 + \sum_{n=1}^{\infty} (A_n\, e^{-in\theta} + B_n\, e^{in\theta})\rho^n. \tag{2.10}$$

Here $\Phi_0 \equiv \Phi(0, \theta)$ and, because $\Phi$ is real, we need consider only $B_n$. An analysis similar to the above gives

$$B_n = -\frac{1}{2}i \int_0^l \mu(s)\zeta'(s)\, e^{-n\zeta(s)}\, ds, \qquad n = 1, 2, 3, \cdots; \tag{2.11}$$

however, to obtain (2.11), it is necessary to assume that $P_0$ lies outside $C$ in order that $\rho' > R$.

**3. Asymptotic behavior of Fourier coefficients.** We now return to the problem of determining the form of $a_n$ for large $n$, by considering the integral (2.9). A prerequisite is knowledge of properties of $\mu$ and $\zeta$. The chief task of this section is to examine $\mu$, which is unknown; this we shall do with the aid of (2.7). But first we shall briefly consider the prescribed function $\zeta$.

**3.1. General properties of $\zeta$.** The function $\zeta$ is defined by (2.8). We note that it has logarithmic branch points at those $s$ for which $W(s) = R_0\, e^{i\theta_0}$. Nevertheless, the integrand is single-valued; it will have zeros at such points. We shall find that these points can be of importance; see § 3 of Part II.

In the subsequent work, we assume that $C$ is a sufficiently smooth curve and that $u(s)$, $v(s)$ and $\varphi(s)$ are real analytic functions of $s$ with fundamental period $l$. In order that continuation be possible, we must assume further that at least part of the segment $0 \leqq s \leqq l$ lies within the domain of analyticity; to avoid complications, we take the entire real axis of $s$ to be in this domain. Then, because $s$ is the arclength parameter,

$$u'(s)^2 + v'(s)^2 = 1 \tag{3.1}$$

at every point of the $s$-plane that can be reached by analytic continuation from a point on the real axis. Consequently, neither $W'(s)\,(\equiv u'(s) + iv'(s))$ nor $\overline{W}'(s)\,(\equiv u'(s) - iv'(s))$ can vanish at an interior point of the domain of analyticity of $W$. Thus $\zeta$

possesses no saddle points; any points at which $\zeta'(s)$ vanishes must be among the singularities of $W$.

On the other hand, were a parameter other than arclength employed, saddle points could well arise. This was the case in [6], and the examples in Part II are of this type.

To summarize, we observe that $\zeta$ will be singular where $W$ is singular, has logarithmic branch points for those $s$ which satisfy $W(s) = R_0 e^{i\theta_0}$ and has no saddle points.

**3.2. General properties of $\mu$.** While the detailed structure of $\mu$ depends on the specific form of $C$, a qualitative idea of its general properties can be gained from (2.7). Here we use the right-hand side of (2.7) to continue $\mu(\tau)$ away from the real axis.

[The method to be described is similar to that which has been developed in quantum field theory and partial differential equations (see, for example [7]) to locate singularities of integrals; however, we find it convenient to fix the path of integration although the end result will not differ from that found by deforming the contour when possible to avoid singularities of the integrand.]

The most important question that then arises is whether the domain into which we continue $\mu$ is sufficiently large for our purposes; that is, does it extend to the relevant singularities of $\mu$ and $\zeta$? We are able to show that it does.

Let us examine the kernel in (2.7):

$$\begin{aligned} \frac{1}{r}\frac{\partial r}{\partial n} &= \frac{[u(s) - u(\tau)]v'(s) - [v(s) - v(\tau)]u'(s)}{[u(s) - u(\tau)]^2 + [v(s) - v(\tau)]^2} \\ &\equiv K(s, \tau), \qquad \text{say.} \end{aligned} \tag{3.2}$$

Here the denominator is equal to $r^2$, the square of the distance between two points on $C$.

By direct calculation, we see that (3.2) is analytic in $\tau$ in a neighborhood of the real $\tau$-axis for every $s$ in $0 \leqq s \leqq l$, and is there continuous in both $\tau$ and $s$. We conclude that the integral in (2.7) is analytic near the real $\tau$-axis. Consequently the right-hand side of (2.7) provides an analytic continuation of $\pi\mu(\tau)$ into a neighborhood of the real $\tau$-axis.

*Remark.* $K(s, \tau)$ is not typical of kernels that arise from more general equations, or in higher dimensions. $K(s, \tau)$ is regular for real $\tau$, while other kernels may be singular on the contour, or surface, of integration. But a singular kernel can still yield an integral which defines an analytic function, and this suffices to make the continuation possible.

What, then, is the extent of the domain $A$ into which we can continue $\mu(\tau)$? Clearly this is determined by the singularities of $\varphi$ and $W$ and the form of the kernel. In general, $\varphi$ is unrestricted, other than to be analytic and periodic. We conclude that the singularities of $\mu$ arise essentially from two different sources: (i) singularities that are prescribed arbitrarily through $\varphi(\tau)$ (and which therefore are not related to the analytic form of $C$), and (ii) singularities arising through the integral in (2.7). We consider all singularities of type (i) to be known, and direct our further attention to (ii).

Generally speaking, the existence of singularities in the integral (2.7) may be attributed to two sources: they may arise because $\tau$ coincides with a singularity of $W$, or they may be related to the vanishing of the denominator in (3.2). To be more precise, we further classify type (ii) singularities into groups (iia) and (iib). Class (iia) singularities are those that arise directly from, and occur at, the singular points of $W$; these are easily located. Singularities of class (iib) arise through the vanishing of the denominator in (3.2), coupled with singularities of $\varphi$ and $W$; they are discussed below.

**3.2.1. The root loci** $r^2 = 0$. We have remarked that type (iib) singularities are associated with vanishing of the denominator in (3.2). The precise connection will be made in § 3.2.2, but it is clear that we must first examine the equation $r^2 = 0$; furthermore, the discussion of the preceding section shows that we may disregard the trivial solution $\tau = s$.

We shall assume that the numerator and denominator vanish simultaneously only in a set of isolated points; if the case were otherwise, some of the subsequent remarks would need modification. In fact, it is not difficult to argue that $\tau = s$ modulo $l$ (this condition will be understood where necessary) is, in general, the only common zero interior to $A$ of both numerator and denominator of $K(s, \tau)$, although isolated common roots might occur.

Therefore we shall only examine the root locus (or loci) $\tau = \tau(s)$ traced by $\tau$ as $s$ describes the interval $0 \leqq s \leqq l$ subject to the conditions $r^2 = 0$, $\tau \neq s$. We see that the root loci are solutions of one or other of the equations

$$u(s) - iv(s) = u(\tau) - iv(\tau), \tag{3.3}$$

$$u(s) + iv(s) = u(\tau) + iv(\tau), \tag{3.4}$$

where $0 \leqq s \leqq l, 0 \leqq \operatorname{Re} \tau \leqq l, \tau \neq s$; here we have confined attention to the periodic strip $0 \leqq \operatorname{Re} \tau \leqq l$.

In the most elementary cases (3.3) and (3.4) have no solutions in the finite $\tau$-plane. For example, if $C$ is a circle of radius $c$ and we choose $w = 0$ to be its center, then

$$u(s) \pm iv(s) = ce^{\pm is/c}$$

and the above equations have no solution (other than the trivial solution $\tau = s$) in the strip. In cases such as these, the continuation of $\mu(\tau)$ can be effected completely by means of (2.7); the only singularities in the finite plane will be of types (i) and (iia).

Next we assume that (3.3) and (3.4) have nontrivial solutions, and we now mention some useful properties of such loci; for simplicity, we confine attention chiefly to (3.3).

We observe that the root loci are analytic curves which can terminate only at singularities or at infinity; this is a consequence of the nonvanishing of $u' \pm iv'$ at points of analyticity (see, for example, [10, p. 245]).

We have already indicated that a sufficiently small neighborhood of the real axis of $\tau$ is free from root loci; in particular, no locus intersects the real axis. Moreover, if (3.3) has more than one solution, they cannot intersect at a point of analyticity because $u' - iv' \neq 0$ and $\tau(s)$ is a single-valued function of $s$.

However, different loci may join smoothly together; this occurred in [6]. The possibility that a solution of (3.3) intersects one of (3.4) does not seem to be excluded; that this has no effect on the analyticity of $\mu$ will be seen below.

**3.2.2. The domain** $A$. We next describe a procedure for finding the part of the domain of analyticity $A$ in $0 \leqq \operatorname{Re} \tau \leqq l$. If $0 \leqq s \leqq l$ and $\tau$ is near the real axis, we know that the integral in (2.7) is analytic in $\tau$; but as $\tau$ approaches a root locus, a pole of the integrand tends to the real axis of $s$ from one side or the other. Let us denote this domain of analyticity by $A_1$. Then (2.7) is valid throughout $A_1$, which is symmetrical with respect to the real axis, and is bounded in $0 \leqq \operatorname{Re} \tau \leqq l$ by root loci.

Since $A_1$ contains the segment $[0, l]$ of the real axis, we can continue $\mu(\tau)$ out of $A_1$ into a larger domain $A_2$. When $\tau$ crosses $\partial A_1$, a pole crosses the contour of integration; we denote this pole (a solution of (3.3) or (3.4)) by $s(\tau)$ ($s(\tau) \in A_1$); it is easy to show that the residue contribution is $\pm\pi\mu(s(\tau))$, the choice of sign being dependent on whether the pole crosses the real axis from above or below. Then from (2.7) we have

$$\mu(\tau) = \pm\mu(s(\tau)) + \frac{1}{\pi}\int_0^l \mu(s)K(s,\tau)\,ds - \frac{1}{\pi}\varphi(\tau) + \frac{1}{\pi}\Phi_\infty, \tag{3.5}$$

$$s(\tau) \in A_1, \quad \tau \in A_2 - A_1.$$

Together, (2.7) and (3.5) determine $\mu(\tau)$ throughout $A_2$. By again invoking the implicit function theorem [10, p. 245] we see that $s(\tau)$ is an analytic function of $\tau$ and, since $s(\tau) \in A_1$, the right-hand side of (3.5) provides an analytic continuation of $\mu(\tau)$ into $A_2 - A_1$.

Because $\mu(s(\tau))$ is determined by (2.7), with $\tau$ replaced therein by $s(\tau)$, (3.5) may be rewritten as

$$\mu(\tau) = \frac{1}{\pi}\int_0^l \mu(s)[K(s,\tau) \pm K(s,s(\tau))]\,ds - \frac{1}{\pi}[\varphi(\tau) + \Phi_\infty \pm \{\varphi(s(\tau)) + \Phi_\infty\}], \tag{3.6}$$

$$s(\tau) \in A_1, \quad \tau \in A_2 - A_1.$$

For the moment, let us assume that there are no solutions to (3.3) and (3.4), other than those bounding $A_1$. Then $\tau$ describes $\partial A_2$ when $s(\tau)$ describes all, or part, of $\partial A_1$. We can continue $\mu(\tau)$ out of $A_2$ into a larger domain $A_3$; in doing so, a pole of $K(s, s(\tau))$ crosses the real axis of $s$, and the appropriate residue must be considered. Then $A_3$ contains all points of analyticity of $\mu(\tau)$ in the entire strip $0 \leqq \operatorname{Re} \tau \leqq l$.

On the other hand, if (3.3) and (3.4) possess solutions other than those bounding $A_1$, we must also consider residues due to poles of $K(s, \tau)$ and $K(s, s(\tau))$ which cross the real axis of $s$ when $\tau$ or $s(\tau)$ penetrates one of these loci.

In general, we see that $\mu$ can be continued analytically to any finite point $\tau$ of analyticity in the strip $0 \leqq \operatorname{Re} \tau \leqq l$, in a finite number of steps.

If a solution of (3.3) intersects with one of (3.4) when $s = s_0$ and $\tau = \tau_0$, continuation is still possible. As $\tau$ approaches such an intersection, two first order poles tend to coalesce on the integration contour at $s_0$. If they approach from the same side, the contour may be deformed away from the real axis (permitting $\tau$ to pass through $\tau_0$) and then deformed back, the two poles being captured in the process; $\mu(\tau)$ is thus continued past $\tau_0$ without incident. If the poles approach the integration contour from opposite sides, and tend to form a "pinch," a similar procedure continues $\mu(\tau)$ past $\tau_0$; here, however, we must use the fact that the numerator in (2.7), also vanishes for $s = s_0$ and $\tau = \tau_0$.

We see that (2.7), (3.6), and possibly additional equations of similar form, determine $\mu(\tau)$ throughout the domain $A$ in which it is analytic. If $u$, $v$ and $\varphi$ are all entire functions, then $\mu$ is entire.

For the case in which there are no solutions to (3.3) and (3.4) other than those bounding $A_1$, these results may be summarized as follows: let $S$ denote the set of singular points of $u(\tau)$, $v(\tau)$ and $\varphi(\tau)$. Then in general the singularities of $\mu(\tau)$ are members of the set $T \equiv \{\tau | \tau \in S \text{ or } s(\tau) \in S\}$ augmented, possibly, by the point at infinity. It is apparent that $\bar{\tau} \in S$ if $\tau \in S$, so the singular points of $\mu(\tau)$ are arranged symmetrically about the real axis. We see that all possible singularities of $\mu$ in the finite plane of its argument can be located directly by reference to the given data of the problem.

**3.3. Bounds for the Fourier coefficients.** We now return to the discussion of $a_n$ (and $B_n$). But first we emphasize that the following treatment is largely qualitative and a number of assertions would require verification in specific cases.

We deform the integration contour in the $s$-plane and require that the exponential be dominant on the deformed contour or contours. The growth of $\mu$ can be estimated from (2.7) or (3.6) (or from a suitable generalization thereof); the form of (3.2) suggests that the integral is bounded at infinity.

Because of its periodicity, the values of the integrand in (2.9) (or (2.11)) for $s = i\sigma$ and $s = l + i\sigma$ $(-\infty < \sigma < \infty)$ are equal. Thus the integration contour $0 \leqq s \leqq l$ may be translated in the direction of the imaginary axis without changing the value of the integral, provided it meets no singularity of the integrand. When a singularity $s = s_0$ is captured, a suitable loop deformation of the contour must be made; the contribution of such to $a_n$ is a term which, for $n \to \infty$, is $O(\exp n[\zeta(s_0) + \varepsilon])$ and to $B_n$ the singularity contributes a term which is $O(\exp\{-n[\zeta(s_0) - \varepsilon]\})$ for every $\varepsilon > 0$ but for no $\varepsilon < 0$. We assume that the deformations are such that $a_n$ and $B_n$ are dominated by one or more of these terms.

We see that the optimal deformations are in directions which minimize the $O$-terms, and one (or more) singularities provide the dominant contribution. In the case of $a_n$, this determines the distance from $P_0$ to the most distant singularity. Continuation of the deformation process yields further contributions from singularities of $\mu(s)$ which correspond to singularities of $\Phi$ closer to $P_0$. A similar interpretation can be placed on the singularities of $\mu(s)$ relevant to $B_n$. For simplicity, we restrict attention to the dominant term. To be specific, let $S_a$ denote the set of singular points captured when the contour in (2.9) is deformed, and let $S_B$ be the corresponding set for (2.11). Then the dominant singularity (or singularities) for $a_n$ is that which maximizes $\operatorname{Re}\zeta(s_0)$, $s_0 \in S_a$; and the dominant singularity for $B_n$ is that which minimizes $\operatorname{Re}\zeta(s_0)$, $s_0 \in S_B$.

Suppose then that $s = s_a$ denotes the dominant singularity or singularities for $a_n$, while $s = s_B$ is dominant for $B_n$; note that $s_a$ and $s_B$ may depend on $R_0$ and $\theta_0$. For simplicity, we assume that there is a finite number of such points in each case for any particular values of $R_0$ and $\theta_0$ in (2.8). As we shall see in § 4, this implies that only a finite number of singularities lie on the circle of convergence $\rho = \rho_m$. Then we see that

$$a_n = O(|[W(s_a) - R_0 e^{i\theta_0}] e^{\varepsilon}|^n), \tag{3.7}$$

$$B_n = O(|e^{\varepsilon}/[W(s_B) - R_0 e^{i\theta_0}]|^n) \tag{3.8}$$

as $n \to \infty$ for every $\varepsilon > 0$ and for no $\varepsilon < 0$. The constants implied are uniformly bounded in $n$.

**4. Convex hull of singularities.** Each of the relations (3.7) and (3.8) suffices to determine $H(D)$; these bounds are quite crude and sharper estimates would be required if more detailed information were desired. The procedure to find $H(D)$ is formally the same, whether we use the interior expansion or the exterior representation; we consider only the latter.

From (2.1) and (3.7), we see that the minimal radius is determined by

$$\rho_m^2 = |W(s_a) - R_0 e^{i\theta_0}|^2, \tag{4.1}$$

which on writing $W(s_a) = \alpha e^{i\beta}$ ($\alpha, \beta$ real) becomes

$$\rho_m^2 = R_0^2 - 2\alpha R_0 \cos(\theta_0 - \beta) + \alpha^2. \tag{4.2}$$

There is at least one singularity of $\Phi$ on the circle $\rho = \rho_m$. To find $H(D)$, we set $\rho_m = R_0 + \delta_m$ in (4.2) and solve for $\delta_m$ by choosing the root which remains finite and becomes independent of $R_0$ as $R_0 \to \infty$. On letting $R_0 \to \infty$ (so the circular arcs which envelop $H(D)$ become straight-line segments), we find that $\delta_m \to \delta(\theta_0)$:

$$\delta(\theta_0) \equiv -\alpha \cos(\theta_0 - \beta). \tag{4.3}$$

The family of tangents to $\partial H(D)$ satisfies

$$u \cos\theta_0 + v \sin\theta_0 = -\delta(\theta_0). \tag{4.4}$$

On eliminating $\theta_0$ between (4.4) and its derivative with respect to $\theta_0$, we obtain the envelope equation.

If there is but one $s_a$ and it does not change as $\theta_0$ varies, we find that $H(D)$ is the isolated point in the $w$-plane determined by

$$w = \alpha e^{i\beta} \equiv u(s_a) + iv(s_a). \tag{4.5}$$

In this case, $\Phi$ possesses just one singularity. If two or more points $s_a$ contribute over disjoint ranges of the interval $0 \leqq \theta_0 \leqq 2\pi$, then $\partial H(D)$ is a convex polygon. The relevant values $W(s_a)$ being $\alpha_n e^{i\beta_n}$, $n = 1, 2, 3, \cdots, N$, the vertices of $\partial H(D)$ are the points $w = w_n$:

$$w_n = \alpha_n e^{i\beta_n}, \qquad n = 1, 2, 3, \cdots, N. \tag{4.6}$$

It may be conjectured that to each isolated singular point of the set $\bigcup_{0 \leqq \theta_0 \leqq 2\pi} S_a$ (see § 3.3) there corresponds an isolated singularity of $\Phi$ which may or may not be a boundary point of $H(D)$. More generally, we see that any part of $\partial H(D)$ which does not consist of straight-line segments must arise from a continuum of singularities (that is, a natural boundary) of $\mu$ or $W$.

**5. Additional remarks.** In the preceding sections, we have developed a method for locating singularities of two-dimensional exterior harmonic functions. The procedure may be regarded as a special case of the integral operator methods of Bergman [8] and Gilbert [7]. We have avoided the use of a complex potential or the reflection method which, though more elegant, are restricted to two-dimensional applications. Our method can be extended to more general elliptic equations and to higher dimensions; it has been illustrated here in its most simple setting.

While we have considered only the Dirichlet boundary condition, a Neumann problem is equally amenable to analysis; here it would be necessary to represent the harmonic function as the potential of a simple layer on $C$.

For each boundary condition, we obtain a Fredholm integral equation of the second kind for the unknown density; this is employed to locate singularities of the density. An alternative approach could be based on the integral equations of the second kind which are satisfied on $C$ by $\partial\Phi/\partial n$ (Dirichlet problem) and $\Phi$ (Neumann problem); these may be obtained with the aid of Green's theorem.

Finally, we note that we could examine the corresponding interior problem for the harmonic function $\Psi$, such that $\Psi - \Phi_\infty$ is generated by the double-layer density $\mu$. Here the effect of not solving (2.7) is to restrict $P_0$ to the interior of $C$; consequently the singularities of $\Psi$, which lie outside $C$, are located with less precision that those of $\Phi$.

In the paper that follows, we show that the method developed here is of more than purely theoretical interest, by applying it to two examples.

**Acknowledgment.** The author wishes to thank A. L. VanKoughnett for his helpful comments in several discussions and for a critical reading of the manuscript.

REFERENCES

[1] P. R. Garabedian, *Partial Differential Equations*, John Wiley, New York, 1964.

[2] V. N. Strakhov, *The analytic continuation of two-dimensional potential fields, with applications to the solution of the inverse problem of magnetic and gravitational exploration. I*, Izv. Akad. Nauk SSSR Ser. Geofiz., (1962), pp. 307–316, English transl., Bull. (Izv.) Acad. Sci. USSR Geophys. Ser., (1962), pp. 209–214.

[3] ———, *The analytic continuation of two-dimensional potential fields, with applications to the solution of the inverse problem of magnetic and gravitational exploration. II*, Ibid., (1962), pp. 336–347; English transl., Ibid., (1962), pp. 227–232.

[4] ———, *The analytic continuation of two-dimensional potential fields with applications to the solution of the inverse problem of magnetic and gravitational exploration. III*, Ibid., (1962), pp. 491–505; English transl., Ibid., 1962, pp. 324–336.

[5] G. M. Voskoboinikov, *Integral transformations and the location of the singularities of a logarithmic potential*, Izv. Akad. Nauk SSSR, Ser. Fiz. Zemli, (1965), pp. 76–89; English transl., Izv. Acad. Sci. USSR Phys. Solid Earth, (1965), no. 1, pp. 42–48.

[6] R. F. Millar, *On the Rayleigh assumption in scattering by a periodic surface*, Proc. Cambridge Philos. Soc., 65 (1969), pp. 773–791.

[7] R. P. GILBERT, *Function Theoretic Methods in Partial Differential Equations*, Academic Press, New York, 1969.

[8] S. BERGMAN, *Integral Operators in the Theory of Linear Partial Differential Equations*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N.F., vol. 23, Springer-Verlag, Berlin, 1961, 145 pp.

[9] I. G. PETROVSKY, *Partial Differential Equations*, Interscience, New York, 1954.

[10] P. DIENES, *The Taylor Series*, Dover, New York, 1957.

SIAM J. MATH. ANAL.
Vol. 1, No. 3, August 1970

# THE LOCATION OF SINGULARITIES OF TWO-DIMENSIONAL HARMONIC FUNCTIONS. II: APPLICATIONS*

R. F. MILLAR†

**Abstract.** The procedure developed in Part I is applied to two examples, the intention in each case being to illustrate some feature of the method. For the first example, a previously considered problem is reexamined; here the boundary $C$ in the $w$-plane is

$$w = \exp - i\kappa(x + ib\cos\kappa x), \qquad 0 \leqq x \leqq 2\pi/\kappa,$$

and the two singularities are located precisely. For the second example, $C$ is taken to be an ellipse; the singularities at the foci are found.

**1. Introduction.** In Part I of the present work we developed a method for locating the singularities of a two-dimensional harmonic function $\Phi$ which satisfies a given Dirichlet-type condition on a suitable boundary $C$. Only the formal aspects of the theory were discussed. In this part we shall apply these notions to locate the singularities in two examples. Each is chosen not necessarily with the intention of obtaining a new result, but rather to illustrate some particular feature of the method.

In § 2, we reexamine a previously considered problem [1]. This involves locating the singularities when the boundary $C$ in the $w$-plane is determined by

$$w = \exp\{-i\kappa(x + ib\cos\kappa x)\}, \qquad 0 \leqq x \leqq 2\pi/\kappa.$$

It illustrates how use of a parameter other than arclength on $C$ may introduce saddle points into the discussion. But there is another important reason for considering this example. While previously we were able only to bound the singularities within a certain circle, our envelope method shows that there are two point singularities and locates them precisely.

In § 3, for a final example we take $C$ to be an ellipse. Here $W(s)$ has singularities in the finite plane of its argument. It is convenient to parametrize $C$ by the eccentric angle $t$, rather than $s$; some of the singularities of $W$ become saddle points in the $t$-plane. We find singularities of $\Phi$ at the two foci of $C$.

Reference to equations and sections in Part I will be prefixed by the letter "I."

**2. An example arising from diffraction theory**. An earlier paper [1] dealt with the scattering of a downcoming scalar plane wave by a periodic surface with the profile

$$y = b\cos\kappa x, \qquad -\infty < x < \infty. \tag{2.1}$$

In the course of this investigation, we were led to determine a condition for which the periodic harmonic Green's function (with pole at infinity) for the surface (2.1) had no singularities (except at infinity) in $y > -b$. We defined $z = x + iy$ and mapped the strip $0 \leqq x \leqq a$ $(\equiv 2\pi/\kappa)$ conformally onto the $w$ $(\equiv u + iv)$-plane by the transformation

$$w = e^{-i\kappa z}; \tag{2.2}$$

* Received by the editors October 23, 1968, and in revised form November 23, 1969.

† Antenna Engineering Section, National Research Council of Canada, Ottawa 7, Ontario.

then $w$ traced out the curve $C$ containing $w = 0$ as $x$ ranged over the interval $[0, a]$ with $y$ given by (2.1). Our transformed problem now called for a condition under which the singularities of $\Phi$ ($\Phi$ being harmonic outside $C$ and equal to $-\frac{1}{2}\log(u^2 + v^2)$ on $C$) lay within the circle $|w| = e^{-\kappa b}$. If

$$\lambda \equiv \exp \kappa(b \cosh \kappa X_0 - X_0), \tag{2.3}$$

with

$$\sinh \kappa X_0 = 1/(\kappa b), \tag{2.4}$$

we found that a necessary and sufficient condition was

$$\lambda < e^{-\kappa b}; \tag{2.5}$$

this implied that $0 \leqq \kappa b < 0.448$.

By using the method developed in Part I, we are able to improve on this result. It is not necessary to restrict $\lambda$ by (2.5) so our conclusion is valid for all $\kappa b$. We shall find that there are two isolated singularities within $C$; we locate them precisely at the points $u = \lambda$, $v = 0$, and $u = v = 0$.

In [1], we found it more convenient to parametrize the curve $C$ by $x$ than by arclength $s$; consequently, instead of (I, 2.7) we found

$$\begin{aligned} \pi\nu(x) + \int_0^a \nu(t)\frac{[U(t) - U(x)]V'(t) - [V(t) - V(x)]U'(t)}{[U(t) - U(x)]^2 + [V(t) - V(x)]^2}\,dt \\ = \kappa b \cos \kappa x + \Phi_\infty. \end{aligned} \tag{2.6}$$

We took $R_0 = 0$ in [1]; the generalization to $R_0 > 0$ gives, in place of (I, 2.9),

$$a_n = \frac{1}{2} i\kappa \int_0^a \nu(x)\xi'(x)\, e^{n\xi(x)}\, dx. \tag{2.7}$$

In these equations, $\nu(x) \equiv \mu(s)$, where $\mu(s)$ is the double layer density at a point on $C$ specified by arclength $s$,

$$s \equiv \int_x^a [1 + (\kappa b \sin \kappa\sigma)^2]^{1/2}\, d\sigma, \tag{2.8}$$

while

$$\begin{aligned} U(x) &\equiv e^{\kappa b \cos \kappa x} \cos \kappa x, \\ V(x) &\equiv -e^{\kappa b \cos \kappa x} \sin \kappa x \end{aligned} \tag{2.9}$$

and, to within an arbitrary integral multiple of $2\pi i$,

$$\xi(x) \equiv \log[\exp \kappa(b \cos \kappa x - ix) - R_0\, e^{i\theta_0}]. \tag{2.10}$$

Because of the formal similarity between (2.6) and (I, 2.7) (see also (I, 3.2)), the arguments of I, § 3.2 may be repeated to determine properties of $\nu$; in particular, we find that at most one root locus is crossed in continuing $\nu(x)$ from an $x$-value on the real axis to any other $x$ (see [1, Fig. 6]).

However, there is one difference introduced by our change of parameter. Let $t(x)$ denote the position of the pole, the analogue of $s(\tau)$ in I, § 3.2.2. Then

$$\frac{dt(x)}{dx} = \frac{U'(x) \pm iV'(x)}{U'(t(x)) \pm iV'(t(x))}, \tag{2.11}$$

where the choice of sign is determined by the relevant solution of

$$[U(t) - U(x)]^2 + [V(t) - V(x)]^2 = 0, \qquad t \neq x;$$

for purposes of discussion, we choose the upper sign. We recall that $s(\tau)$ was an analytic function of $\tau$ when $s(\tau) \in A_1$, the domain containing the real axis within which the integral in (I, 2.7) was analytic in $\tau$. On the other hand, $t(x)$ is not necessarily analytic throughout the corresponding domain $A_1'$, for (2.11) may be unbounded therein. To be precise, if $U'(x_0) + iV'(x_0) = 0$ then, in general, $t(x)$ is not an analytic function of $x$ near $t(x) = x_0$. Such points $x = x_0$ exist in the present problem; they are saddle points of $\xi(x)$ and are among the points at which (2.8) is singular. If we choose the lower sign in (2.11), we see that $t(x)$ is not analytic near $t(x) = \bar{x}_0$; $s$ is also singular for $x = \bar{x}_0$.

Thus the essential difference brought about by the change of parameter is the introduction of saddle points $x = x_0$ into the exponent of the integral for $a_n$ and a possible singularity in $v$ when its argument $x$ is such that $t(x) = x_0$ or $t(x) = \bar{x}_0$; the saddle points correspond to singularities in the arclength representation (2.8) of $C$. Because $U(x)$, $V(x)$ and $\kappa b \cos \kappa x$ are entire functions of $x$, we conclude that all singularities of $v(x)$, for Im $x$ finite, will be found among the $x$ for which $t(x) = x_0$ or $t(x) = \bar{x}_0$; we shall soon see that there are no such $x$.

We assume that $R_0 \gg 1$ so that the argument of the logarithm in (2.10) does not vanish in a sufficiently wide strip containing the real $x$-axis. The saddle points of $\xi(x)$ are independent of $R_0$ (and coincident with those for $R_0 = 0$); in the strip $0 \leqq \operatorname{Re} x \leqq a$, there are saddle points at $x = -iX_0$, $\frac{1}{2}a + iX_0$ and $a - iX_0$, with $X_0\,(> 0)$ determined by (2.4).

It may be verified that there is no $x$ that satisfies the equation

$$[U(t(x)) - U(x)]^2 + [V(t(x)) - V(x)]^2 = 0$$

when $t(x) = \pm iX_0 \,(\mathrm{mod}\, a)$ or $t(x) = \frac{1}{2}a \pm iX_0 \,(\mathrm{mod}\, a)$ other than the trivial solution $x = t(x)$. In the light of the above remarks, we conclude that $v(x)$ is an entire function.

To determine which saddle points are relevant, we examine the curves Im $\xi$ $(\equiv \xi_2) =$ const. through each and the level curves Re $\xi\,(\equiv \xi_1) =$ const. These depend on $\theta_0$ which, for reasons of symmetry, we are able to confine to the interval $[0, \pi]$. Because $R_0 \gg 1$, we find that the steepest paths through $x = -iX_0$ and in a sufficiently wide strip containing the real axis are

$$e^{\kappa\gamma} \sin(\kappa\delta + \theta_0) = \lambda \sin \theta_0; \tag{2.12}$$

here $\lambda$ is defined by (2.3),

$$\gamma \equiv b \cos \kappa x_1 \cosh \kappa x_2 + x_2, \tag{2.13}$$

$$\delta \equiv b \sin \kappa x_1 \sinh \kappa x_2 + x_1, \tag{2.14}$$

with $x \equiv x_1 + ix_2$. The steepest paths through $x = a - iX_0$ are merely the translation of (2.12) through a distance $a$, while those through $x = \frac{1}{2}a + iX_0$ are

$$e^{\kappa\gamma} \sin(\kappa\delta + \theta_0) = -(1/\lambda)\sin\theta_0. \tag{2.15}$$

The level curves satisfy

$$e^{2\kappa\gamma} + R_0^2 - 2R_0 e^{\kappa\gamma}\cos(\kappa\delta + \theta_0) = \text{const}. \tag{2.16}$$

or, for $R_0 \gg 1$,

$$e^{\kappa\gamma}\cos(\kappa\delta + \theta_0) = \text{const}. \tag{2.17}$$

It is not difficult to show that the logarithmic branch points of $\xi$ are determined by

$$e^{\kappa\gamma} = R_0, \qquad \kappa\delta + \theta_0 = 2m\pi, \qquad m = 0, \pm 1, \pm 2, \cdots. \tag{2.18}$$

These play no essential role in the subsequent analysis.

By comparing $\xi_1(-iX_0)$ and $\xi_1(\frac{1}{2}a + iX_0)$, and by examining the form of the steepest paths and level curves as $\theta_0$ varies, we arrive at the following conclusions.

(i) The saddle point at $x = \frac{1}{2}a + iX_0$ is not relevant.

(ii) The saddle points at $x = -iX_0$ and $x = a - iX_0$ are not relevant if $R_0 \cos\theta_0 > \frac{1}{2}\lambda$, the geometrical significance of which will become apparent later. In these circumstances, it is possible to deform the contour in (2.7) into one of finite length on which

$$\xi_1(x) \leqq \log(R_0 + \varepsilon) \tag{2.19}$$

for every $\varepsilon > 0$; for $\varepsilon = 0$ the contour is infinite in length, and no such contour exists for any $\varepsilon < 0$. See Fig. 1a.

(iii) If $R_0 \cos\theta_0 < \frac{1}{2}\lambda$, it is not possible to deform the contour in (2.7) into one on which (2.19) obtains. In this case the saddle points at $x = -iX_0$ and $x = a - iX_0$ are relevant, and

$$\xi_1(-iX_0) > \log R_0. \tag{2.20}$$

As is apparent from Fig. 1b, the optimal deformation is into a contour linking the two saddle points and lying wholly within the domain in which

$$\xi_1(x) < \xi_1(-iX_0). \tag{2.21}$$

Consequently we obtain the following bounds for $a_n$:

$$a_n = O[(R_0 + \varepsilon)^n], \qquad R_0\cos\theta_0 > \tfrac{1}{2}\lambda, \tag{2.22}$$

for every $\varepsilon > 0$, and

$$a_n = O(\exp n[\xi(-iX_0) + \varepsilon]), \qquad R_0\cos\theta_0 < \tfrac{1}{2}\lambda. \tag{2.23}$$

By using (2.10), we may rewrite (2.23) as

$$a_n = O(|\lambda - R_0 e^{i\theta_0}|^n e^{n\varepsilon}), \qquad \varepsilon > 0, \quad R_0\cos\theta_0 < \tfrac{1}{2}\lambda, \tag{2.24}$$

which is the analogue of (I, 3.7).

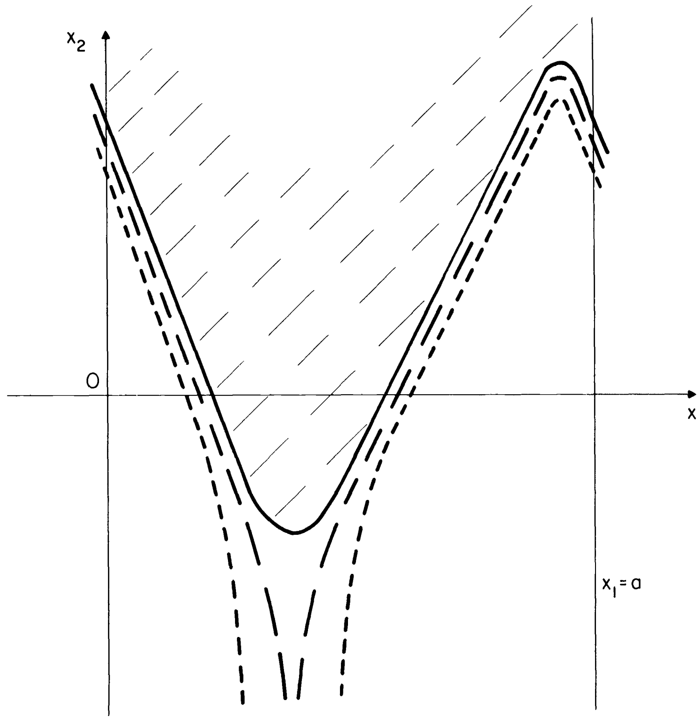


FIG. 1a. *Qualitative form of level curves relevant to evaluation of* (2.7) *when* $R_0 \cos\theta_0 > \frac{1}{2}\lambda$: ——— $\xi_1 = \log(R_0 + \varepsilon)$ $(\varepsilon > 0)$; — — $\xi_1 = \log R_0$; - - - - $\xi_1 = \log(R_0 + \varepsilon)$ $(\varepsilon < 0)$. *In the shaded domain,* $\xi_1 \geqq \log(R_0 + \varepsilon)$ $(\varepsilon > 0)$.

We may now repeat the arguments of I, § 4 and find that $\Phi$ has two isolated singularities. Equation (2.22) determines a singularity at the origin

$$u = v = 0, \tag{2.25}$$

while (2.24) corresponds to a singularity at

$$u = \lambda, \qquad v = 0; \tag{2.26}$$

in the $z$-plane, (2.26) determines an infinity of isolated singularities at

$$\begin{aligned} x &= na, \quad n = 0, \pm 1, \pm 2, \cdots, \\ y &= b \cosh \kappa X_0 - X_0, \end{aligned} \tag{2.27}$$

while (2.25) corresponds to $y = -\infty$. The points (2.27) lie in $y < -b$ if $\kappa b < 0.448$.

The significance of the conditions $R_0 \cos\theta_0 \gtrless \frac{1}{2}\lambda$ in (ii) and (iii) above is now apparent. When $R_0 \cos\theta_0 > \frac{1}{2}\lambda$, $\Phi$ is harmonic outside the circle centered on $(R_0, \theta_0)$ which passes through $u = v = 0$; when $R_0 \cos\theta_0 < \frac{1}{2}\lambda$, $\Phi$ is harmonic outside the circle centered on $(R_0, \theta_0)$ which passes through $u = \lambda$, $v = 0$. In each case, the radius of the circle is minimal.

The result expressed by (2.25) and (2.26) is an improvement upon that given earlier. In [1], it was merely established that all singularities of $\Phi$ lay in the disc

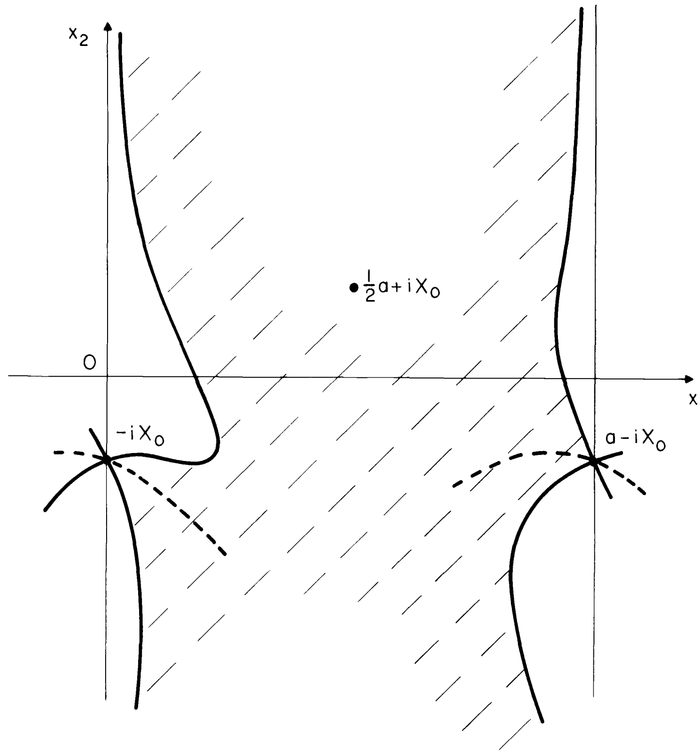


FIG. 1b. *Qualitative form of relevant configuration when* $R_0 \cos\theta_0 < \frac{1}{2}\lambda$: ——— $\xi_1 = \xi_1(-iX_0)$; – – – – *initial arcs of paths of steepest descent. In the shaded domain,* $\xi_1 \leqq \xi_1(-iX_0)$.

$|w| < e^{-\kappa b}$ provided that $\lambda < e^{-\kappa b}$; here we have removed this restriction on $\lambda$ and have shown that there are two singularities which we have located explicitly.

We have already mentioned that the transformation (2.8) is singular when $x$ coincides with a saddle point; we thus see that half of these singularities in the arclength representation of $C$ give rise to a singularity in $\Phi$ at the point determined by (I, 4.5).

**3. The elliptical boundary.** As a second example, we take $C$ to be an ellipse with semiaxes $a$ and $b$ ($a > b$). The origin is at its center, and the foci lie on the $u$-axis. In this case, as in § 2, $W(s)$ possesses singularities in the finite plane; we shall see that these correspond to singularities of $\Phi$ at the foci of $C$.

A point on the ellipse is specified by

$$u(s) = a\cos t, \qquad v(s) = b\sin t, \qquad 0 \leqq t \leqq 2\pi. \tag{3.1}$$

Here $t = t(s)$, with

$$s = a\int_0^t (1 - e^2\cos^2\theta)^{1/2}\, d\theta, \tag{3.2}$$

$e$ denoting the eccentricity. Then $ds/dt$, vanishes when $\cos t = \pm 1/e$ ($e < 1$) and $W$ is singular when its argument $s$ takes the corresponding values.

Here it is convenient to use $t$, rather than $s$, as the complex parameter. Then $\zeta(s) \to \xi(t)$, and there exists the possibility that $\xi$ possesses saddle points. If $W(s) \to \tilde{W}(t)$, it is apparent from (3.1) that $\tilde{W}$ is an entire function.

Under the transformation (3.2), let $\mu(s) \to \nu(t)$. Because $\nu(t)$ satisfies an equation similar to (2.6), we can continue $\nu$ away from the real axis in the manner described in I, § 3.2. Hence we examine the root loci, which here are solutions (other than $x = t \pmod{2\pi}$) of

$$a \cos t \pm ib \sin t = a \cos x \pm ib \sin x, \qquad 0 \leqq \operatorname{Re} t \leqq 2\pi. \tag{3.3}$$

It is easy to see that all remaining roots are of the form

$$x = -t \pm 2i \cosh^{-1}(1/e) + 2n\pi, \qquad n = 0, \pm 1, \pm 2, \cdots,$$

and of these

$$x = -t \pm 2i \cosh^{-1}(1/e) + 2\pi \tag{3.4}$$

lie in $0 \leqq \operatorname{Re} x \leqq 2\pi$.

Let us assume that the prescribed boundary values $\tilde{\varphi}(t)$ are the values of an entire function $\tilde{\varphi}$. Because $\tilde{W}$ is entire, and because $dx/dt \neq 0$ or $\infty$ in (3.4), we may apply the results summarized near the end of I, § 3.2.2 to conclude that $\nu$ is an entire function. (The analysis of I, § 3.2 assumed that arclength was the parameter, but is valid here also.)

The coefficient $a_n$ is given by

$$a_n = -\frac{1}{2}\frac{i}{n}\int_0^{2\pi} \nu'(t)\, e^{n\xi(t)}\, dt \tag{3.5}$$

and

$$\xi(t) = \log(a \cos t + ib \sin t - R_0 e^{i\theta_0}). \tag{3.6}$$

The saddle points of $\xi$ (and of $\tilde{W}$) are the roots $t'_m$ of $\tan t = ib/a$, so

$$t'_m = i \cosh^{-1}(1/e) + m\pi, \qquad m = 0, \pm 1, \pm 2, \cdots. \tag{3.7}$$

The corresponding points in the $s$-plane form part of the set of singularities of $W$ which we mentioned subsequently to (3.2); this illustrates again the correspondence between certain singularities in the $s$-plane and saddle points in the plane of a different parameter.

There are three saddle points and two logarithmic branch points of $\xi$ in $0 \leqq \operatorname{Re} t \leqq 2\pi$. Let us assume that $0 < \theta_0 < \pi$. If $R_0 \gg 1$, the branch points are located in the $t$-plane approximately at $\theta_0 - i \log[2R_0/(a+b)]$ and $2\pi - \theta_0 + i \log[2R_0/(a-b)]$. It is now not difficult to determine qualitatively the form of the steepest paths; they are illustrated in Fig. 2. Because $\operatorname{Re} \xi(t) = -\infty$ at the branch points, we see that the paths of steepest descent pass through them; the arrows indicate the directions in which $\operatorname{Re} \xi(t)$ decreases.

Evidently the integration contour may be deformed to coincide with paths of steepest descent. Furthermore,

$$\begin{aligned} \operatorname{Re} \xi(t'_0) &= \tfrac{1}{2} \log[(ae - R_0 \cos\theta_0)^2 + R_0^2 \sin^2\theta_0] \\ &= \operatorname{Re} \xi(t'_2) \end{aligned} \tag{3.8}$$

and

$$\text{(3.9)} \qquad \operatorname{Re} \xi(t_1') = \tfrac{1}{2} \log \left[ (ae + R_0 \cos \theta_0)^2 + R_0^2 \sin^2 \theta_0 \right].$$

Consequently, the dominant saddle point is at $t = t_1'$ if $0 < \theta_0 < \frac{1}{2}\pi$, while $t = t_0'$ and $t = t_2'$ dominate when $\frac{1}{2}\pi < \theta_0 < \pi$.

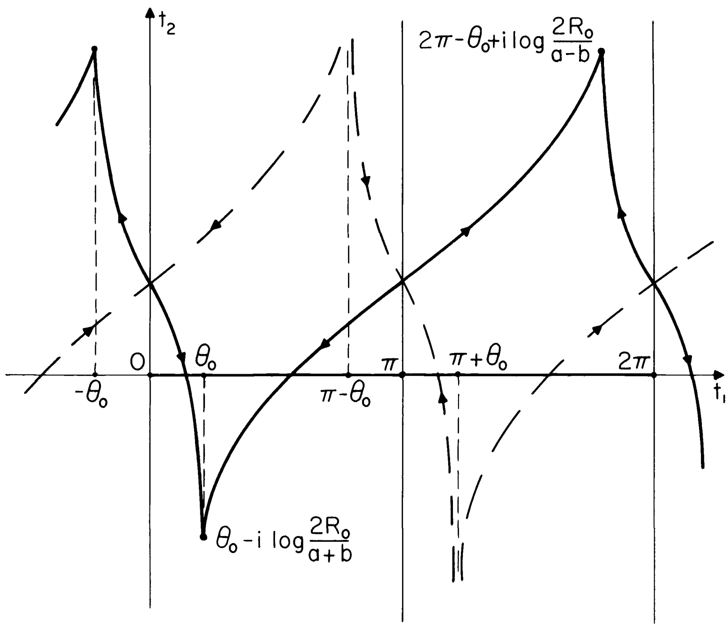


FIG. 2. *Paths of steepest descent (solid curves) and steepest ascent (broken curves) through saddle points in $t(\equiv t_1 + it_2)$-plane when C is an ellipse. Arrows indicate direction in which* $\operatorname{Re} \xi(t)$ *decreases.*

From this we can find the order of magnitude of $a_n$, the minimal radius $\rho_m$ and the envelope of the minimal circles as $\theta_0$ varies. As we have shown in I, § 4, the saddle point at $t = t_1'$ corresponds to a point singularity of $\Phi$ at $w = a \cos t_1' + ib \sin t_1'$, that is, at $w = -ae + i0$. The saddle points at $t = t_0'$, $t = t_2'$ correspond to a singularity at $w = ae + i0$. Thus, as $\theta_0$ ranges from 0 to $\frac{1}{2}\pi$ (and, by symmetry, from $-\frac{1}{2}\pi$ to 0), the focus $(-ae, 0)$ is enveloped in part while the focus $(ae, 0)$ is partially enveloped when $\theta_0$ ranges from $\frac{1}{2}\pi$ to $\frac{3}{2}\pi$. The convex hull of singularities is the interfocal segment of $v = 0$.

For simplicity, we have assumed $\tilde{\varphi}$ to be an entire function. If we permit $\tilde{\varphi}$ to possess singularities in the finite plane, then these will be reflected in singularities of $\Phi$.

**4. Concluding remarks.** In Parts I and II of the present work we have developed and applied a procedure for locating the convex hull of singularities of a two-dimensional harmonic function $\Phi$ which is harmonic outside an analytic curve $C$. On $C$, $\Phi$ satisfies a Dirichlet-type boundary condition. Rather than introduce the familiar complex potential, we have employed techniques which hold promise of applicability in higher dimensions as well as to solutions of more general equations, such as the Helmholtz equation; it is hoped to carry out these extensions at a later date.

The key to the method is the representation of $\Phi$ as the potential of a double layer on $C$, a type of representation which is not inherently limited either to two dimensions or to harmonic functions. The double layer density $\mu$ and the potential $\Phi$ may be regarded as transforms of one another, connected by integral operators. Thus our work has much in common with theory developed by Bergman [2] and, more recently, by Gilbert [3]. However, the present approach seems better suited to the treatment of boundary value problems.

The connection between the present approach and an analysis based on the extension of the solution into the domain of complex values of the independent variables (as described in [4, Chap. 16]) is more tenuous than its relationship to the integral operator methods. Although it leads to the results in an apparently devious manner, our analysis possesses a conceptual advantage in that it involves only functions of a single complex variable. On the other hand, in two dimensions it may be of less generality than the theory developed in [4].

## REFERENCES

[1] R. F. MILLAR, *On the Rayleigh assumption in scattering by a periodic surface*, Proc. Cambridge Philos. Soc., 65 (1969), pp. 773–791.

[2] S. BERGMAN, *Integral Operators in the Theory of Linear Partial Differential Equations*, Ergebnisse der Mathematik und ihrer Grenzgebiete, N.F., vol. 23, Springer-Verlag, Berlin, 1961, 145 pp.

[3] R. P. GILBERT, *Function Theoretic Methods in Partial Differential Equations*, Academic Press, New York, 1969.

[4] P. R. GARABEDIAN, *Partial Differential Equations*, John Wiley, New York, 1964.

SIAM J. MATH. ANAL.
Vol. 1, No. 3, August 1970

# SOME ISOPERIMETRIC INEQUALITIES FOR HARMONIC FUNCTIONS*

L. E. PAYNE†

**1. Introduction.** In this paper we present a new isoperimetric lower bound for the first nonzero eigenvalue $p_2$ in the Stekloff problem [13] for a bounded convex domain $D$ in $R_2$. Such an inequality is of interest in itself and is also useful for determining a priori error bounds in the Neumann problem for second order elliptic equations (see [3]).

The first isoperimetric inequality for $p_2$ to appear in the literature was that due to Weinstock [15], i.e.,

$$p_2 \leqq 2\pi/L, \tag{1.1}$$

where $L$ is the length of $\partial D$. Lower bounds of various types (generally somewhat complicated) have been computed by Bramble and Payne [2], Kuttler and Sigillito [9], [10], [11], and Bandle [1]. Other results are due to Troesch [14] and to Hersch and Payne [5]. In this note we show that for convex domains,

$$p_2 \geqq K_{\min}, \tag{1.2}$$

where $K_{\min}$ denotes the minimum curvature of $\partial D$. We obtain, in fact, the two-sided bound

$$K_{\max} \geqq p_2 \geqq K_{\min}. \tag{1.3}$$

In § 3 we consider the eigenvalue problem characterized by

$$\nu_1 = \inf_{\Delta h = 0 \text{ in } D} \frac{\oint_{\partial D} h^2\, ds}{\int_D h^2\, dx} \tag{1.4}$$

for a convex domain $D$ with Lipschitz boundary $\partial D$. We establish the inequality

$$\nu_1 \geqq 2K_{\min}. \tag{1.5}$$

We shall henceforth refer to $\nu_1$ as the first Dirichlet eigenvalue.

**2. The Stekloff problem.** The first nonzero Stekloff eigenvalue $p_2$ is characterized as follows:

$$p_2 = \inf_{\oint_{\partial D} \phi\, ds = 0} \frac{\int_D |\text{grad } \phi|^2\, dx}{\oint_{\partial D} \phi^2\, ds}, \qquad \phi \in H_1(D). \tag{2.1}$$

* Received by the editors November 18, 1969.

† Center for Applied Mathematics, Cornell University, Ithaca, New York 14850. This work was supported in part by the National Science Foundation under Grant GP-11038 while the author was on leave at the Eidgenössische Technische Hochschule in Zürich.

It can be shown that if $\partial D$ is a Lipschitz boundary then the minimizing function $H$ exists and satisfies

$$\Delta H = 0 \quad \text{in } D,$$
$$\frac{\partial H}{\partial n} - p_2 H = 0 \quad \text{on } \partial D, \tag{2.2}$$
$$\oint_{\partial D} H\, ds = 0.$$

Here $\partial/\partial n$ denotes the outward normal derivative on $\partial D$.

Let us assume for the moment that $\partial D \in C^\infty$. Then the differential equation will be satisfied on the boundary. We now set

$$v = |\text{grad } H|^2. \tag{2.3}$$

Clearly $v$ is subharmonic in $D$ and hence $v$ takes its maximum value at a point $P$ of $\partial D$. But by Hopf's second principle [8] either

$$v \equiv \text{const.} \quad \text{in } D \tag{2.4}$$

or

$$\frac{\partial v}{\partial n}(P) > 0. \tag{2.5}$$

Since the tangential derivative of $v$ must vanish at $P$ we thus have the following two conditions at $P$ (assuming $v \not\equiv$ const. in $D$):

$$2\left[\frac{\partial H}{\partial n}\frac{\partial^2 H}{\partial n^2} + \frac{\partial H}{\partial s}\frac{\partial}{\partial s}\left(\frac{\partial H}{\partial n}\right) - K\left(\frac{\partial H}{\partial s}\right)^2\right] > 0, \tag{2.6a}$$

$$2\left[\frac{\partial H}{\partial n}\frac{\partial}{\partial s}\left(\frac{\partial H}{\partial n}\right) + \frac{\partial H}{\partial s}\frac{\partial^2 H}{\partial s^2}\right] = 0. \tag{2.6b}$$

Here (2.6a) is obtained by rewriting (2.5) and $K$ denotes the curvature at $P$. The differential equation in normal coordinates is given by

$$\frac{\partial^2 H}{\partial n^2} + K\frac{\partial H}{\partial n} + \frac{\partial^2 H}{\partial s^2} = 0 \quad \text{at } P. \tag{2.7}$$

Inserting the expression for $\partial^2 H/\partial n^2$ from (2.7) into (2.6) we obtain

$$\frac{\partial H}{\partial s}\frac{\partial}{\partial s}\left(\frac{\partial H}{\partial n}\right) - \frac{\partial H}{\partial n}\frac{\partial^2 H}{\partial s^2} - K|\text{grad } H|^2 > 0. \tag{2.8}$$

Let us assume for the moment that $\partial H/\partial s \neq 0$ at $P$. Then (2.6b) implies

$$\frac{\partial^2 H}{\partial s^2}(P) = -p_2\frac{\partial H}{\partial n}(P). \tag{2.9}$$

Insertion of (2.9) into (2.8) and use of the boundary condition in (2.2) lead to

$$(p_2 - K)|\text{grad } H|^2 > 0 \tag{2.10}$$

which implies

$$p_2 > K(P) \geqq K_{\min}. \tag{2.11}$$

Thus if $\partial D \in C^\infty$, $v \not\equiv$ const. in $D$, and $\partial H(P)/\partial s \neq 0$, the desired result has been obtained. If $\partial H(P)/\partial s = 0$, we use the fact that at $P$

$$\partial^2 v/\partial s^2 \leqq 0 \tag{2.12}$$

to obtain

$$p_2^2 H \frac{\partial^2 H}{\partial s^2} + \left(\frac{\partial^2 H}{\partial s^2}\right)^2 \leqq 0 \tag{2.13}$$

at $P$. Also in this case (2.8) becomes

$$H \frac{\partial^2 H}{\partial s^2} + K p_2 H^2 < 0. \tag{2.14}$$

Multiplying (2.14) by $p_2^2$ and adding to (2.13) we have

$$\left(\frac{\partial^2 H}{\partial s^2} + p_2^2 H\right)^2 + p_2^3(K - p_2)H^2 < 0 \tag{2.15}$$

which again implies (2.11). We must now investigate the consequence of $v \equiv$ const.

If $H$ is harmonic and the square of its gradient is constant, then $H$ must be a linear function of $x$ and $y$, i.e.,

$$H = ax + by + c, \qquad a, b, c, \text{ arbitrary constants}. \tag{2.16}$$

The boundary conditions give

$$\begin{aligned} &\frac{\partial H}{\partial n} - p_2 H \equiv a n_x + b n_y - p_2(ax + by + c) = 0 \quad \text{on } \partial D. \\ &\oint_{\partial D} (ax + by + c)\, ds = 0. \end{aligned} \tag{2.17}$$

If we now set

$$\xi = ax + by, \tag{2.18}$$

then (2.17) becomes

$$\begin{aligned} &n_\xi - p_2(\xi + c) = 0 \quad \text{in } \partial D, \\ &\oint_{\partial D} (\xi + c)\, ds = 0, \end{aligned} \tag{2.19}$$

which is clearly satisfied only for the circle (let $y = f(x)$ describe an arc of $\partial D$ and use (2.19)) in which case $c = 0$ if the origin is taken at the center of the circle. But for the circle we know that

$$p_2 = K = K_{\min}. \tag{2.20}$$

Thus we have proved that if $\partial D \in C^\infty$, $p_2 \geqq K_{\min}$. We remark now that if $\partial D$ is not in $C^\infty$ we may approximate it by $C^\infty$ curves, take the limit, and observe that the result remains valid for any Lipschitz boundary. (Note if $\partial D$ is convex, then at a corner $K$ tends to $+\infty$ so that $K_{\min}$ would not occur there. On the other hand, if the boundary has any straight-line segments, the inequality just states the obvious fact that $p_2 > 0$. Also the result would be of no interest though true if the domain were nonconvex.)

We now show that

$$p_2 \leqq K_{\max}. \tag{2.21}$$

To do this we observe that from Weinstock's inequality (1.1) and the Gauss–Bonnet formula it follows that

$$p_2 L \leqq 2\pi = \int_{\partial D} K\, ds \leqq K_{\max} L. \tag{2.22}$$

We have thus established the following theorem.

THEOREM 1. *If $D$ is a simply connected bounded domain in $R_2$ with Lipschitz boundary $\partial D$, then the first nonzero Stekloff eigenvalue $p_2$ satisfies* (1.3) *with the equality signs holding if and only if the domain is in the interior of a circle.*

The lower bound is of course of no interest unless $\partial D$ is convex.

**3. The Dirichlet eigenvalue problem.** In this section we wish to establish (1.5). It can be shown (see, e.g., Fichera [4]) that if the boundary $\partial D$ is sufficiently smooth, the eigenvalue $\nu_1$ satisfies the following system:

$$\begin{aligned} &\Delta^2 B = 0 \quad \text{in } D,\\ &B = 0 \quad \text{on } \partial D,\\ &\Delta B - \nu_1 \frac{\partial B}{\partial n} = 0 \quad \text{on } \partial D, \end{aligned} \tag{3.1}$$

and that the minimizing function $h_1$ of (1.4) and $B$ are related through the identity

$$h_1 = \Delta B. \tag{3.2}$$

(See also Bramble and Payne [2], Hersch and Payne [6].)

Let us again assume for the moment that $\partial D \in C^\infty$. Then clearly (3.1) holds. But Miranda [12] has shown that the quantity

$$W = |\text{grad } B|^2 - B\Delta B \tag{3.3}$$

assumes its maximum value on the boundary. This follows from the fact that

$$\begin{aligned} \Delta W &= 2\sum_{i,j=1}^{2}\left(\frac{\partial^2 B}{\partial x_i \partial x_j}\right)^2 - (\Delta B)^2\\ &= \left(\frac{\partial^2 B}{\partial x^2} - \frac{\partial^2 B}{\partial y^2}\right)^2 + 4\left(\frac{\partial^2 B}{\partial x \partial y}\right)^2 \geqq 0. \end{aligned} \tag{3.4}$$

Thus at the point $P_1$ on $\partial D$ where $W$ assumes its maximum value we have (if $W \not\equiv$ const. in $D$)

$$\partial W/\partial n > 0. \tag{3.5}$$

Since $B$ vanishes on $\partial D$ this expression may be written

$$2\frac{\partial B}{\partial n}\frac{\partial^2 B}{\partial n^2} - \frac{\partial B}{\partial n}\Delta B > 0 \quad \text{at } P_1. \tag{3.6}$$

However, at $P_1$

$$\begin{aligned} \Delta B &= v_1 \frac{\partial B}{\partial n}, \\ \frac{\partial^2 B}{\partial n^2} &= \Delta B - K\frac{\partial B}{\partial n} = (v_1 - K)\frac{\partial B}{\partial n}. \end{aligned} \tag{3.7}$$

Inserting these expressions into (3.6), we obtain

$$(v_1 - 2K)\left(\frac{\partial B}{\partial n}\right)^2 > 0 \quad \text{at } P_1. \tag{3.8}$$

This clearly implies

$$v_1 > 2K(P) \geqq 2K_{\min}. \tag{3.9}$$

If $W \equiv$ const. in $D$, it follows that

$$W = \left(\frac{\partial B}{\partial n}\right)^2 = \text{const.} \quad \text{on } \partial D. \tag{3.10}$$

This in turn implies by (3.1) that $\Delta B$ is constant on $\partial D$ and hence that $B$ is proportional to the torsion function in $D$. Thus it follows that if $W \equiv$ const. in $D$,

$$\begin{aligned} \Delta B &= k \quad \text{in } D, \\ B &= 0 \quad \text{on } \partial D, \\ \frac{\partial B}{\partial n} &= \frac{kA}{L} \quad \text{on } \partial D, \end{aligned} \tag{3.11}$$

where $A$ denotes the area of $D$ and $L$ the length of its perimeter.

Now if $W =$ const., then at every point of $\partial B$ it follows from the arguments of (3.5) through (3.8) with the inequality replaced by an equality sign that either

$$\partial B/\partial n = 0 \tag{3.12}$$

or

$$v_1 = 2K = \text{const.} \tag{3.13}$$

Clearly the condition (3.12) implies $B \equiv 0$ and hence we would be led to the inadmissible trivial solution. Since the only closed domain for which $K =$ const. is the circle, it then follows that the only smooth domain for which the equality sign in (1.5) can hold is the interior of a circle.

Again if $\partial D \notin C^\infty$, we may approximate $\partial D$ by $C^\infty$ curves and take the limit. We thus obtain the following theorem.

THEOREM 2. *If $D$ is a bounded two-dimensional domain with convex Lipschitz boundary $\partial D$, then the first Dirichlet eigenvalue $v_1$ satisfies* (1.5) *with equality if and only if $D$ is the interior of a circle.*

**Acknowledgment.** The author wishes to express his thanks to Professor H. F. Weinberger for helpful suggestions on the proof of Theorem 2 and to the referee for suggestions on improving the presentation.

## REFERENCES


[1] C. BANDLE, *Ueber das Stekloffsche Eigenwertproblem: Isoperimetrische Ungleichungen für symmetrische Gebiete*, ZAMP, 19 (1968), pp. 627–637.

[2] J. H. BRAMBLE AND L. E. PAYNE, *Bounds for solutions of second order elliptic partial differential equations*, Contributions to Differential Equations, 1 (1963), pp. 95–127.

[3] ———, *Bounds in the Neumann problem for second order uniformly elliptic operators*, Pacific J. Math., 12 (1962), pp. 823–833.

[4] G. FICHERA, *Su un principio de dualità per talune formole di maggiorazione relative alle equazioni differenziali*, Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat., 19 (1955), pp. 411–418.

[5] J. HERSCH AND L. E. PAYNE, *Extremal principles and isoperimetric inequalities for some mixed problems of Stekloff type*, ZAMP, 19 (1968), pp. 802–817.

[6] ———, *On one dimensional auxiliary problems and a priori inequalities*, to appear.

[7] E. HOPF, *Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus*, Sitz. d. Preuss. Akad. Wiss., 19 (1927), pp. 147–152.

[8] ———, *A remark on linear elliptic differential equations of second order*, Proc. Amer. Math. Soc., 3 (1952), pp. 791–793.

[9] J. R. KUTTLER AND V. G. SIGILLITO, *Lower bounds for Stekloff and free membrane eigenvalues*, SIAM Rev., 10 (1968), pp. 368–370.

[10] ———, *Inequalities for membrane and Stekloff eigenvalues*, J. Math. Anal. Appl., 23 (1968), pp. 148–160.

[11] ———, *An inequality for a Stekloff eigenvalue by the method of defect*, Proc. Amer. Math. Soc., 20 (1969), pp. 357–360.

[12] C. MIRANDA, *Formole di maggiorazione e teorema di esistenza per le funzioni biarmoniche di due variabili*, Giorn. Mat. Battaglini, 78 (1948–49), pp. 97–118.

[13] N. W. STEKLOFF, *Sur les problèmes fondamentaux de la physique mathématique*, Ann. Sci. École Norm. Sup., 19 (1902), pp. 455–490.

[14] B. A. TROESCH, *Free oscillations of a fluid in a container*, Boundary Problems in Differential Equations, University of Wisconsin Press, 1960, pp. 274–299.

[15] R. WEINSTOCK, *Inequalities for a classical eigenvalue problem*, J. Rational Mech. Anal., 3 (1954), pp. 745–753.

SIAM J. MATH. ANAL.
Vol. 1, No. 3, August 1970

# ON A LAPLACE INTEGRAL INVOLVING LOGARITHMS*

R. WONG†

In a recent paper, A. Erdélyi [1] has given a proof of the following result. The function

(1) $$M(z, \lambda, \mu) = \int_c^\infty t^{\lambda-1} (\log t)^\mu e^{-zt}\, dt,$$

where $\mu$ is real, $c > 1$, $\lambda > 0$, $z > 0$, has the generalized asymptotic expansion

(2) $$M(z, \lambda, \mu) \sim \sum_{n=0}^\infty \binom{\mu}{n} \Gamma^{(n)}(\lambda) z^{-\lambda}(-\log z)^{\mu-n}; \ \{z^{-\lambda}(\log z)^{\mu-n}\}$$

as $z \to 0+$, the meaning of (2) being

(3) $$M(z, \lambda, \mu) = z^{-\lambda}(-\log z)^\mu \left[ \sum_{n=0}^N \binom{\mu}{n} \Gamma^{(n)}(\lambda)(-\log z)^{-n} + o((\log z)^{-N}) \right]$$

as $z \to 0+$, for every fixed integer $N \geqq 0$.

The purpose of this note is to give an alternative proof and also to extend the result to complex values of $\lambda$, $\mu$ and $z$, restricted by $\operatorname{Re} \lambda > 0$ and $z \in S(\Delta)$. Here $S(\Delta)$ is the sector

$$|\operatorname{ph} z| \leqq \tfrac{1}{2}\pi - \Delta,$$

$\Delta$ being an arbitrary positive number.

To illustrate our procedure, we first consider a special case. The integral

$$\int_0^\infty t^{\lambda-1}(\log t)^m e^{-zt}\, dt,$$

where $m$ is a nonnegative integer and $\operatorname{Re} \lambda > 0$, converges for $\operatorname{Re} z > 0$. If $z$ is real and $z \neq 1$, the substitution $u = zt$ gives

$$\int_0^\infty t^{\lambda-1}(\log t)^m e^{-zt}\, dt = z^{-\lambda}(-\log z)^m \int_0^\infty u^{\lambda-1}\left(1 - \frac{\log u}{\log z}\right)^m e^{-u}\, du.$$

Using the binomial expansion, we have

(4) $$\int_0^\infty t^{\lambda-1}(\log t)^m e^{-zt}\, dt = \sum_{n=0}^m \binom{m}{n} \Gamma^{(n)}(\lambda) z^{-\lambda}(-\log z)^{m-n}.$$

By analytic continuation, (4) continues to hold when $z$ is complex and $z \in S(\Delta)$.

If $\operatorname{Re} \lambda > 0$ and $\operatorname{Re} \mu > -1$, then

(5) $$\int_0^c t^{\lambda-1}(\log t)^\mu e^{-zt}\, dt = O(1) \qquad \text{as } z \to 0 \text{ in } S(\Delta),$$

---

* Received by the editors December 18, 1969, and in revised form March 6, 1970.

† Department of Mathematics and Astronomy, University of Manitoba, Winnipeg, Manitoba.

for any fixed $c > 1$. Therefore, for any nonnegative integer $m$, we have, from (4),

$$M(z, \lambda, m) = \sum_{n=0}^{m} \binom{m}{n} \Gamma^{(n)}(\lambda) z^{-\lambda}(-\log z)^{m-n} + O(1) \tag{6}$$

as $z \to 0$ in $S(\Delta)$; and consequently, for every integer $N$ in $0 \leqq N \leqq m$,

$$M(z, \lambda, m) = \sum_{n=0}^{N} \binom{m}{n} \Gamma^{(n)}(\lambda) z^{-\lambda}(-\log z)^{m-n} + o(z^{-\lambda}(\log z)^{m-N}) \tag{7}$$

as $z \to 0$ in $S(\Delta)$, since the $O$-term in (6) can be included in the $o$-term in (7). This establishes (3) in case when $\mu$ is a nonnegative integer.

Although the proof of the general case is more involved, it follows essentially the same pattern. We assume throughout that

$$c > 1, \qquad \operatorname{Re} \lambda > 0, \qquad \mu \text{ arbitrary}, \qquad z \in S(\Delta). \tag{8}$$

We require the following three lemmas.

LEMMA 1. *For every fixed $\delta$ in $0 < \delta < 1$, there exists a fixed $\rho > 0$ such that if $a = a(z) = |z|^{-\delta}$ then*

$$\int_c^a t^{\lambda-1}(\log t)^{\mu} e^{-zt}\, dt = O(z^{-\lambda+\rho}(\log z)^{\mu}) \qquad \textit{as } z \to 0 \textit{ in } S(\Delta). \tag{9}$$

*Proof.* Let $\alpha = \operatorname{Re} \lambda$, $\beta = \operatorname{Re} \mu$ and $x = \operatorname{Re} z$. Since $e^{-xt} \leqq 1$ and for any $\eta > 0$ there is an $M = M(\eta)$ such that $(\log t)^{\beta} \leqq Mt^{\eta}$ for $t \geqq c$, the integral in (9) is dominated by

$$M \int_c^a t^{\alpha+\eta-1}\, dt \leqq \frac{M}{\alpha + \eta} a^{\alpha+\eta}.$$

Furthermore, $a = |z|^{-\delta}$ so that

$$M \int_c^a t^{\alpha+\eta-1}\, dt \leqq \frac{M}{\alpha + \eta} \cdot \frac{|z^{-\lambda}|}{|z^{\eta\delta - \lambda(1-\delta)}|}.$$

Since $\operatorname{Re} \lambda > 0$ and $\eta$ is arbitrary, there exists a fixed $\rho > 0$ such that

$$\int_c^a t^{\lambda-1}(\log t)^{\mu} e^{-zt}\, dt = O(z^{-\lambda+2\rho}) \qquad \text{as } z \to 0 \text{ in } S(\Delta).$$

The conclusion of the lemma now follows.

LEMMA 2. *If $b = b(z) = |z|^{-2+\delta}$, where $0 < \delta < 1$, then there exists some fixed $\varepsilon > 0$ such that*

$$\int_b^{\infty} t^{\lambda-1}(\log t)^{\mu} e^{-zt}\, dt = O(z^{-\lambda} \exp(-\varepsilon |z|^{\delta-1})) \qquad \textit{as } z \to 0 \textit{ in } S(\Delta). \tag{10}$$

*Proof.* Again we write $\alpha = \operatorname{Re} \lambda$ and $\beta = \operatorname{Re} \mu$. For any $z \in S(\Delta)$, we have $|e^{-zt}| \leqq e^{-\varepsilon'|z|t}$ for some $\varepsilon' > 0$. Furthermore, for all $t \geqq c$, $(\log t)^{\beta} \leqq Mt$ for some $M > 0$. Therefore, the integral in (10) is dominated by

$$M \int_b^{\infty} t^{\alpha} e^{-\varepsilon'|z|t}\, dt = M|z^{-\lambda-1}| \int_{b|z|}^{\infty} u^{\alpha} e^{-\varepsilon' u}\, du. \tag{11}$$

Moreover, $u^\alpha \leqq K \exp(\varepsilon' u/2)$ so that the last integral is in its turn dominated by

$$K \int_{b|z|}^{\infty} e^{-\varepsilon' u/2}\, du = \frac{2K}{\varepsilon'} \exp\left(-\frac{\varepsilon'}{2}|z|^{\delta-1}\right).$$

This, together with (11), proves Lemma 2.

LEMMA 3. *With $a = a(z)$ and $b = b(z)$ defined as above, we have, for any integer $n \geqq 0$,*

$$\int_{az}^{bz} u^{\lambda-1}(\log u)^n e^{-u}\, du = \Gamma^{(n)}(\lambda) + o((\log z)^{-N}) \tag{12}$$

*as $z \to 0$ in $S(\Delta)$, for every integer $N \geqq 0$.*

*Proof.* Using the argument of Lemma 2, it is easy to show that

$$\int_{bz}^{\infty e^{i\theta}} u^{\lambda-1}(\log u)^n e^{-u}\, du = O(\exp(-\varepsilon|z|^{\delta-1})) \tag{13}$$

as $z \to 0$ in $S(\Delta)$, where $\theta = \arg z$. Furthermore, the proof of Lemma 1 can be used to give

$$\int_{0}^{az} u^{\lambda-1}(\log u)^n e^{-u}\, du = O(z^\rho) \tag{14}$$

as $z \to 0$ in $S(\Delta)$, for some fixed $\rho > 0$. Since the $O$-term in (13) may be included in that of (14), we obtain

$$\int_{az}^{bz} u^{\lambda-1}(\log u)^n e^{-u}\, du = \int_{0}^{\infty e^{i\theta}} u^{\lambda-1}(\log u)^n e^{-u}\, du + O(z^\rho) = \Gamma^{(n)}(\lambda) + O(z^\rho)$$

as $z \to 0$ in $S(\Delta)$, for some fixed $\rho > 0$, thus establishing Lemma 3.

With the aid of these three preliminary results, we are now in a position to state and prove our principal result.

THEOREM. *For any fixed complex numbers $\lambda$ and $\mu$ with* $\operatorname{Re} \lambda > 0$,

$$M(z, \lambda, \mu) \sim z^{-\lambda}(-\log z)^\mu \left[ \sum_{n=0}^{\infty} \binom{\mu}{n} \Gamma^{(n)}(\lambda)(-\log z)^{-n};\ \{(\log z)^{-n}\} \right] \tag{15}$$

*as $z \to 0$ in $S(\Delta)$. The result holds uniformly in the approach of $z \to 0$.*

*Proof.* Set

$$M(z, \lambda, \mu) = \int_{c}^{\infty} t^{\lambda-1}(\log t)^\mu e^{-zt}\, dt = I_1(z) + I_2(z) + I_3(z),$$

where the integrals $I_1, I_2, I_3$ correspond respectively to the intervals $(c, a)$, $(a, b)$, $(b, \infty)$. Here $a = |z|^{-\delta}$ and $b = |z|^{-2+\delta}$ for some fixed $\delta$ in $0 < \delta < 1$. Estimates of $I_1$ and $I_3$ are given in Lemmas 1 and 2.

Consider the integral $I_2(z)$. Replacement of $zt$ by $u$ gives

$$I_2(z) = z^{-\lambda}(-\log z)^\mu \int_{az}^{bz} u^{\lambda-1}\left(1 - \frac{\log u}{\log z}\right)^\mu e^{-u}\, du.$$

On the path of integration, $|z|^{1-\delta} \leqq |u| \leqq |z|^{-1+\delta}$ and hence

$$\left|\frac{\log u}{\log z}\right| \leqq 1 - \delta_1$$

for some fixed $0 < \delta_1 < 1$. Therefore, for every fixed integer $N \geqq 0$,

$$\left(1 - \frac{\log u}{\log z}\right)^{\mu} = \sum_{n=0}^{N} (-1)^n \binom{\mu}{n} \frac{(\log u)^n}{(\log z)^n} + O\left(\frac{(\log u)^{N+1}}{(\log z)^{N+1}}\right)$$

as $|z| \to 0$. Since the integral

$$\int_{az}^{bz} |u^{\lambda-1}(\log u)^{N+1} e^{-u}\, du|$$

exists and is uniformly bounded as $z \to 0$ in $S(\Delta)$, one must have

$$I_2(z) = z^{-\lambda}(-\log z)^{\mu}\left[\sum_{n=0}^{N} \binom{\mu}{n} (-\log z)^{-n} \int_{az}^{bz} u^{\lambda-1}(\log u)^n e^{-u}\, du + O((\log z)^{-N-1})\right]$$

as $z \to 0$ in $S(\Delta)$. It now follows from Lemma 3 that

$$I_2(z) = z^{-\lambda}(-\log z)^{\mu}\left[\sum_{n=0}^{N} \binom{\mu}{n} \Gamma^{(n)}(\lambda)(-\log z)^{-n} + o((\log z)^{-N})\right]$$

as $z \to 0$ in $S(\Delta)$. Furthermore, by Lemmas 1 and 2 the same is true of $M(z, \lambda, \mu)$. Therefore, for every fixed integer $N \geqq 0$,

$$M(z, \lambda, \mu) = z^{-\lambda}(-\log z)^{\mu}\left[\sum_{n=0}^{N} \binom{\mu}{n} \Gamma^{(n)}(\lambda)(-\log z)^{-n} + o((\log z)^{-N})\right]$$

as $z \to 0$ in $S(\Delta)$.

Although we have not said anything about the uniformity of our results, they are indeed quite independent of the approach of $z \to 0$ in the sector $S(\Delta)$. This completes the proof of our theorem.

*Remark* 1. In view of (5), we have from (15) that if $\operatorname{Re} \lambda > 0$ and $\operatorname{Re} \mu > -1$ then

$$\int_0^{\infty} t^{\lambda-1}(\log t)^{\mu} e^{-zt}\, dt \sim z^{-\lambda}(-\log z)^{\mu}\left[\sum_{r=0}^{\infty} \binom{\mu}{r} \Gamma^{(r)}(\lambda)(-\log z)^{-r}; \quad \{(\log z)^{-r}\}\right] \tag{16}$$

as $z \to 0$ in $S(\Delta)$. It is interesting to note that this result (16) is also valid when $|z| \to \infty$. A detailed proof is given in [2].

*Remark* 2. In the case when $\mu$ is a nonnegative integer, the series (16) terminates and therefore converges. In all other cases it diverges. To show this, we remark that the power series

$$\Gamma(\lambda + z) = \sum_{n=0}^{\infty} \frac{\Gamma^{(n)}(\lambda)}{n!} z^n$$

has $|\lambda|$ as its radius of convergence. Hence, by Cauchy–Hadamard's formula,

$$\limsup_{n\to\infty} \left|\frac{\Gamma^{(n)}(\lambda)}{n!}\right|^{1/n} = \frac{1}{|\lambda|}.$$

It now follows from Stirling's formula that, for any fixed value of log $z$,

$$\limsup_{n\to\infty}\left|\frac{\Gamma^{(n)}(\lambda)}{n!}\,\frac{\Gamma(n-\mu)}{\Gamma(-\mu)}\,\frac{1}{(\log z)^n}\right|^{1/n} > 1,$$

and in consequence the expansion (16) diverges.

**Acknowledgment.** The writer wishes to thank Professor F. W. J. Olver for many suggestions, which greatly improved the presentation of this material.

## REFERENCES

[1] A. Erdélyi, *General asymptotic expansions of Laplace integrals*, Arch. Rational Mech. Anal., 7 (1961), pp. 1–20.

[2] R. Wong and M. Wyman, *A generalization of Watson's lemma*, to appear.

SIAM J. MATH. ANAL.
Vol. 1, No. 3, August 1970

# THE REAL ZEROS OF STRUVE'S FUNCTION*

J. STEINIG†

**1. Introduction.** Struve's function $\mathbf{H}_\nu(z)$ of complex order $\nu$ and complex argument $z$ is defined for all $\nu$, and for all $z$ with $-\pi < \arg z \leqq \pi$, by the expansion

$$\mathbf{H}_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{\nu+2n+1}}{\Gamma(n+3/2)\Gamma(\nu+n+3/2)}. \tag{1}$$

For $\mathrm{Re}(\nu) > -\frac{1}{2}$, it has the representation

$$\mathbf{H}_\nu(z) = \frac{2(z/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_0^1 (1-t^2)^{\nu-1/2} \sin(zt)\,dt \tag{2}$$

[14, § 10.4, (1)], which is akin to Poisson's integral [14, § 3.3, (2)] for $J_\nu(z)$, the Bessel function of the first kind, of order $\nu$: if $\mathrm{Re}(\nu) > -\frac{1}{2}$,

$$J_\nu(z) = \frac{2(z/2)^\nu}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_0^1 (1-t^2)^{\nu-1/2} \cos(zt)\,dt. \tag{3}$$

Struve's function $\mathbf{H}_\nu(z)$ is a particular solution of the nonhomogeneous differential equation

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + (z^2-\nu^2)y = \frac{4(z/2)^{\nu+1}}{\Gamma(1/2)\Gamma(\nu+1/2)} \tag{4}$$

[14, § 10.4, (10)]. Since the associated homogeneous equation is Bessel's equation, the general solution of (4) may be written as $\alpha J_\nu(z) + \beta Y_\nu(z) + \mathbf{H}_\nu(z)$, where $Y_\nu(z)$ is the Bessel function of the second kind, of order $\nu$, and $\alpha$, $\beta$ are arbitrary complex constants.

For large $|z|$, we have the asymptotic expansion

$$\mathbf{H}_\nu(z) = Y_\nu(z) + \frac{1}{\pi}\sum_{n=0}^{p-1} \frac{\Gamma(n+1/2)}{\Gamma(\nu+1/2-n)}\left(\frac{1}{2}z\right)^{\nu-2n-1} + O(|z|^{\nu-2p-1}), \tag{5}$$

valid for all $\nu$, which is obtained from the identity

$$\mathbf{H}_\nu(z) - Y_\nu(z) = \frac{(z/2)^{\nu-1}}{\Gamma(1/2)\Gamma(\nu+1/2)} \int_0^{\infty \exp(i\beta)} e^{-u}\left(1+\frac{u^2}{z^2}\right)^{\nu-1/2} du, \tag{6}$$

where $-\pi/2 < \beta < \pi/2$ and $-\pi/2 + \beta < \arg z < \pi/2 + \beta$ (see [14, § 10.41, (4)]).

Several conclusions about the behavior of $\mathbf{H}_\nu(x)$ for real $\nu$ and for $x \geqq 0$ can be drawn from (1), (5) and (6). Its behavior as $x \to 0+$ may be inferred from (1). If $\nu + 3/2$ is not zero or a negative integer, then

$$\mathbf{H}_\nu(x) \sim \frac{2(x/2)^{\nu+1}}{\Gamma(1/2)\Gamma(\nu+3/2)}, \quad x \to 0+; \tag{7}$$

if $\nu + 3/2 = -n$, $n = 0, 1, 2, \cdots$, then

$$\mathbf{H}_\nu(x) \sim \frac{(-1)^{n+1}(x/2)^{n+3/2}}{\Gamma(n+5/2)}, \quad x \to 0+.$$

* Received by the editors December 29, 1969.

† Department of Mathematics, University of Illinois, Urbana, Illinois 61801.

Consequently, $\mathbf{H}_\nu(0+) = 0$ if $\nu > -1$, or if $\nu = -n - 3/2$, $n = 0, 1, 2, \cdots$ ; and $\mathbf{H}_{-1}(0+) = 2/\pi$. Otherwise, $\mathbf{H}_\nu(0+)$ is infinite, and the sign of $\mathbf{H}_\nu(x)$ near the origin is that of $\Gamma(\nu + 3/2)$.

Since $Y_\nu(x) \to 0$ as $x \to +\infty$, it follows from (5) that as $x \to +\infty$, $\mathbf{H}_\nu(x) \to +\infty$ if $\nu > 1$, whereas $\mathbf{H}_\nu(x) \to 0$ if $\nu < 1$, and $\mathbf{H}_1(x) \to 2/\pi$.

If $\nu = -n + 1/2$, $n = 1, 2, \cdots$, then (4) reduces to Bessel's equation, and (6) shows that

$$\mathbf{H}_{-n+1/2}(x) = Y_{-n+1/2}(x) = (-1)^{n-1} J_{n-1/2}(x), \qquad n = 1, 2, \cdots. \tag{8}$$

If $\nu \leqq 1$, and $\nu \neq -n + 1/2$, $n = 1, 2, \cdots$, it follows from (6) that for $x > 0$, $\Gamma(\nu + \frac{1}{2})[\mathbf{H}_\nu(x) - Y_\nu(x)]$ is a positive, monotone decreasing function of $x$.

It is easily verified with (1) that $\mathbf{H}_\nu(z)$ satisfies the recurrence formulas

$$\frac{d}{dz}\{z^\nu \mathbf{H}_\nu(z)\} = z^\nu \mathbf{H}_{\nu-1}(z) \tag{9}$$

and

$$\frac{d}{dz}\{z^{-\nu} \mathbf{H}_\nu(z)\} = \frac{1}{2^\nu \Gamma(1/2)\Gamma(\nu + 3/2)} - z^{-\nu}\mathbf{H}_{\nu+1}(z) \tag{10}$$

[14, § 10.4]. It follows from (8) and (10) that $\mathbf{H}_\nu(z)$ can be expressed in finite form in terms of elementary functions when $2\nu$ is an odd integer, since the same is true of $J_\nu(z)$ (see[14, § 3.4]). For instance,

$$\mathbf{H}_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z, \qquad \mathbf{H}_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} (1 - \cos z).$$

Obviously, $\mathbf{H}_{1/2}(x)$ has an infinity of zeros, but no changes of sign. Some information concerning the sign of $\mathbf{H}_\nu(x)$ for real $\nu \neq \frac{1}{2}$ can be deduced from (5). As $x^{-\nu}\mathbf{H}_\nu(x)$ is an odd function of $x$, we restrict ourselves to $x > 0$. Since

$$Y_\nu(x) = \left(\frac{2}{\pi x}\right)^{1/2} [\sin(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) + O(1/x)]$$

as $x \to +\infty$, the dominant term on the right-hand side of (5) is

$$\left(\frac{2}{\pi x}\right)^{1/2} \sin(x - \tfrac{1}{2}\nu\pi - \tfrac{1}{4}\pi) \quad \text{or} \quad \frac{(x/2)^{\nu-1}}{\Gamma(1/2)\Gamma(\nu + 1/2)},$$

according as $\nu < \frac{1}{2}$ or $\nu > \frac{1}{2}$. Therefore, $\mathbf{H}_\nu(x)$ has an infinity of changes of sign if $\nu < \frac{1}{2}$, but is certainly positive for sufficiently large $x$ if $\nu > \frac{1}{2}$ (see [14, § 10.45]). In fact, more is true: when $\nu > \frac{1}{2}$, $\mathbf{H}_\nu(x) > 0$ for *all* $x > 0$. Watson gives two proofs of this property of $\mathbf{H}_\nu(x)$ (see[14, § 10.45, § 13.47, (9)]), which is also easily deduced from (2) on observing that for $\nu > \frac{1}{2}$, $(1 - t^2)^{\nu - 1/2}$ is a positive, decreasing function of $t$ in $0 < t < 1$.

Little else seems to be known about the real zeros of Struve's function for $\nu < \frac{1}{2}$. It appears to have escaped notice that a general theorem of Pólya's [10] on the zeros of a class of entire functions of the form $\int_0^1 f(t)\cos(zt)\,dt$, or

$\int_0^1 f(t)\sin(zt)\,dt$, when applied to (2) and (3), implies[1] that if $-\frac{1}{2} < \nu < \frac{1}{2}$ and if $\lambda, \mu$ are real numbers, not both zero, then the zeros of $\lambda z^{-\nu}J_\nu(z) - \mu z^{-\nu}\mathbf{H}_\nu(z)$ are all real and simple. Moreover, in this range of $\nu$, the positive zeros of $\mathbf{H}_\nu(x)$ are interlaced with those of $J_\nu(x)$, and lie one each in the intervals $(k\pi, (k+1)\pi)$, $k = 1, 2, \cdots$. In other words if $j_{\nu,r}$ and $h_{\nu,r}$ denote the $r$th positive zero of $J_\nu(x)$ and of $\mathbf{H}_\nu(x)$, respectively, then, for $-\frac{1}{2} < \nu < \frac{1}{2}$,

(11) $j_{\nu,2r-1} < (2r-1)\pi < h_{\nu,2r-1} < j_{\nu,2r} < 2r\pi < h_{\nu,2r} < j_{\nu,2r+1}, \quad r = 1, 2, \cdots.$

Pólya's proof uses Hurwitz's theorem on the zeros of the uniform limit of a sequence of analytic functions [8, § 1.1] and the Eneström–Kakeya theorem for polynomials [8, § 7.30].

In this paper we shall establish certain properties of the zeros of $\mathbf{H}_\nu(x)$ which are suggested by Pólya's results or by familiar properties of solutions of Bessel's equation. We shall show that for $\nu < \frac{1}{2}$, the positive zeros of $\mathbf{H}_\nu(x)$ are simple, and separate those of $Y_\nu(x)$ in pairs, if $\nu \neq -n + \frac{1}{2}$, $n = 1, 2, \cdots$; that the positive zeros of $J_\nu(x)$ are interlaced with those of $\mathbf{H}_\nu(x)$; and that the positive zeros of $\mathbf{H}_{\nu-1}(x)$ alternate with those of $\mathbf{H}_\nu(x)$. Furthermore, if $\lambda$, $\mu$ are real, and $\lambda\mu \neq 0$, then all positive zeros of $\lambda J_\nu(x) - \mu\mathbf{H}_\nu(x)$ are simple, with the possible exception of the first one. This zero is simple if $|\nu| < \frac{1}{2}$, or if $-n \leqq \nu \leqq -n + \frac{1}{2}, n = 1, 2, \cdots$. For each other $\nu < -\frac{1}{2}$, there is a unique choice of $\lambda/\mu$ for which the first positive zero of $\lambda J_\nu(x) - \mu\mathbf{H}_\nu(x)$ is of multiplicity 2.

We begin by proving these results for $|\nu| < \frac{1}{2}$ in § 3, and then in § 4 extend them to smaller values of $\nu$ by induction arguments. For $|\nu| < \frac{1}{2}$ we obtain the following chain of inequalities, which subsumes (11):

$$y_{\nu,2r-1} < j_{\nu,2r-1} < y_{\nu,2r} < h_{\nu,2r-1} \leqq 2j_{\nu,r} < j_{\nu,2r} < 2r\pi < h_{\nu,2r} < y_{\nu,2r+1},$$

$r = 1, 2, \cdots$; here $y_{\nu,r}$ denotes the $r$th positive zero of $Y_\nu(x)$.

Our method of proof is different from Pólya's. In the range $|\nu| < \frac{1}{2}$, we appeal to a theorem of ours on the zeros of certain fractional integrals (Theorem A) and to a comparison theorem for nonhomogeneous linear second order differential equations (Theorem B). To extend our results to smaller values of $\nu$, we apply G. D. Birkhoff's separation theorem (Theorem C) to a third order homogeneous linear differential equation which is obtained from (4) by a standard device [1, § 1.7]. In § 5, we indicate another approach to the main results of § 3, which rests on the inequality

$$\int_0^\xi x^\nu J_\nu(x)\,dx > 0, \qquad \xi > 0, \quad -\tfrac{1}{2} < \nu < \tfrac{1}{2}.$$

It should be noted that our method, which avoids the use of the complex variable, does not yield the result, implicit in Pólya's theorem, that $\mathbf{H}_\nu(z)$ has only real zeros for $-\frac{1}{2} < \nu < \frac{1}{2}$.

---

[1] The second example on p. 363 of [10] is $\frac{1}{2}\pi\mathbf{H}_0(z)$.

**2. Preliminaries.** In § 3 we shall apply a result of ours on fractional integrals [11], which we state as

THEOREM A. *Suppose that g is real-valued, and is continuous on every finite interval* $(\alpha, A)$, *where* $\alpha$ *is fixed, and that g is not identically zero in any interval* $(A, \infty)$. *Let* $0 < \mu < 1$, *and let*

$$g_\mu(x) = \frac{1}{\Gamma(\mu)} \int_x^\infty g(t)(t-x)^{\mu-1}\,dt, \qquad x \geqq \alpha,$$

*the integral being supposed convergent at the upper limit. If* $g_\mu$ *attains its maximum and its minimum on* $[a, \infty]$ *at* $x_1, x_2$ *respectively, where* $\alpha \leqq a \leqq x_1 < x_2 < \infty$, *then g has an odd number of changes of sign in* $(x_1, x_2]$, *unless it has infinitely many. A similar result holds if* $a \leqq x_2 < x_1$.

In applying Theorem A, we shall require

LEMMA 1. *If* $\nu > -\frac{1}{2}$, *the successive relative maxima of* $x^{-\nu/2}|J_\nu(x^{1/2})|$, *as x increases from* 0 *to* $\infty$, *form a decreasing sequence.*

*Proof.* Apply the Sonine–Pólya theorem [13, § 7.31] to the differential equation $(x^{\nu+1}y')' + \frac{1}{4}x^\nu y = 0$, which has $x^{-\nu/2}J_\nu(x^{1/2})$ as a particular solution [14, § 4.31, (19)–(20)].

The next result is a comparison theorem for two linear differential equations of the second order, one of which may be nonhomogeneous.

THEOREM B. *Let* $f_1, f_2$ *and F be continuous on* $(a, b)$ *and let* $f_1 \geqq f_2$ *and* $F \geqq 0$ *throughout this interval. Let* $u_1$ *and* $u_2$ *be nontrivial solutions of the differential equations* $u_1'' + f_1 u_1 = 0$ *and* $u_2'' + f_2 u_2 = F$. *If* $x'$ *and* $x''$ *are consecutive zeros of* $u_2$ *in* $(a, b)$, *if* $u_2(x) > 0$ *for* $x' < x < x''$, *and if either* $f_1 \not\equiv f_2$ *or* $F \not\equiv 0$ *on* $(x', x'')$, *then* $u_1$ *changes sign in* $(x', x'')$. *The same conclusion holds for* $x' = a$, *if* $u_2(a+0) = 0$, *on condition that*

$$\lim_{x \to a+0} \{u_1 u_2' - u_1' u_2\} = 0. \tag{12}$$

*An analogous result holds if* $F \leqq 0$ *throughout* $(a, b)$ *and* $u_2(x) < 0$ *for* $x' < x < x''$.

*Proof.* Use the identity

$$[u_1 u_2' - u_1' u_2]_{x'}^{x''} = \int_{x'}^{x''} u_1 u_2 (f_1 - f_2)\,dx + \int_{x'}^{x''} u_1 F\,dx,$$

and proceed as in the proof of Sturm's comparison theorem.[2]

With Theorem B, we can prove

LEMMA 2. *If* $\nu < \frac{1}{2}$, *but* $\nu \neq -n + \frac{1}{2}$, $n = 1, 2, \cdots$, *and if* $x'$ *and* $x''$ *are consecutive positive zeros of* $\mathbf{H}_\nu(x)$, *such that* $\Gamma(\nu + \frac{1}{2})\mathbf{H}_\nu(x) > 0$ *for* $x' < x < x''$, *then* $Y_\nu(x)$ *has an even number of zeros (at least two) in* $(x', x'')$. *And if* $-\frac{1}{2} < \nu < \frac{1}{2}$, $Y_\nu(x)$ *has an even number* $(\geqq 2)$ *of zeros in* $(0, h_{\nu,1})$.

*Proof.* The substitution $y = x^{-1/2}u$ in Bessel's equation and in (4) shows that $u_1(x) = x^{1/2}Y_\nu(x)$ and $u_2(x) = x^{1/2}\mathbf{H}_\nu(x)$ are solutions of $x^2u_1'' + (x^2 - \nu + \frac{1}{4})u_1 = 0$ and $x^2u_2'' + (x^2 - \nu + \frac{1}{4})u_2 = 2^{1-\nu}x^{\nu-1/2}/(\Gamma(\frac{1}{2})\Gamma(\nu + \frac{1}{2}))$. Then, by Theorem B, $Y_\nu(x)$ vanishes at least once in $(x', x'')$. But $\Gamma(\nu + \frac{1}{2})[\mathbf{H}_\nu(x) - Y_\nu(x)] > 0$, by (6);

[2] If $F \equiv 0$, Theorem B reduces to a form of Sturm's theorem given by Szegö [13, § 1.82].

hence $Y_\nu(x)$ must have an even number of zeros in $(x', x'')$. For $|\nu| < \frac{1}{2}$, $\mathbf{H}_\nu(0) = 0$, and $\Gamma(\nu + \frac{1}{2})\mathbf{H}_\nu(x) > 0$ for $0 < x < h_{\nu,1}$, by (7). Here (12) is satisfied, not by $u_1(x) = x^{1/2}Y_\nu(x)$, but by $u_1(x) = x^{1/2}J_\nu(x)$; it follows that $j_{\nu,1} < h_{\nu,1}$. Since[3] $y_{\nu,1} < j_{\nu,1}$ for $\nu > -\frac{1}{2}$, we can conclude as above.

In § 4, we shall need the following separation theorem of G. D. Birkhoff [2], [12, Theorem 4.41].

THEOREM C (G. D. Birkhoff). *Let $u_1, u_2, u_3$ be linearly independent solutions of the differential equation $u''' + a_1(x)u'' + a_2(x)u' + a_3(x)u = 0$, where $a_n$ is a real-valued, continuous function of class $C^{3-n}(a, b)$, $n = 1, 2, 3$. Let*

$$z_1 = u_2u_3' - u_2'u_3, \qquad z_2 = u_3u_1' - u_3'u_1, \qquad z_3 = u_1u_2' - u_1'u_2.$$

*If $(i, j, k)$ is any permutation of $(1, 2, 3)$, then between any two consecutive zeros of $u_i$ (or of $z_i$) in $(a, b)$, the number of zeros of $u_j$ plus the number of zeros of $z_k$ is odd, multiple zeros being counted according to their multiplicity.*

*Remark* 2.1. A zero of $z_k$ which is not a zero of $u_i$, $i \neq k$, is simple. Indeed, if $z_k = u_iu_j' - u_i'u_j$ and $z_k' = u_iu_j'' - u_i''u_j$ both vanished at $x_0$, and if $u_i(x_0) \neq 0$, then we would have $u_i'u_j'' - u_i''u_j' = 0$ at $x_0$. But then the Wronskian of $u_1, u_2, u_3$ would vanish at $x_0$, which is impossible, since these solutions are linearly independent.

Later, Birkhoff's theorem will be combined with

LEMMA 3 (see [1, §3.5]). *If $\nu \neq -n + \frac{1}{2}$, $n = 1, 2, \cdots$, then $J_\nu(x)$, $Y_\nu(x)$ and $\mathbf{H}_\nu(x)$ are linearly independent solutions of the differential equation*

$$z^3y''' + (2 - \nu)z^2y'' + [z^3 - \nu(\nu + 1)z]y' + [(1 - \nu)z^2 + \nu^2(\nu + 1)]y = 0. \tag{13}$$

From results of Watson [14, § 15.33] and the fact that the zeros of $J_\nu(x)$ and $Y_\nu(x)$ interlace [14, § 15.24] we deduce

LEMMA 4. *If $-\frac{1}{2} < \nu < \frac{1}{2}$, the positive zeros of $J_\nu(x)$ lie in the intervals $(m\pi + \frac{3}{4}\pi + \frac{1}{2}\nu\pi,\ m\pi + \frac{7}{8}\pi + \frac{1}{4}\nu\pi)$, $m = 0, 1, 2, \cdots$, one to each interval. The positive zeros of $Y_\nu(x)$ lie one each in the intervals $(m\pi + \frac{1}{4}\pi + \frac{1}{2}\nu\pi, m\pi + \frac{3}{8}\pi + \frac{1}{4}\nu\pi)$, $m = 0, 1, 2, \cdots$.*

**3. The results for** $-\frac{1}{2} < \nu < \frac{1}{2}$. We begin by establishing

PROPOSITION 1. *For $|\nu| < \frac{1}{2}$ and $x > 0$, the changes of sign of $\mathbf{H}_\nu(x)$ occur singly in the intervals $(m\pi, (m + 1)\pi)$, $m = 1, 2, \cdots$.*

*Proof.* By (2), $\mathbf{H}_\nu(x) = \phi_\nu(x) \int_0^1 f_\nu(t) \sin(xt)\, dt$, where $f_\nu(t) = (1 - t^2)^{\nu - 1/2}$ and $\phi_\nu(x) = 2(\frac{1}{2}x)^\nu/(\Gamma(\frac{1}{2})\Gamma(\nu + \frac{1}{2}))$. For $\nu > -\frac{1}{2}$ and $x > 0$, $\phi_\nu(x) > 0$; for $|\nu| < \frac{1}{2}$ and $0 \leqq t < 1$, $f_\nu(t)$ is a continuous, positive, strictly increasing function of $t$. Obviously, $\mathbf{H}_\nu(x) > 0$ for $0 < x \leqq \pi$. Furthermore, $\mathbf{H}_\nu((2m + 1)\pi) > 0$, $m = 1, 2, \cdots$, as can be seen by expressing the corresponding integral as a sum of integrals over half-periods of $\sin((2m + 1)\pi t)$. This sum has terms of alternating sign and increasing

---

[3] For $\nu > 0$, see [14, § 15.3, (10)]. For $-\frac{1}{2} < \nu \leqq 0$, note that $J_\nu(x) \to +\infty$, $Y_\nu(x) \to -\infty$, as $x \to 0+$. Then, since $J_\nu(x)Y_\nu'(x) - J_\nu'(x)Y_\nu(x) > 0$ (see [14, § 3.63, (1)]), we have $Y_\nu(j_{\nu,1}) > 0$, whence $y_{\nu,1} < j_{\nu,1}$. Similar reasoning shows that $j_{\nu,1} < y_{\nu,1}$ for $-n < \nu \leqq -n + \frac{1}{2}$, and $y_{\nu,1} < j_{\nu,1}$ for $-n - \frac{1}{2} < \nu \leqq -n$, $n = 1, 2, \cdots$.

absolute value. Hence, it has the sign of its last term, which is positive, since the number of terms is odd. The same argument will show that $\mathbf{H}_\nu(2m\pi) < 0$, $m = 1, 2, \cdots$. Therefore, since $\mathbf{H}_\nu \in C^\infty(0, \infty)$, it has an odd number of changes of sign in each of the intervals $I_m = (m\pi, (m+1)\pi)$, $m = 1, 2, \cdots$. In fact, it has exactly one, for otherwise, by Lemma 2, $Y_\nu(x)$ would have two zeros in $I_m$, in contradiction with Lemma 4.

With this result, we can prove

PROPOSITION 2. *For $-\frac{1}{2} < \nu < \frac{1}{2}$, the positive zeros of $H_\nu(x)$ are simple.*

*Proof.* Since $\mathbf{H}_\nu$ is a solution of (4), its positive zeros are of multiplicity at most 2. As the right-hand side of (4) is positive for real, positive $z$ and $-\frac{1}{2} < \nu < \frac{1}{2}$, $\mathbf{H}_\nu(x)$ would have to be positive in the neighborhood of a double zero. Thus if $\mathbf{H}_\nu(x)$ had a double zero in some interval $I'_m = [m\pi, (m+1)\pi]$, $m \geqq 1$, it would have two zeros in $I'_m$, by Proposition 1, and it would be positive between them. But then, by Lemma 2, $Y_\nu(x)$ would also have two zeros in $I'_m$, in contradiction with Lemma 4.

Propositions 1 and 2 and Lemma 4 lead to

PROPOSITION 3. *For $|\nu| < \frac{1}{2}$ and $x \geqq 0$, the zeros of $H_\nu(x)$ alternate in pairs with those of $Y_\nu(x)$, with two zeros of $Y_\nu(x)$ in each interval bounded by consecutive zeros of $\mathbf{H}_\nu(x)$, on which $\mathbf{H}_\nu(x) > 0$.*

*Proof.* If $|\nu| < \frac{1}{2}$, $\mathbf{H}_\nu(x) > 0$ for $0 < x < h_{\nu,1}$, by (7). Hence, by Proposition 2, $\mathbf{H}_\nu(x) > 0$ on the intervals $(h_{\nu,2r}, h_{\nu,2r+1})$, $r \geqq 0$ $(h_{\nu,0} = 0)$, and $\mathbf{H}_\nu(x) \leqq 0$ elsewhere. Since $\mathbf{H}_\nu(x) - Y_\nu(x) > 0$ for $|\nu| < \frac{1}{2}$, $Y_\nu(x) \neq 0$ for $h_{\nu,2r-1} \leqq x \leqq h_{\nu,2r}$. By Lemma 2, the intervals $(0, h_{\nu,1})$ and $(h_{\nu,2r}, h_{\nu,2r+1})$ each contain an even number of zeros of $Y_\nu(x)$. But by Proposition 1, $h_{\nu,2r+1} - h_{\nu,2r} < 2\pi$. And by Lemma 4, $Y_\nu(x)$ cannot have 4 zeros in an interval of length $2\pi$.

From Theorem A, Proposition 1 and Lemma 1, we deduce

PROPOSITION 4. *If $|\nu| < \frac{1}{2}$, each half-open interval $(j_{\nu,r}, j_{\nu,r+1}]$ bounded by consecutive positive zeros of $J_\nu(x)$ contains exactly two zeros of $\mathbf{H}_\nu(2x)$. And there is exactly one zero of $\mathbf{H}_\nu(2x)$ in $(0, j_{\nu,1}]$.*

*Proof.* We apply Theorem A to Meijer's [9, p. 141] fractional integral

$$(14)\ \frac{1}{\Gamma(\mu)} \int_y^\infty x^{\mu/2-3/4} \mathbf{H}_{1/2-\mu}(x^{1/2})(x-y)^{\mu-1}\, dx = \pi^{1/2}(2y)^{\mu-1/2}\left[J_{1/2-\mu}\left(\frac{1}{2}y^{1/2}\right)\right]^2,$$

$y > 0, 0 < \mu < 1$.

By Lemma 1, the successive relative maxima of the right-hand side of (14) form a decreasing sequence if $\mu < 1$. Also, the stationary values of $x^{-\nu}J_\nu(x)$, $x > 0$, are the positive zeros of $J_{\nu+1}(x)$, of which there is precisely one between two consecutive positive zeros of $J_\nu(x)$ (see [14, § 15.22]). For $\nu > -1$, the first positive zero of $J_\nu(x)$ is nearer the origin than that of $J_{\nu+1}(x)$, by a theorem of Bôcher's [3, Theorem IX].

It now follows from Theorem A that each of the intervals $(0, j_{\nu,1}]$, $(j_{\nu,r}, j_{\nu+1,r}]$ and $(j_{\nu+1,r}, j_{\nu,r+1}]$, $r \geqq 1$, contains an odd number of zeros of $\mathbf{H}_\nu(2x)$. Thus each interval $(j_{\nu,r}, j_{\nu,r+1}]$ contains an even number ($\geqq 2$) of zeros of $\mathbf{H}_\nu(2x)$. But if one of these intervals contained four zeros of $\mathbf{H}_\nu(2x)$, its length would exceed $\pi$, by Proposition 1. This is impossible, by Lemma 4. Similarly, $(0, j_{\nu,1}]$ contains exactly one zero of $\mathbf{H}_\nu(2x)$.

We are now in a position to prove

PROPOSITION 5. *If* $|\nu| < \frac{1}{2}$, *there is exactly one zero of* $J_\nu(x)$ *between two consecutive positive zeros of* $\mathbf{H}_\nu(x)$, *and exactly one in the interval* $(0, h_{\nu,1})$.

*Proof.* From Propositions 1, 3 and 4 and Lemma 4 we infer that for $r = 1, 2, \cdots$,

$$y_{\nu,2r-1} < y_{\nu,2r} < h_{\nu,2r-1} \leqq 2j_{\nu,r} < 2\pi r < h_{\nu,2r} < y_{\nu,2r+1} < y_{\nu,2r+2}.$$

The zeros of $J_\nu(x)$ and $Y_\nu(x)$ are interlaced, and $j_{\nu,1} > y_{\nu,1}$, whence $y_{\nu,2r-1} < j_{\nu,2r-1} < y_{\nu,2r}$. It is easily deduced from Lemma 4 that $2j_{\nu,r} < j_{\nu,2r} < 2\pi r$. Hence, for $|\nu| < \frac{1}{2}$ and $r = 1, 2, \cdots$, we have

$$(15)\quad y_{\nu,2r-1} < j_{\nu,2r-1} < y_{\nu,2r} < h_{\nu,2r-1} \leqq 2j_{\nu,r} < j_{\nu,2r} < 2\pi r < h_{\nu,2r} < y_{\nu,2r+1}.$$

Lemma 4 and (15) give the following refinement of Proposition 1.

COROLLARY 3.1. *If* $|\nu| < \frac{1}{2}$, *the positive zeros of* $\mathbf{H}_\nu(x)$ *lie in the intervals* $(2m\pi - \frac{3}{4}\pi + \frac{1}{2}\nu\pi, 2m\pi - \frac{1}{4}\pi + \frac{1}{2}\nu\pi)$ *and* $(2m\pi, 2m\pi + \frac{3}{8}\pi + \frac{1}{4}\nu\pi)$, *one to each interval* $(m = 1, 2, \cdots)$.

**4. Results for** $\nu < \frac{1}{2}$. A simple induction argument, based on Lemma 2, yields the following extensions of Propositions 2 and 3.

THEOREM 1. *For* $\nu < \frac{1}{2}$, *the positive zeros of* $\mathbf{H}_\nu(x)$ *are simple.*

THEOREM 2. *For* $\nu < \frac{1}{2}$, $\nu \neq -n + \frac{1}{2}$, $n = 1, 2, \cdots$, *and* $x \geqq 0$, *the zeros of* $\mathbf{H}_\nu(x)$ *alternate in pairs with those of* $Y_\nu(x)$, *two zeros of* $Y_\nu(x)$ *lying in each interval bounded by consecutive zeros of* $\mathbf{H}_\nu(x)$, *on which* $\Gamma(\nu + \frac{1}{2})\mathbf{H}_\nu(x) > 0$.

*Proof.* The case $|\nu| < \frac{1}{2}$ of both theorems is disposed of by Propositions 2 and 3. For $\nu = -n + \frac{1}{2}$, $n = 1, 2, \cdots$, Theorem 1 asserts a familiar property of $Y_\nu(x)$ (see[14, § 15.21]).

We shall complete the proof by showing that the validity of either of Theorems 1 and 2 for some $\nu < -\frac{1}{2}$, $\nu \neq -n + \frac{1}{2}$, is implied by the validity of both for $\nu + 1$.

First, we observe that as $\mathbf{H}_\nu$ is a solution of (4), we would have $\Gamma(\nu + \frac{1}{2})\mathbf{H}_\nu(x) > 0$ in a neighborhood of a double zero. Therefore, $h_{\nu,1}$ is simple if $\nu < -\frac{1}{2}$, for then $\Gamma(\nu + \frac{1}{2})\mathbf{H}_\nu(x) < 0$ on $(0, h_{\nu,1})$, by (7). And if $h_{\nu,r}$, $r > 1$, were a double zero, $(h_{\nu,r-1}, h_{\nu,r+1})$ would contain four zeros of $Y_\nu(x)$, by Lemma 2, hence three zeros of $Y_{\nu+1}(x)$ (see [14, § 15.22]), and therefore, by the induction hypothesis, two zeros of $\mathbf{H}_{\nu+1}(x)$. But then, in virtue of Rolle's theorem, applied to (9), $\mathbf{H}_\nu$ would change sign in $(h_{\nu,r-1}, h_{\nu,r+1})$, which is absurd.

Second, $Y_\nu(x) \neq 0$ if $\Gamma(\nu + \frac{1}{2})\mathbf{H}_\nu(x) < 0$, since $\Gamma(\nu + \frac{1}{2})[\mathbf{H}_\nu(x) - Y_\nu(x)] > 0$. Accordingly, let $h'$, $h''$ be positive zeros of $\mathbf{H}_\nu(x)$, such that $\Gamma(\nu + \frac{1}{2})\mathbf{H}_\nu(x) > 0$ for $h' < x < h''$. By Lemma 2, $Y_\nu(x)$ has an even number of zeros in $(h', h'')$. If it had four, $(h', h'')$ would contain three zeros of $Y_{\nu+1}(x)$, hence two of $\mathbf{H}_{\nu+1}(x)$, by the induction hypothesis. But again, this is impossible, for it would imply that $\mathbf{H}_\nu(x)$ changes sign in $(h', h'')$.

This completes the proof of Theorems 1 and 2.

It follows at once from this proof that for $\nu \leqq -\frac{1}{2}$,

$$(16)\qquad h_{\nu,2r-1} \leqq y_{\nu,2r-1} < y_{\nu,2r} \leqq h_{\nu,2r}, \qquad r = 1, 2, \cdots,$$

with equality if and only if $\nu = -n + \frac{1}{2}$, $n = 1, 2, \cdots$.

We now give a common proof for the following two results.

THEOREM 3. *If* $\nu < \frac{1}{2}$, *the positive zeros of* $\mathbf{H}_\nu(x)$ *and of* $J_\nu(x)$ *separate one another.*

THEOREM 4. *If* $\nu < \frac{1}{2}$, *then* $J_\nu(x)\,\mathbf{H}'_\nu(x) - J'_\nu(x)\,\mathbf{H}_\nu(x) \neq 0$ *for* $x \geqq j_{\nu,1}$, *and also for* $0 < x \leqq j_{\nu,1}$, *if* $|\nu| < \frac{1}{2}$.

*Proof.* We first show that the validity of Theorem 4, for a given $\nu < \frac{1}{2}$, $\nu \neq -n + \frac{1}{2}$, implies that of Theorem 3 for $\nu - 1$.

Since $j_{\nu-1,1} < y_{\nu-1,2}$, we infer from (16) that $j_{\nu-1,1} < h_{\nu-1,2}$; hence we have only to show that each interval $(j_{\nu-1,r}, j_{\nu-1,r+1})$ contains exactly one zero of $\mathbf{H}_{\nu-1}(x)$. From (9) and the cognate relation for $J_\nu(x)$ (see[14, § 3.2]) we get

$$J_\nu(x)\mathbf{H}'_\nu(x) - J'_\nu(x)\mathbf{H}_\nu(x) = J_\nu(x)\mathbf{H}_{\nu-1}(x) - J_{\nu-1}(x)\mathbf{H}_\nu(x). \tag{17}$$

Thus our hypothesis on $\nu$ implies that $J_\nu(x)\mathbf{H}_{\nu-1}(x) - J_{\nu-1}(x)\mathbf{H}_\nu(x) \neq 0$ for $x \geqq j_{\nu-1,1}$, since $j_{\nu-1,1} > j_{\nu,1}$ if $\nu \leqq 0$, by Bôcher's theorem [3, Theorem IX]. It is now clear that $\mathbf{H}_{\nu-1}(x)$ and $J_{\nu-1}(x)$ have no common zero. Moreover, since the positive zeros of $J_\nu(x)$ are simple and alternate with those of $J_{\nu-1}(x)$, the continuity and nonvanishing of $J_\nu(x)\mathbf{H}_{\nu-1}(x) - J_{\nu-1}(x)\mathbf{H}_\nu(x)$ for $x \geqq j_{\nu-1,1}$ imply that $\mathbf{H}_{\nu-1}(x)$ has an odd number of zeros between two consecutive positive zeros of $J_{\nu-1}(x)$. But it cannot have three, for then, by Theorem 2, $Y_{\nu-1}(x)$ would have two zeros between two consecutive zeros of $J_{\nu-1}(x)$, which is impossible.

Next, we show that the validity of Theorem 3 for some $\nu < \frac{1}{2}$, $\nu \neq -n + \frac{1}{2}$, implies that of Theorem 4 for the same $\nu$. We apply Theorem C to (13) and deduce that in any interval $(j_{\nu,r}, j_{\nu,r+1})$, $\mathbf{H}_\nu(x)$ and $J_\nu(x)\mathbf{H}'_\nu(x) - J'_\nu(x)\mathbf{H}_\nu(x)$ have together an odd number of zeros, if each is counted according to its multiplicity. With our hypothesis on $\nu$, Theorem 1 and Remark 2.1, this implies that $J_\nu(x)\mathbf{H}'_\nu(x) - J'_\nu(x)\mathbf{H}_\nu(x)$ has an even number of zeros, all simple, in $(j_{\nu,r}, j_{\nu,r+1})$. But if it had two zeros there, we could apply Theorem C again: between two consecutive zeros of $J_\nu(x)\mathbf{H}'_\nu(x) - J'_\nu(x)\mathbf{H}_\nu(x)$, the number of zeros of $J_\nu(x)$ plus the number of zeros of $J_\nu(x)Y'_\nu(x) - J'_\nu(x)Y_\nu(x)$ is odd. Now $J_\nu(x) \neq 0$ in $(j_{\nu,r}, j_{\nu,r+1})$; and $J_\nu(x)Y'_\nu(x) - J'_\nu(x)Y_\nu(x)$ never vanishes, since it is the Wronskian of two linearly independent solutions of Bessel's equation. Finally, $J_\nu(x)$ and $J_\nu(x)\mathbf{H}'_\nu(x) - J'_\nu(x)\mathbf{H}_\nu(x)$ have no common zero, since all zeros of $J_\nu(x)$ are simple. Hence $J_\nu(x)\mathbf{H}'_\nu(x) - J'_\nu(x)\mathbf{H}_\nu(x) \neq 0$ for $x \geqq j_{\nu,1}$. And if $|\nu| < \frac{1}{2}$, it is easily verified with (7) and (17) that $J_\nu(x)\mathbf{H}'_\nu(x) - J'_\nu(x)\mathbf{H}_\nu(x) > 0$ both at $j_{\nu,1}$ and in some sufficiently small interval $(0, \varepsilon)$.

It now follows by induction from Proposition 5 that Theorems 3 and 4 hold for $|\nu + n| < \frac{1}{2}, n = 0, 1, 2, \cdots$. And for $\nu = -n + \frac{1}{2}, n = 1, 2, \cdots$, these theorems assert familiar properties of solutions of Bessel's equation. This completes our proof.

From (16) and Theorem 3 we deduce that for $-n - \frac{1}{2} < \nu \leqq -n, n = 1, 2, \cdots$,

$$h_{\nu,2r-1} < y_{\nu,2r-1} < j_{\nu,2r-1} < y_{\nu,2r} < h_{\nu,2r} < j_{\nu,2r} < h_{\nu,2r+1}, \tag{18}$$

$r = 1, 2, \cdots$, since $y_{\nu,1} < j_{\nu,1}$ in this case.[3] Similarly, for $-n < \nu < -n + \frac{1}{2}$, we have

$$j_{\nu,2r-1} < h_{\nu,2r-1} < y_{\nu,2r-1} < j_{\nu,2r} < y_{\nu,2r} < h_{\nu,2r} < j_{\nu,2r+1}. \tag{19}$$

With this remark, we can complete Theorem 4 as follows.

THEOREM 4′. *If* $|\nu| < \frac{1}{2}$, *or if* $-n \leqq \nu \leqq -n + \frac{1}{2}$, $n = 1, 2, \cdots$, *then* $J_\nu(x)\mathbf{H}_\nu'(x) - J_\nu'(x)\mathbf{H}_\nu(x) > 0$ *for* $x > 0$. *If* $-n - \frac{1}{2} < \nu < -n$, *then* $J_\nu(x)\mathbf{H}_\nu'(x) - J_\nu'(x)\mathbf{H}_\nu(x) > 0$ *for* $x \geqq h_{\nu,1}$, *and this function has exactly one zero, which is simple, in* $(0, h_{\nu,1})$.

*Proof.* For $\nu = -n + \frac{1}{2}$, the theorem asserts a familiar property of the Wronskian of $J_\nu(x)$ and $Y_\nu(x)$ (see [14, § 3.63, (1)]).

From (7) and the corresponding relation for $J_\nu(x)$, we can determine the sign of $\mathbf{H}_\nu'(h_{\nu,1})$, and of $J_\nu'(j_{\nu,1})$. By combining this information with (15), (18) and (19), it is not difficult to see that at $x = h_{\nu,1}$, $J_\nu(x)\mathbf{H}_\nu'(x) - J_\nu'(x)\mathbf{H}_\nu(x) > 0$ for all $\nu < \frac{1}{2}$.

Furthermore, it follows from (17), using (7), that as $x \to 0+$, $J_\nu(x)\mathbf{H}_\nu'(x) - J_\nu'(x)\mathbf{H}_\nu(x) \sim (\frac{1}{2}x)^{2\nu}/(\Gamma(\frac{1}{2})\Gamma(\nu + 1)\Gamma(\nu + \frac{3}{2}))$, if $\nu \neq -n - \frac{1}{2}$, $\nu \neq -n \, (n = 1, 2, \cdots)$, and that if $\nu = -n \, (n = 1, 2, \cdots)$, this function is $\sim 2(-1)^\nu/(\Gamma(\frac{1}{2})\Gamma(1 - \nu)\Gamma(\nu + \frac{1}{2}))$. Hence, $J_\nu(x)\mathbf{H}_\nu'(x) - J_\nu'(x)\mathbf{H}_\nu(x) < 0$ near $x = 0$ if and only if $-n - \frac{1}{2} < \nu < -n$, $n = 1, 2, \cdots$. The theorem then follows, if we remember from the proof of Theorem 4 that $J_\nu(x)\mathbf{H}_\nu'(x) - J_\nu'(x)\mathbf{H}_\nu(x)$ can have at most one (simple) zero.

In analogy to Bôcher's theorem, we have

PROPOSITION 6. *If* $|\nu| < \frac{1}{2}$, *the smallest positive zero of* $\mathbf{H}_{\nu-1}(x)$ *is nearer the origin than that of* $\mathbf{H}_\nu(x)$. *The situation is reversed if* $\nu \leqq -\frac{1}{2}$.

*Proof.* The case $\nu = -n + \frac{1}{2}$, $n = 1, 2, \cdots$, is covered by Bôcher's theorem [3, Theorem IX], which implies that $y_{\nu-1,1} < y_{\nu,1}$ if $\nu > 0$, and $y_{\nu-1,1} > y_{\nu,1}$ if $\nu \leqq 0$.

If $|\nu| < \frac{1}{2}$, it follows from (7), (9) and Rolle's theorem that $\mathbf{H}_{\nu-1}(x)$ changes sign in $(0, h_{\nu,1})$. Hence $h_{\nu-1,1} < h_{\nu,1}$ in this case.

For $\nu < -\frac{1}{2}$, $\nu \neq -n + \frac{1}{2}$, we use (10) and Rolle's theorem. Together with (7), they imply that $\Gamma(\nu + \frac{1}{2})\mathbf{H}_\nu(x) - g_\nu(x)$ has at least one zero in $(0, h_{\nu-1,1})$, where $g_\nu(x) = (\frac{1}{2}x)^{\nu-1}/\Gamma(\frac{1}{2})$. Now $g_\nu(x) > 0$ for $x > 0$, and $\Gamma(\nu + \frac{1}{2})\mathbf{H}_\nu(x) < 0$ on $(0, h_{\nu,1})$ when $\nu < -\frac{1}{2}$, $\nu \neq -n + \frac{1}{2}$, by (7). Hence, $h_{\nu,1} < h_{\nu-1,1}$.

We use Proposition 6 and Theorem 4′ to establish the following analogue of a familiar property of Bessel functions.

THEOREM 5. *For* $\nu < \frac{1}{2}$, *the positive zeros of* $\mathbf{H}_\nu(x)$ *and* $\mathbf{H}_{\nu-1}(x)$ *separate each other.*

*Proof.* First, we note that $\mathbf{H}_\nu(x)$ and $\mathbf{H}_{\nu-1}(x)$ have no common zero, since this would be a double zero of $\mathbf{H}_\nu(x)$, by (9).

Next, since $J_\nu(x)\mathbf{H}_{\nu-1}(x) - J_{\nu-1}(x)\mathbf{H}_\nu(x) \neq 0$ for $x \geqq h_{\nu,1}$, and since the positive zeros of $J_\nu(x)$ and $\mathbf{H}_\nu(x)$ are simple and interlaced, there is an odd number of zeros of $\mathbf{H}_{\nu-1}(x)$ between two consecutive positive zeros of $\mathbf{H}_\nu(x)$. On the other hand, Theorem 4′ and Proposition 6 ensure that $J_\nu(x)\mathbf{H}_{\nu-1}(x) - J_{\nu-1}(x)\mathbf{H}_\nu(x) \neq 0$ for $x \geqq h_{\nu-1,1}$ and $\nu < \frac{1}{2}$; it follows that $\mathbf{H}_\nu(x)$ has an odd number of zeros between two consecutive positive zeros of $\mathbf{H}_{\nu-1}(x)$, and the theorem is proved.

We consider now functions $\lambda J_\nu(z) - \mu\mathbf{H}_\nu(z)$ for real, positive $z$ and for $\nu < \frac{1}{2}$, where $\lambda$, $\mu$ are real and $\lambda\mu \neq 0$. Since $\lambda\mu^{-1}J_\nu(z) - \mathbf{H}_\nu(z)$ is a solution of (4), the positive zeros of such a function are of multiplicity at most 2. The case $|\nu| < \frac{1}{2}$ of the following theorem is implicit in a theorem of Pólya's [10].

THEOREM 6. *Let* $\lambda$, $\mu$ *be real and* $\lambda\mu \neq 0$. *Let* $x > 0$, *and* $\nu < \frac{1}{2}$. *Then* $\lambda J_\nu(x) - \mu\mathbf{H}_\nu(x)$ *has only simple zeros if* $|\nu| < \frac{1}{2}$, *or if* $-n \leqq \nu \leqq -n + \frac{1}{2}$, $n = 1, 2, \cdots$. *Otherwise,* $\lambda J_\nu(x) - \mu\mathbf{H}_\nu(x)$ *has at most one double zero, say* $x_0$, *situated in* $(0, h_{\nu,1})$.

*In fact, $x_0$ is the zero of $J_\nu(x)\mathbf{H}'_\nu(x) - J'_\nu(x)\mathbf{H}_\nu(x)$ whose existence is asserted in Theorem 4′, and $\lambda J_\nu(x) - \mu\mathbf{H}_\nu(x)$ has a double zero at $x_0$ if and only if*

$$\frac{\lambda}{\mu} = \frac{\mathbf{H}_\nu(x_0)}{J_\nu(x_0)}.$$

*Proof.* At $x_0$, $\lambda J_\nu(x) - \mu\mathbf{H}_\nu(x)$ has a double zero if and only if $\lambda J_\nu(x_0) - \mu\mathbf{H}_\nu(x_0) = \lambda J'_\nu(x_0) - \mu\mathbf{H}'_\nu(x_0) = 0$. As this is possible with $\lambda\mu \neq 0$ if and only if $J_\nu(x_0)\mathbf{H}'_\nu(x_0) - J'_\nu(x_0)\mathbf{H}_\nu(x_0) = 0$, the conclusion of Theorem 6 follows immediately from Theorem 4′.

*Remark* 4.1. The proof of Theorems 3 and 4 can easily be adapted to show that if $\mathscr{C}_\nu(x)$ is a real solution of Bessel's equation whose positive zeros are interlaced with those of $\mathbf{H}_\nu(x)$ for some $\nu < \frac{1}{2}$, then the zeros of $\mathscr{C}_{\nu-n}(x)$ and $\mathbf{H}_{\nu-n}(x)$ are interlaced for $n = 1, 2, \cdots$.

**5. Another approach for $|\nu| < \frac{1}{2}$.** We shall briefly indicate how Propositions 2 and 5 could be proved without appealing to our Lemma 1, or to Pólya's general theorem.

We have the indefinite integral [1, (3.72)]

$$\int x^\nu J_\nu(x)\,dx = 2^{\nu-1}\Gamma(\tfrac{1}{2})\Gamma(\nu+\tfrac{1}{2})x[J_\nu(x)\mathbf{H}'_\nu(x) - J'_\nu(x)\mathbf{H}_\nu(x)]\,; \tag{20}$$

if $\nu > -\frac{1}{2}$, we may choose 0 as lower limit of integration, since $J_\nu(x) = O(x^\nu)$ as $x \to 0+$. As the right-hand side of (20) vanishes at $x = 0$ if $\nu > -\frac{1}{2}$, we have

$$\int_0^\xi x^\nu J_\nu(x)\,dx = 2^{\nu-1}\Gamma(\tfrac{1}{2})\Gamma(\nu+\tfrac{1}{2})\xi[J_\nu(\xi)\mathbf{H}'_\nu(\xi) - J'_\nu(\xi)\mathbf{H}_\nu(\xi)],$$

$\xi \geqq 0, \nu > -\frac{1}{2}$. Hence, in order to prove Propositions 2 and 5, it would suffice to show that

$$\int_0^\xi x^\nu J_\nu(x)\,dx > 0, \qquad \xi > 0, \quad |\nu| < \tfrac{1}{2}. \tag{21}$$

To establish (21), one can appeal to results of R. G. Cooke and E. Makai. Cooke has shown [4, in particular pp. 178–185], [5] that for $\nu > -1$,

$$\left|\int_{j_{\nu,r-1}}^{j_{\nu,r}} J_\nu(x)\,dx\right| > \left|\int_{j_{\nu,r}}^{j_{\nu,r+1}} J_\nu(x)\,dx\right|, \qquad r = 1, 2, \cdots,$$

where $j_{\nu,0} = 0$ and the other $j_{\nu,r}$ are the positive zeros of $J_\nu(x)$, arranged in increasing order. Since $x^\nu$, $\nu < 0$, is a positive, monotonically decreasing function on $(0, \infty)$, it follows that $\int_0^\xi x^\nu J_\nu(x)\,dx > 0$ for $\xi > 0$ and $-\frac{1}{2} < \nu \leqq 0$. For $0 < \nu < \frac{1}{2}$ one can use the fact that $x^\nu J_\nu(x)$ is a solution of the differential equation $(x^{1-2\nu}y')' +$

$x^{1-2\nu}y = 0$ (see[14, § 4.31, (19)–(20)]). By an argument similar to Makai's[4] [7], this can be shown to imply[5] that for $0 < \nu < \frac{1}{2}$,

$$\left| \int_{j_{\nu,r-1}}^{j_{\nu,r}} x^\nu J_\nu(x)\, dx \right| > \left| \int_{j_{\nu,r}}^{j_{\nu,r+1}} x^\nu J_\nu(x)\, dx \right|, \qquad r = 1, 2, \cdots.$$

It follows that $\int_0^\xi x^\nu J_\nu(x)\, dx > 0$ for $\xi > 0$ and $0 < \nu < \frac{1}{2}$, so that (21) holds as stated.

Although this approach does not yield inequality (15), it can be combined with Remark 4.1 to obtain further solutions of Bessel's equation whose zeros interlace with those of $\mathbf{H}_\nu(x)$.

## REFERENCES


[1] A. W. BABISTER, *Transcendental Functions Satisfying Nonhomogeneous Differential Equations*, Macmillan, New York, 1967.

[2] G. D. BIRKHOFF, *On solutions of ordinary linear homogeneous differential equations of the third order*, Ann. of Math. (2), 12 (1911), pp. 103–127.

[3] M. BÔCHER, *On certain pairs of transcendental functions whose roots separate each other*, Trans. Amer. Math. Soc., 2 (1901), pp. 428–436.

[4] R. G. COOKE, *Gibbs's phenomenon in Fourier-Bessel series and integrals*, Proc. London Math. Soc. (2), 27 (1928), pp. 171–192.

[5] ———, *A monotonic property of Bessel functions*, J. London Math. Soc., 12 (1937), pp. 180–185.

[6] L. LORCH, *Comparison of two formulations of Sonin's theorem and of their respective applications to Bessel functions*, Studia Sci. Math. Hungar., 1 (1966), pp. 141–145.

[7] E. MAKAI, *On a monotonic property of certain Sturm-Liouville functions*, Acta Math. Acad. Sci. Hungar., 3 (1952), pp. 165–171.

[8] M. MARDEN, *Geometry of Polynomials*, Mathematical Surveys, no. 3, rev. ed., American Mathematical Society, Providence, R.I., 1966.

[9] C. S. MEIJER, *Ueber Besselsche, Struvesche und Lommelsche Funktionen (zweite Mitteilung)*, Nederl. Akad. Wetensch. Indag. Math., 2 (1940), pp. 136–148.

[10] G. PÓLYA, *Ueber die Nullstellen gewisser ganzer Funktionen*, Math. Z., 2 (1918), pp. 352–383.

[11] J. STEINIG, *The changes of sign of fractional integrals*, Math. Z., to appear.

[12] C. A. SWANSON, *Comparison and Oscillation Theory of Linear Differential Equations*, Academic Press, New York, 1968.

[13] G. SZEGÖ, *Orthogonal Polynomials*, Colloquium Publications, vol. 23, rev. ed., American Mathematical Society, Providence, R.I., 1959.

[14] G. N. WATSON, *A treatise on the theory of Bessel functions*, second ed., Cambridge University Press, Cambridge, 1944.


---

[4] Makai considers equations of the form $y'' + \phi(x)y = 0$, where $\phi$ is positive and monotonic. However, it is not difficult to prove similar results for equations of the form $(k(x)y')' + \phi(x)y = 0$, where $k\phi$ is positive and monotonic.

[5] The proof of the first inequality (for $r = 1$) depends on the fact that for $\nu > 0$, $x^\nu J_\nu(x) \to 0$ as $x \to 0+$ (see also L. Lorch's footnote (4) in [6]). The interval $(0, j_{\nu,1})$ also requires special treatment in Cooke's proof [5].

SIAM J. MATH. ANAL.
Vol. 1, No. 3, August 1970

# PERTURBATIONS IN FULLY NONLINEAR SYSTEMS*

L. E. MAY†

**1. Introduction.** In this paper a system of nonlinear differential equations is considered subject to a small perturbation. In the principal case the unperturbed system is assumed to possess a solution $x(t)$ which is bounded over the entire real line. Using assumptions which essentially concern the behavior of the linear variational equation of the unperturbed system, we show that the perturbed system possesses a solution which is "close" to $x(t)$ over the entire real line.

Our work is motivated strongly by knowledge of perturbed linear systems. We use a variant of the classical variation of constants formula introduced by Alekseev [1]. This leads us to consider an improper nonlinear integral equation, and it is not clear that a solution of the integral equation is also a solution of the perturbed differential equation. In the perturbed linear case this difficulty does not arise because the unknown does not appear in the kernel function, and the only problem is to show that the integral equation has a solution. In the fully nonlinear case it appears to be necessary to introduce additional hypotheses to insure that any solution of the integral equation is also a solution of the differential equation. In [5], Marlin and Struble are led to consider a related problem, and the techniques used here are similar to those in that paper.

Consideration is given to the case when the unperturbed system is periodic. The solutions of the perturbed system are also discussed on a half-line. When the unperturbed system is autonomous, uniqueness is considered and one theorem is given which applies to the case when the solution $x(t)$ is almost periodic.

**2. Preliminary considerations.** Consider the two systems of differential equations

$$x' = f(t, x), \tag{1}$$

$$y' = f(t, y) + \beta g(t, y). \tag{2}$$

We assume that $f$ and $g$ are continuous $n$-functions defined on $I \times \Omega$, where $I$ is either a half-line or the entire real line $R$ and $\Omega$ is an open connected subset of $R_n$. In (2), $\beta$ denotes a scalar parameter. We further suppose that $f_x(t, x) = (\partial f/\partial x)(t, x)$ exists and is continuous on $I \times \Omega$. The solutions of (1) with initial values in $\Omega$ are then locally unique. (See, for example, [4].) Let $t_0 \in I$, $x_0 \in \Omega$. The solution $x(t)$ of (1) which satisfies $x(t_0) = x_0$ is denoted by $x(t, t_0, x_0)$. We also assume that solutions of (2) with initial values in $\Omega$ exist and are locally unique, and we denote them in a similar way. The principal matrix solution $(\partial x/\partial x_0)(t, t_0, x_0)$ of

$$\frac{dZ}{dt} = f_x(t, x(t, t_0, x_0))Z$$

* Received by the editors January 8, 1970.

† Department of Mathematics, Carleton University, Ottawa, Ontario, Canada. This work was supported by the U.S. Army Research Office, Durham, North Carolina. This paper is based on the author's doctoral thesis prepared under the direction of Professor R. A. Struble, North Carolina State University at Raleigh.

is denoted by $Q(t, t_0, x_0)$ and satisfies $Q(t_0, t_0, x_0) = I$. Alekseev [1] has proved the following generalization of the classical variation of constants formula which relates the solutions of (1) and (2). (The proof can also be found in [3].)

THEOREM 1. *Let $t_0 \in I$ and $x_0 \in \Omega$. Then, for all $t \in I$ for which $x(t, t_0, x_0) \in \Omega$ and $y(t, t_0, x_0) \in \Omega$, we have*

$$y(t, t_0, x_0) = x(t, t_0, x_0) + \beta \int_{t_0}^{t} Q(t, s, y(s, t_0, x_0))g(s, y(s, t_0, x_0))\, ds.$$

Let $D$ be a bounded convex subregion of $\Omega$ such that the closure of $D$ is contained in $\Omega$. Henceforth, throughout this work $x(t)$ denotes a fixed bounded solution of (1) or $R$ which lies in $D$ and which has no limit points on the boundary of $D$. Then there exists $d > 0$ such that

$$\{x_0 : |x_0 - x(t)| \leqq d \text{ for some } t \in R\} \subseteq D.$$

Let $\mathscr{C}$ denote the set of all continuous $n$-functions defined on $R$, and let

$$\mathscr{S} = \{y \in C : \|y - x\| \leqq d\}.$$

(In the above, and throughout this work, vertical bars denote any appropriate vector or matrix norms, and double vertical bars denote the appropriate sup norm.) We suppose that (1) may be written as a pair of uncoupled equations

$$x_1' = f_1(t, x_1), \tag{3}$$

$$x_2' = f_2(t, x_2), \tag{4}$$

where $f_1$ is a $k$-function and $f_2$ an $(n - k)$-function. Throughout this work the subscript 1 refers to a $k$-vector and 2 to an $(n - k)$-vector (or, in the case of matrices, to a $k \times k$ matrix and $(n - k) \times (n - k)$ matrix, respectively). We assume that, for arbitrary $t_0 \in R$ and $z \in \bar{D}$, the solutions $x_1(t, t_0, z_1)$ and $x_2(t, t_0, z_2)$ of (3) and (4) exist and are defined on a right half-line $\{t \in R : t \geqq t_0\}$ and a left half-line $\{t \in R : t \leqq t_0\}$, respectively. We assume that $x_1(t, t_0, z_1) \in \Omega_1 = \left\{y_1 : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \Omega \text{ for some } y_2\right\}$ for $t \geqq t_0$, and that $x_2(t, t_0, z_2) \in \Omega_2 = \left\{y_2 : \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \Omega, \text{ for some } y_1\right\}$ for $t \leqq t_0$. Thus the solutions $Q_1(t, t_0, z_1)$ and $Q_2(t, t_0, z_2)$ of the corresponding linear variational equations exist in the same circumstances. The Alekseev formula and also the work for the linear case in [4, p. 330, et seq.] suggest that, under appropriate hypotheses, a solution $y$ of the integral equation,

$$\begin{aligned} y(t) = x(t) - \beta \int_t^{\infty} &\begin{pmatrix} Q_1(t, s, y_1(s)) & 0 \\ 0 & 0 \end{pmatrix} g(s, y(s))\, ds \\ + \beta \int_{-\infty}^{t} &\begin{pmatrix} 0 & 0 \\ 0 & Q_2(t, s, y_2(s)) \end{pmatrix} g(s, y(s))\, ds, \end{aligned} \tag{5}$$

will also satisfy (2). (The partitioned matrices in (5) are, of course, of order $n \times n$.) A variant of the Alekseev formula, similar to the above, appears first to have been used by Brauer [3]. However a more comprehensive study of this type of formula has been undertaken in [5]. We may also consider solutions of (2) defined on a half-line $\{t : t \geqq \alpha\}$. In this case we are led to consider the improper integral equation

$$y(t) = \bar{x}(t) - \beta \int_t^\infty \begin{pmatrix} Q_1(t, s, y_1(s)) & 0 \\ 0 & 0 \end{pmatrix} g(s, y(s))\, ds \tag{6}$$

$$+ \beta \int_\alpha^t \begin{pmatrix} 0 & 0 \\ 0 & Q_2(t, s, y_2(s)) \end{pmatrix} g(s, y(s))\, ds,$$

where $t \geqq \alpha$, and where $\bar{x}(t)$ is a bounded solution of (1) defined on $\{t : t \geqq \alpha\}$.

In the next section we show that, under an appropriate hypothesis, (5) possesses a solution $y \in \mathscr{S}$. To this end we consider the metric function on $\mathscr{C}$ defined for $y, z \in \mathscr{C}$ by

$$\rho(y, z) = \sup_{T > 0} \min \left\{ \max_{|t| \leqq T} |y(t) - z(t)|, 1/T \right\}.$$

It is easily seen that the pair $(\mathscr{C}, \rho)$ is a complete metric space. Furthermore, if $\varepsilon > 0$, we have $\rho(y, z) \leqq \varepsilon$ if and only if $|y(t) - z(t)| \leqq \varepsilon$ for all $t \in [-1/\varepsilon, 1/\varepsilon]$. Convergence in $(\mathscr{C}, \rho)$ is thus easily seen to be uniform convergence on compact subsets of $R$. Using the Ascoli–Arzela theorem, we can easily see that the following is a characterization of relative compactness in $(\mathscr{C}, \rho)$.

(C) A set $K$ in $(\mathscr{C}, \rho)$ is relatively compact if for each finite interval $I$ and $\varepsilon > 0$ there exist $M > 0$ and $\delta > 0$ such that

$$|y(t)| \leqq M$$

for all $t \in I$ and $y \in K$, and

$$|y(t_1) - y(t_2)| \leqq \varepsilon$$

for all $y \in K$ and all $t_1, t_2 \in I$ with $|t_1 - t_2| < \delta$.

The metric space $(\mathscr{C}, \rho)$ is discussed for the one-dimensional case in [7, pp. 512–513].

Schauder's Theorem 2 [8] clearly holds in $(\mathscr{C}, \rho)$. The proof given by Nemyckii [6] of Mazur's lemma in Banach spaces is easily modified to show that the lemma holds in $(\mathscr{C}, \rho)$. It then follows that the theorem given below, Schauder's Theorem 3, holds in $(\mathscr{C}, \rho)$.

THEOREM 2. *Let $H$ be a closed convex subset of $(\mathscr{C}, \rho)$ and let $\mathscr{T}$ be a continuous map of $H$ into itself. Furthermore suppose $\mathscr{T}H$ is relatively compact. Then there exists $x_0 \in H$ such that $\mathscr{T}x_0 = x_0$.*

**3. Existence of solutions of the improper integral equations.** The following hypothesis is basic and is used to show that (5) has a solution and is later used in showing that any solution $y \in \mathscr{S}$ of (5) satisfies (2).

($H_1$) There exist functions $J_1(T, t)$ and $J_2(T, t)$ defined for $T \geqq t$ and $T \leqq t$, respectively, with the following properties:

(i)
$$\int_T^{\infty} |Q_1(t, s, y_1(s))g_1(s, y(s))|\, ds \leqq J_1(T, t) \quad \text{for } T \geqq t,$$

and

$$\int_{-\infty}^{T} |Q_2(t, s, y_2(s))g_2(s, y(s))|\, ds \leqq J_2(T, t) \quad \text{for } T \leqq t$$

and for all $y \in \mathscr{S}$.

(ii)
$$\lim_{T \to \infty} J_1(T, t) = 0 \quad \text{and} \quad \lim_{T \to -\infty} J_2(T, t) = 0,$$

uniformly for $t$ in compact subsets of $R$.

(iii) There exists a constant $J$ such that

$$J_1(t, t) \leqq J \quad \text{and} \quad J_2(t, t) \leqq J$$

for all $t \in R$.

THEOREM 3. *Let* ($H_1$) *be satisfied. Then for all* $\beta$ *with* $|\beta|$ *sufficiently small,* (5) *possesses a solution* $y \in \mathscr{S}$.

*Proof.* For simplicity of proof, we assume that $f(t, x) = f_1(t, x)$ and we omit the subscript 1 in this proof. We regard $\mathscr{S}$ as a subset of $(\mathscr{C}, \rho)$ and define a mapping $\mathscr{T}$ of $y \in \mathscr{S}$ by

$$\mathscr{T}y(t) = x(t) - \beta \int_t^{\infty} Q(t, s, y(s))g(s, y(s))\, ds. \tag{7}$$

We apply Theorem 2 to show the existence of a fixed point of $\mathscr{T}$.

Let $y \in \mathscr{S}$. Then, by ($H_1$),

$$\|\mathscr{T}y - x\| \leqq |\beta| J,$$

so that $\mathscr{T}\mathscr{S} \subseteq \mathscr{S}$ if $|\beta| \leqq d/J$. Hereafter, we assume $|\beta| \leqq d/J$.

Using criterion (C), we now show that $\mathscr{T}\mathscr{S}$ is $\rho$-relatively compact. Let $I = [a, b]$ be any interval and let $\varepsilon > 0$. If $y \in \mathscr{S}$, then

$$\|\mathscr{T}y\| \leqq \|x\| + d,$$

whence the functions of $\mathscr{T}\mathscr{S}$ are uniformly bounded. Let $t_1, t_2 \in I$ and consider

$$\begin{aligned} |\mathscr{T}y(t_1) - \mathscr{T}y(t_2)| \leqq |x(t_1) - x(t_2)| + |\beta| \bigg| &\int_{t_1}^{\infty} Q(t_1, s, y(s))g(s, y(s))\, ds \\ &- \int_{t_2}^{\infty} Q(t_2, s, y(s))g(s, y(s))\, ds \bigg|. \end{aligned} \tag{8}$$

Since $x$ is continuous, it is uniformly so on compact sets and thus there exists $\delta_1 > 0$ such that $|x(t_1) - x(t_2)| \leqq \varepsilon/4$ whenever $t_1, t_2 \in I$ with $|t_1 - t_2| < \delta_1$.

Then, using ($H_1$), we have for any $T > t_1$,

$$
\begin{aligned}
&|\beta| \left| \int_{t_1}^{\infty} Q(t_1, s, y(s))g(s, y(s))\, ds - \int_{t_2}^{\infty} Q(t_2, s, y(s))g(s, y(s))\, ds \right| \\
&\quad \leqq |\beta| \Big\{ J(T, t_1) + J(T, t_2) \\
&\qquad + \int_{t_1}^{T} |[Q(t_1, s, y(s)) - Q(t_2, s, y(s))]g(s, y(s))|\, ds \\
&\qquad + \int_{t_2}^{t_1} |Q(t_2, s, y(s))g(s, y(s))|\, ds \Big\}.
\end{aligned} \tag{9}
$$

By virtue of ($H_1$), we may fix $T$ so that for all $t_1, t_2 \in I$ we have $(d/J)(J(T, t_1) + J(T, t_2)) \leqq \varepsilon/4$. The function $|Q(t, s, x)g(s, x)|$ is continuous on the compact set $I \times [a, T] \times \bar{D}$ and thus is uniformly continuous on this set. Thus we may find $\delta_2 > 0$ such that the integrand of the first integral of the right-hand side of (9) is less than $J\varepsilon/(4d(T - a))$, whenever $t_1, t_2 \in I$ with $|t_1 - t_2| < \delta_2$. Now let $C$ be a bound for the continuous function $|Q(t, s, x)g(s, x)|$ on the compact set $I \times [a, T] \times \bar{D}$ and let $\delta_3 = J\varepsilon/(4dC)$. Then it is clear that the final integral on the right-hand side of (9) is less than $\varepsilon/4$, whenever $t_1, t_2 \in I$ with $|t_1 - t_2| < \delta_3$.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. From (8) and (9) and the discussion in the above paragraph, we see that if $y \in \mathscr{S}$ then

$$|\mathscr{T}y(t_1) - \mathscr{T}y(t_2)| \leqq \varepsilon,$$

whenever $t_1, t_2 \in I$ with $|t_1 - t_2| < \delta$. Since the functions of $\mathscr{T}\mathscr{S}$ are uniformly bounded it follows from (C) that $\mathscr{T}\mathscr{S}$ is $\rho$-relatively compact.

We show that $\mathscr{T}$ is a $\rho$-continuous map of $\mathscr{S}$. Let $y, z \in \mathscr{S}$ and let $\varepsilon > 0$, and consider, for $t \in [-1/\varepsilon, 1/\varepsilon]$,

$$|\mathscr{T}y(t) - \mathscr{T}z(t)| = |\beta| \left| \int_t^{\infty} Q(t, s, y(s))g(s, y(s)) - Q(t, s, z(s))g(s, z(s))\, ds \right|.$$

Using ($H_1$), we fix $T \geqq 1/\varepsilon$ such that $J(T, t) < J\varepsilon/(4d)$ for all $t \in [-1/\varepsilon, 1/\varepsilon]$. Then

$$|\mathscr{T}y(t) - \mathscr{T}z(t)| \leqq |\beta| \left| \int_t^{T} Q(t, s, y(s))g(s, y(s)) - Q(t, s, z(s))g(s, z(s))\, ds \right| + \frac{\varepsilon}{2}.$$

Since $Q(t, s, x)g(s, x)$ is uniformly continuous on compact sets we can find $\delta^* > 0$ such that

$$|Q(t, s, v)g(s, v) - Q(t, s, u)g(s, u)| \leqq \frac{J\varepsilon}{2d(T + 1/\varepsilon)},$$

whenever $v, u \in \bar{D}$ with $|v - u| < \delta^*$ and $t \in [-1/\varepsilon, 1/\varepsilon]$ and $t \leqq s \in [-1/\varepsilon, T]$. Now let $\delta' > 0$ be such that $|y(s) - z(s)| < \delta^*$ for $s \in [-1/\varepsilon, T]$, whenever $\rho(y, z) < \delta'$. Then if $\rho(y, z) < \delta'$, we have

$$|\mathscr{T}y(t) - \mathscr{T}z(t)| \leqq \varepsilon$$

for all $t \in [-1/\varepsilon, 1/\varepsilon]$. That is, $\rho(\mathscr{T}y, \mathscr{T}z) \leqq \varepsilon$, whenever $\rho(y, z) < \delta'$. Thus $\mathscr{T}$ is a $\rho$-continuous map of $\mathscr{S}$.

It is not difficult to see that $\mathscr{S}$ is $\rho$-closed and convex. We therefore see from the above and from Theorem 2 that $\mathscr{T}$ possesses a fixed point $y \in \mathscr{S}$ for all $\beta$ with $|\beta|$ sufficiently small.

The theorem is therefore proved.

For the periodic case the following theorem holds.

THEOREM 4. *If* $(H_1)$ *holds and if* $f(t, x)$, $g(t, x)$ *and* $x(t)$ *are periodic in t of common period P, then for all* $\beta$ *with* $|\beta|$ *sufficiently small, there exists a solution* $y \in \mathscr{S}$ *of* (5) *which has period P.*

*Proof.* Let

$$\mathscr{S}^* = \{y \in \mathscr{S} : y(t) = y(t + P) \text{ for all } t \in R\}.$$

Clearly $\mathscr{S}^*$ is $\rho$-closed and convex. We therefore see, from the proof of Theorem 3, that it is only necessary to prove that $\mathscr{T}\mathscr{S}^* \subseteq \mathscr{S}^*$. We again assume, for convenience, that $f(t, x) = f_1(t, x_1)$. Let $z \in \bar{D}$, $t_0 \in R$ and consider the solution $x(t, t_0, z)$ of (1). We have, using the periodicity of $f$,

$$\begin{aligned}\frac{dx}{dt}(t + P, t_0 + P, z) &= f(t + P, x(t + P, t_0 + P, z))\\ &= f(t, x(t + P, t_0 + P, z)).\end{aligned}$$

Thus $x(t, t_0, z)$ and $x(t + P, t_0 + P, z)$ satisfy one and the same differential equation and have the same value for $t = t_0$. Since the solutions of (1) are unique, we have $x(t, t_0, z) = x(t + P, t_0 + P, z)$ and therefore $Q(t, t_0, z) = Q(t + P, t_0 + P, z)$. Using this fact and also using the periodicity of $f$, $g$ and $x$, we have, if $y \in \mathscr{T}^*$,

$$\begin{aligned}\mathscr{T}y(t + P) &= x(t + P) - \beta \int_{t+P}^{\infty} Q(t + P, s, y(s))g(s, y(s))\, ds\\ &= x(t) - \beta \int_{t}^{\infty} Q(t + P, s + P, y(s + P))g(s + P, y(s + P))\, ds\\ &= x(t) - \beta \int_{t}^{\infty} Q(t, s, y(s))g(s, y(s))\, ds\\ &= \mathscr{T}y(t).\end{aligned}$$

Thus $\mathscr{T}\mathscr{S}^* \subseteq \mathscr{S}^*$ for $|\beta|$ sufficiently small, and the theorem is proved.

We now suppose that the right-hand members of (1) and (2) are defined on a set $H \times \Omega$ where $H$ is the half-line $\{t \in R : t \geqq \alpha\}$. We let $\mathscr{C}_\alpha^+$ be the set of continuous $n$-functions defined on $H$, and for $y \in \mathscr{C}_\alpha^+$ we let $\|y\|_\alpha = \sup_{t \geq \alpha} |y(t)|$. Let $\bar{x}(t)$ be a bounded solution of (1) which is defined on $H$ and has values in $D$ and no limit points on the boundary of $D$. Let $\mathscr{S}_\alpha^+ = \{y \in \mathscr{C}_\alpha^+ : \|y - \bar{x}\|_\alpha \leqq d\}$, where $d > 0$ is defined in the same way as in § 2. For each $z \in \bar{D}$ and $t_0 \in H$ we suppose that $\bar{x}_1(t, t_0, z_1)$ and $\bar{x}_2(t, t_0, z_2)$ are defined for $t_0 \geqq t \geqq \alpha$ and $t \geqq t_0 \geqq \alpha$, respectively. Thus $Q_1(t, t_0, z_2)$ and $Q_2(t, t_0, z_2)$ are defined in the same circumstances. We introduce the following hypothesis.

($H_1^*$) There exists a function $J_1(T, t)$ satisfying the conditions given in ($H_1$), whenever $T \geqq t \geqq \alpha$, and there exists a constant $K$ such that

$$\int_\alpha^t |Q_2(t, s, z_2(s))g_2(s, z(s))| \, ds \leqq K$$

for all $z \in \mathscr{S}_\alpha^+$ and $t \geqq \alpha$.

The following theorem is then proved in a similar way to Theorem 3.

THEOREM 5. *Let* ($H_1^*$) *be satisfied. Then for all* $\beta$ *with* $|\beta|$ *sufficiently small,* (6) *possesses a solution* $y \in \mathscr{S}_\alpha^+$.

We may similarly define a class $\mathscr{S}_\alpha^-$ of functions and obtain a similar theorem for a left half-line $t \leqq \alpha$.

If ($H_1$) is satisfied, then it is easily seen that ($H_1^*$) is satisfied, where $\alpha$ is any real number. In the linear, constant coefficient case the bounded solution $x(t) = 0$ is unique and there is an $(n - k)$-parameter family of solutions $\bar{x}(t)$ bounded on the right. We now investigate conditions under which this behavior persists in the nonlinear case and consider a version of Theorem 5 as applied to this situation.

Since $Q_1(t, T, z_1)$ and $Q_2(t, T, z_2)$ are continuous for $z \in \bar{D}$ and $T \geqq t$ and $T \leqq t$, respectively, we may define bounding functions

$$M_1(T, t) = \sup_{z \in \bar{D}} |Q_1(t, T, z_1)|$$

for $T \geqq t$, and

$$M_2(T, t) = \sup_{z \in \bar{D}} |Q_2(t, T, z_2)|$$

for $T \leqq t$. (The functions $M_1$ and $M_2$ are continuous in their domains.) We have the following lemma.

LEMMA. *Let* $y, z \in \bar{D}$. *Then*

$$|x_1(t, T, y_1) - x_1(t, T, z_1)| \leqq M_1(T, t)|y_1 - z_1|$$

*for* $T \geqq t$, *and*

$$|x_2(t, T, y_2) - x_2(t, T, z_2)| \leqq M_2(T, t)|y_2 - z_2|$$

*for* $T \geqq t$.

*Proof.* Let

$$D_1 = \left\{ x : \begin{pmatrix} x \\ x_2 \end{pmatrix} \in D \text{ for some } x_2 \right\}.$$

Then $D_1$ is convex. We have, for $T \geqq t$,

$$\oint_{y_1}^{z_1} Q_1(t, T, w) \, dw = x_1(t, T, z_1) - x_1(t, T, y_1),$$

where the line integral may be taken along any ray in $D_1$ which connects $y_1$ and $z_1$. The first inequality is immediate from the above equation.

The second inequality is proved in a similar way.

Let us now suppose that, for each fixed $t$, $\lim_{T \to \infty} M_1(T, t) = 0$ and $\lim_{T \to -\infty} M_2(T, t) = 0$. From the lemma we easily see that $x(t)$ is the only solution of (1) in $\mathscr{S}$. For, let $x^*(t)$ be another bounded solution of (1) in $\mathscr{S}$. Then, for arbitrary $t'$, we have, for $-T \leqq t' \leqq T$,

$$
\begin{aligned}
|x(t') - x^*(t')| &= \left\| \begin{pmatrix} x_1(t', T, x_1(T)) - x_1^*(t', T, x_1^*(T)) \\ x_2(t', -T, x_2(-T)) - x_2^*(t', -T, x_2^*(-T)) \end{pmatrix} \right\| \\
&\leqq M_1(T, t')|x_1(T) - x_1^*(T)| + M_2(-T, t')|x_2(-T) - x_2^*(-T)|.
\end{aligned} \tag{10}
$$

Thus, since $x, x^* \in \mathscr{S}$, we have

$$|x(t') - x^*(t')| \leqq d(M_1(T, t') + M_2(-T, t')).$$

Letting $T \to \infty$, we obtain $x(t') = x^*(t')$, and since $t'$ is arbitrary, it follows that $x$ is the only solution of (1) in $\mathscr{S}$.

Now suppose that, for fixed $\alpha$, $\sup_{t \geqq \alpha} M_2(\alpha, t) = B_2$ is finite. Let $a_2 \in D_2 = \left\{ a : \begin{pmatrix} a_1 \\ a \end{pmatrix} \in D \text{ for some } a_1 \right\}$ and consider the solution $\bar{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t, \alpha, a_2) \end{pmatrix}$ of (1). Then

$$
\begin{aligned}
|\bar{x}(t) - x(t)| &= |x_2(t, \alpha, a_2) - x_2(t)| \\
&= |x_2(t, \alpha, a_2) - x_2(t, \alpha, x_2(\alpha))| \\
&\leqq M_2(\alpha, t)|a_2 - x_2(\alpha)|,
\end{aligned}
$$

where we use the lemma. We therefore have

$$|\bar{x}(t) - x(t)| \leqq B_2|a_2 - x_2(\alpha)|.$$

Therefore, for all $a_2$ which satisfy $|a_2 - x_2(\alpha)| \leqq d/B_2 = \bar{\delta}$, we have $\bar{x} \in \mathscr{S}_\alpha^+$. Thus there is an $(n - k)$-parameter of solutions of (1) which lies in $\mathscr{S}_\alpha^+$.

THEOREM 6. *Let $\alpha$ be fixed and suppose* ($H_1^*$) *holds and that* $\sup_{t \geqq \alpha} M_2(\alpha, t) = B_2$ *is finite. Let $\bar{x}$ and $\bar{\delta}$ be defined as above. Then, if $a_2$ satisfies $|a_2 - x_2(\alpha)| \leqq \bar{\delta}/2$, there exists $\beta_0 > 0$ (independent of $a_2$) such that if $|\beta| \leqq \beta_0$, then* (6) *possesses a solution $y^{(a_2)} \in S_\alpha^+$*

*Proof.* Consider the mapping $\mathscr{T}_{a_2}$ of $y \in \mathscr{S}_\alpha^+$ defined by

$$
\begin{aligned}
\mathscr{T}_{a_2} y(t) = \bar{x}(t) &- \beta \int_t^\infty \begin{pmatrix} Q_1(t, s, y_1(s)) & 0 \\ 0 & 0 \end{pmatrix} g(s, y(s))\, ds \\
&+ \beta \int_\alpha^t \begin{pmatrix} 0 & 0 \\ 0 & Q_2(t, s, y_2(s)) \end{pmatrix} g(s, y(s))\, ds.
\end{aligned}
$$

Arguing as above, and using ($H_1^*$), we have

$$|\mathscr{T}_{a_2} y(t) - x(t)| \leqq d/2 + |\beta|(J + K)$$

for $t \geqq \alpha$. Thus $\mathscr{T}_{a_2}\mathscr{S}_\alpha^+ \subseteq \mathscr{S}_\alpha^+$, if $|\beta| \leqq d/(2(J + K)) = \beta_0$. Arguing as in Theorem 3, we see that $\mathscr{T}_{a_2}$ has a fixed point $y^{(a_2)}$ for all $\beta$ which satisfy $|\beta| \leqq \beta_0$ and for all $a_2$ which satisfy $|a_2 - x_2(\alpha)| \leqq \delta/2$.

**4. Equivalence of the improper integral equations and the perturbed differential equation.** We first consider the equivalence of (2) and (5).

THEOREM 7. *Let $y \in \mathscr{S}$ satisfy* (2) *and assume that for each fixed $t$, $\lim_{T \to \infty} M_1(T, t) = 0$ and $\lim_{T \to -\infty} M_2(T, t) = 0$. Then $y$ satisfies* (5).

*Proof.* Fix $t$ and let $T$ be such that $-T \leqq t \leqq T$. Consider

$$y(t) + \beta \int_t^T \begin{pmatrix} Q_1(t,s,y_1(s)) & 0 \\ 0 & 0 \end{pmatrix} g(s, y(s))\, ds - \beta \int_{-T}^t \begin{pmatrix} 0 & 0 \\ 0 & Q_2(t,s,y_2(s)) \end{pmatrix} g(s, y(s))\, ds$$

$$= y(t) + \int_t^T \begin{pmatrix} \dfrac{dx_1}{ds}(t,s,y_1(s)) \\ 0 \end{pmatrix} ds - \int_{-T}^t \begin{pmatrix} 0 \\ \dfrac{dx_2}{ds}(t,s,y_2(s)) \end{pmatrix} ds$$

$$= \begin{pmatrix} x_1(t, T, y_1(T)) \\ x_2(t, -T, y_2(-T)) \end{pmatrix}.$$

Arguing as for the inequality (10), we see that

$$\lim_{T\to\infty} \begin{pmatrix} x_1(t, T, y_1(T)) \\ x_2(t, -T, y_2(-T)) \end{pmatrix} = x(t),$$

and therefore $y$ satisfies (5).

To show that a solution $y \in \mathscr{S}$ of (5) satisfies (2) it appears necessary to introduce a somewhat more involved hypothesis.

Let $K(t) = \sup\{|f_x(t,x)| : x \in \bar{D}\}$ and let $L(t)$ be a Lipschitz "constant" for $f_x$ relative to $\bar{D}$; i.e., for all $x, y \in \bar{D}$ we have $|f_x(t,x) - f_x(t,y)| \leqq L(t)|x - y|$. We note that $K(t)$ is also a bound and $L(t)$ a Lipschitz constant for $\partial f_1/\partial x_2$ and $\partial f_2/\partial x_2$. We introduce the following hypothesis.

($H_2$) There exists a function $N(t)$ such that the following integrals exist and satisfy

$$\int_t^\infty |Q_1(t,s,z_1(s))|\, ds \leqq N(t)$$

and

$$\int_{-\infty}^t |Q_2(t,s,z_2(s))|\, ds \leqq N(t)$$

for all $t \in R$ and $z \in \mathscr{S}$. Further, there exist positive constants $\lambda_1$, $\lambda_2$, $\mu$ and $\nu$ such that the following conditions are satisfied:

(i) $$M_1(T,t)J_2(T,T) \leqq \lambda_1$$

if $T \geqq t$, and

$$M_2(T,t)J_2(T,T) \leqq \lambda_2$$

if $T \leqq t$.

(ii) For each fixed $t$,

$$\lim_{T\to\infty} M_1(T,t)J_1(T,T) = 0$$

and

$$\lim_{T\to-\infty} M_2(T,t)J_2(T,T) = 0.$$

(iii) $N(t)L(t) \leqq \mu$ for all $t \in R$.

(iv) $(J_1(t,t) + J_2(t,t))(K(t) + L(t)) \leqq \nu$ for all $t \in R$.

THEOREM 8. *Suppose hypotheses* $(\mathrm{H}_1)$ *and* $(\mathrm{H}_2)$ *hold. Then there exists* $\beta_0 > 0$ *such that if* $y \in \mathscr{S}$ *is a solution of* (5), *with* $|\beta| < \beta_0$, *then* $y$ *is also a solution of* (2).

*Proof.* For simplicity of proof we again assume that $f(t, x) = f_1(t, x_1)$ and we omit the subscript 1 in the proof.

By the lemma in § 3, since $y \in \mathscr{S}$ and since $(\mathrm{H}_1)$ holds, we have

$$\begin{aligned}|x(t, T, y(T)) - x(t)| &\leqq M(T, t)|y(T) - x(T)| \\ &\leqq |\beta| M(T, t) J(T, T)\end{aligned} \tag{11}$$

if $T \geqq t$. Applying $(\mathrm{H}_2)$ (i) we obtain

$$|x(t, T, y(T)) - x(t)| \leqq |\beta|\lambda,$$

if $T \geqq t$. Therefore, if $|\beta| \leqq d/\lambda$, we see that $x(t, T, y(T)) \in \bar{D}$, if $T \geqq t$. Thus, for each fixed $t, f_x(t, x(t, s, y(s)))$ is bounded for $s \geqq t$. From this and from $(\mathrm{H}_1)$ we see that the derivative

$$y'(t) = f(t, x(t)) + \beta g(t, y(t)) - \beta \int_t^\infty f_x(t, x(t, s, y(s))) Q(t, s, y(s)) g(s, y(s))\, ds \tag{12}$$

exists, since the improper integral in the equation is uniformly convergent for $t$ in compact sets. From (11) and $(\mathrm{H}_2)$ (ii) we have

$$x(t) = \lim_{T \to \infty} x(t, T, y(T)).$$

Furthermore, since

$$\frac{dx}{ds}(t, s, y(s)) = Q(t, s, y(s))[y'(s) - f(s, y(s))],$$

we see that

$$\begin{aligned} y(t) - x(t) &= \lim_{T \to \infty} [y(t) - x(t, T, y(T))] \\ &= -\lim_{T \to \infty} \int_t^T \frac{dx}{ds}(t, s, y(s))\, ds \\ &= -\int_t^\infty Q(t, s, y(s))[y'(s) - f(s, y(s))]\, ds. \end{aligned}$$

Thus, using (11), we obtain

$$\int_t^\infty Q(t, s, y(s)) w(s) = 0, \tag{13}$$

where

$$w(s) = \beta g(s, y(s)) + f(s, y(s)) - y'(s).$$

Now (12) may also be written in the form

$$\begin{aligned} w(t) = {} & f(t, y(t)) - f(t, x(t)) \\ & + \beta \int_t^\infty [f_x(t, x(t, s, y(s))) - f_x(t, x(t))] Q(t, s, y(s)) g(s, y(s))\, ds \\ & + \beta f_x(t, x(t)) \int_t^\infty Q(t, s, y(s)) g(s, y(s))\, ds, \end{aligned}$$

whence using the fact that $f_x$ is Lipschitzian, the mean value theorem, (11), ($H_1$), ($H_2$) (i) and ($H_2$) (iv), it follows that $w$ is bounded. From

$$\frac{df}{ds}(t, x(t, s, y(s))) = f_x(t, x(t, s, y(s)))Q(t, s, y(s))[y'(s) - f(s, y(s))],$$

we easily obtain, using (12),

$$w(t) = \int_t^\infty f_x(t, x(t, s, y(s)))Q(t, s, y(s))w(s)\, ds. \tag{14}$$

If we multiply (13) on the left by $f_x(t, x(t))$ and combine the result with (14), we have

$$w(t) = \int_t^\infty [f_x(t, x(t, s, y(s))) - f_x(t, x(t))]Q(t, s, y(s))w(s)\, ds.$$

Using the fact that $f_x$ is Lipschitzian, we see that it follows from the lemma in § 3 and from ($H_1$) and ($H_2$) (i) that

$$\begin{aligned} |w(t)| &\leqq \lambda L(t)|\beta| \int_t^\infty |Q(t, s, y(s))w(s)|\, ds \\ &\leqq \lambda L(t)|\beta| N(t)\, \|w\|. \end{aligned}$$

Thus, using ($H_2$) (iii), we conclude that $w$ satisfies the inequality

$$\|w\| \leqq |\beta|\lambda\mu\|w\|.$$

Clearly this implies that $\|w\| = 0$, if $|\beta| < 1/(\lambda\mu)$. Thus, combining this with the previous restriction on $\beta$, we see that for $|\beta| < \min\{d/\lambda, 1/(\lambda\mu)\}$, the theorem follows from the definition of $w$.

COROLLARY 1. *Let the differential equation*

$$x' = f(t, x)$$

*possess a solution $x(t)$ which is bounded for all $t$. Let $g$ be a function for which hypotheses* ($H_1$) *and* ($H_2$) *are satisfied. Then, for all $\beta$ with $|\beta|$ sufficiently small, the differential equation*

$$y' = f(t, y) + \beta g(t, y)$$

*also possesses a bounded solution.*

*Proof.* See Theorems 3 and 8.

When working on a half-line it is again easy to obtain a theorem similar to Theorem 7. The following hypothesis enables us to prove a theorem similar to Theorem 8.

($H_2^*$) There exists a function $N(t)$ such that the following integral exists and satisfies

$$\int_t^\infty |Q_1(t, s, z_1(s))|\, ds \leqq N(t)$$

for all $z \in \mathscr{S}_\alpha^+$ and $t \geqq \alpha$, for some fixed $\alpha$. Furthermore, there exist constants $\lambda^*$, $\mu^*$ and $\nu^*$ such that the following conditions are satisfied:

(i) $M_1(T, t)J_1(T, T) \leqq \lambda^*$ for all $T \geqq t \geqq \alpha$.

(ii) For each fixed $t \geqq \alpha$, $\lim_{T\to\infty} M_1(T, t)J_1(T, T) = 0$.

(iii) $N(t)L(t) \leqq \mu^*$ for all $t \geqq \alpha$.

(iv) $J_1(t, t)(K(t) + L(t)) \leqq \nu^*$ for all $t \geqq \alpha$.

THEOREM 9. *Suppose* $(H_1^*)$ *and* $(H_2^*)$ *hold. Then there exists* $\beta_0 > 0$ *such that if* $y \in \mathscr{S}_\alpha^+$ *is a solution of* (6) *with* $|\beta| < \beta_0$, *then* $y$ *is also a solution of* (2).

*Proof.* The proof is essentially a combination of the proofs of Theorems 1 and 8. We can in fact show that $\beta_0 = \min\{d/\lambda^*, 1/(\lambda^*\mu^*)\}$ suffices.

COROLLARY 2. *Let* (1) *possess a solution* $\bar{x}(t)$ *which is bounded for all* $t \geqq \alpha$. *Let* $g$ *be a function for which hypotheses* $(H_1^*)$ *and* $(H_2^*)$ *are satisfied. Then for all* $\beta$ *with* $|\beta|$ *sufficiently small,* (2) *also possesses a solution which is bounded for all* $t \geqq \alpha$.

*Proof.* See Theorems 5 and 9.

COROLLARY 3. *Let* $(H_1)$ *and* $(H_2)$ *hold and assume* $\sup_{t\geqq\alpha} M_1(\alpha, t) = B_1$ *and* $\sup_{t\leqq\alpha} M_2(\alpha, t) = B_2$ *are finite. Then, for every fixed* $\alpha$, (1) *possesses at least an* $(n - k)$*-dimensional* ($k$*-dimensional*) *family of solutions bounded on the right* (*left*) *of* $\alpha$; *and, for all* $|\beta|$ *sufficiently small,* (2) *possesses at least an* $(n - k)$*-dimensional* ($k$*-dimensional*) *family of solutions bounded on the right* (*left*) *of* $\alpha$.

*Proof.* The families of solutions associated with (1) exist by the discussion preceding Theorem 6.

Since $(H_1)$ and $(H_2)$ are satisfied so are $(H_1^*)$ and $(H_2^*)$ for any fixed $\alpha$. By Theorem 6, for all $\beta$ sufficiently small, there exists at least an $(n - k)$-parameter family of solutions $\mathscr{F}_\alpha^+$ of (6) which are bounded for $t \geqq \alpha$. (We note that we are, in fact, considering a family of equations like (6).) By Theorem 9, each $y \in \mathscr{F}_\alpha^+$ satisfies (2), where $\beta$ satisfies $|\beta| < \beta_0$ (where $\beta_0$ is *independent* of $y \in \mathscr{F}_\alpha^+$). We may similarly assert the existence of at least a $k$-parameter family of solutions $\mathscr{F}_\alpha^-$ of (2) bounded for $t \leqq \alpha$. Thus, for $|\beta|$ sufficiently small, both families $\mathscr{F}_\alpha^+$ and $\mathscr{F}_\alpha^+$ exist, and the corollary is proved.

**5. The autonomous case.** Throughout this section we assume that (1) is autonomous:

$$x' = f(x). \tag{1'}$$

Then we have $x(t, t_0, x_0) = x(t - t_0, 0, x_0)$, and we write $x(t, t_0, x_0) = x(t - t_0, x_0)$, and similarly we write $Q(t, t_0, x_0) = Q(t - t_0, x_0)$, $M_1(T, t) = M_1(t - T)$ and $M_2(T, t) = M_2(t - T)$.

Let us suppose that for $x \in \bar{D}$, $f(x)$, $f_x(x)$ and $g(t, x)$ are Lipschitzian in $x$ with Lipschitz constant $M$ and that $g$ is bounded by $M$. We assume that there exists a constant $N$ such that

$$\int_t^\infty |Q_1(t - s, z_1(s))|\, ds = \int_0^\infty |Q_1(-u, z_1(u + t))|\, du \leqq N$$

and

$$\int_{-\infty}^t |Q_2(t - s, z_2(s))|\, ds = \int_{-\infty}^0 |Q_2(-u, z_2(u + t))|\, du \leqq N$$

for all $z \in \mathscr{S}$. Since $f(x)$ is Lipschitzian in $x$, the solution $x_1(t - t_0, y_1)$ is Lipschitzian in $y_1$, although the Lipschitz constant may well be time dependent. Thus, since $f_x$ is Lipschitzian in $x$, the right-hand member of the matrix equation,

$$\frac{dZ}{dt} = \frac{\partial f_1}{\partial x_1}(x_1(t - t_0, y_1))Z,$$

is Lipschitzian in $y_1$. Hence the solution $Q_1(t - t_0, y_1)$ of this equation is Lipschitzian in $y_1$ with Lipschitz constant $\lambda_1(t - t_0)$, say. Similarly $Q_2(t - t_0, y_2)$ is Lipschitzian in $y_2$ with Lipschitz constant $\lambda_2(t - t_0)$, say. We also assume that

$$\max\left\{\int_t^\infty \lambda_1(t-s)\,ds;\int_{-\infty}^t \lambda_2(t-s)\,ds\right\} = \max\left\{\int_0^\infty \lambda_1(-u)\,du;\int_{-\infty}^0 \lambda_2(-u)\,du\right\} = C$$

is finite. Furthermore, we suppose that $\lim_{u\to-\infty} M_1(u) = 0 = \lim_{u\to\infty} M_2(u)$. (Then, by the discussion in § 3, $x(t)$ is the only solution in $\mathscr{S}$ of (1). Also $B_1 = \sup_{u \leqq 0} M_1(u)$ and $B_2 = \sup_{u \geqq 0} M_2(u)$ are finite.)

It is easy to see that if $(\mathrm{H}_1)$ and the above conditions hold, then $(\mathrm{H}_2)$ holds. The following theorem is strongly motivated by [4, Theorem 4.1].

THEOREM 10. *Let* $(\mathrm{H}_1)$ *and the above assumptions hold. Then for any* $\alpha$ *there exists a continuous real* $(n - k)$*-dimensional manifold* $\mathscr{M}^+$ *(and a continuous real* $k$*-dimensional manifold* $\mathscr{M}^-$*) such that any solution* $y$ *of* (2) *with* $y(\alpha) \in \mathscr{M}^+$ $(y(\alpha) \in \mathscr{M}^-)$ *satisfies* $y \in \mathscr{S}_\alpha^+$ $(y \in \mathscr{S}_\alpha^-)$. *Moreover any solution* $y$ *of* (2) *near* $x(\alpha)$ *but not on* $\mathscr{M}^+$ $(\mathscr{M}^-)$ *at* $t = \alpha$ *cannot satisfy* $y \in \mathscr{S}_\alpha^+$ $(y \in \mathscr{S}_\alpha^-)$.

*Proof.* Let $\alpha$ be fixed. Let $\bar{\delta}$ be defined as in § 3. Consider the space of all $n$-functions $y(a_2, t)$ which are continuous on $E = \{a_2 : |a_2 - x_2(\alpha)| \leqq \bar{\delta}/2\} \times \{t : t \geqq \alpha\}$. For two such functions $y$, $z$ we define

$$\bar{\rho}(y, z) = \sup_{(a_2,t)\in E} |y(a_2, t) - z(a_2, t)|.$$

Let $\mathscr{P}$ be the space of all continuous $n$-functions on $E$ which satisfy $\bar{\rho}(x, y) \leqq d$. Then $(\mathscr{P}, \bar{\rho})$ is a complete metric space. For $y \in \mathscr{P}$, let

$$\begin{aligned}\mathscr{T}y(a_2, t) = {} & \begin{pmatrix} x_1(t) \\ x_2(t - \alpha, a_2)\end{pmatrix} - \beta\int_t^\infty \begin{pmatrix} Q_1(t - s, y_1(a_2, s)) & 0 \\ 0 & 0\end{pmatrix} g(s, y(a_2, s))\,ds \\ & + \beta\int_\alpha^t \begin{pmatrix} 0 & 0 \\ 0 & Q_2(t - s, y_2(a_2, s))\end{pmatrix} g(s, y(a_2, s))\,ds.\end{aligned} \tag{15}$$

We apply the Banach fixed-point theorem to show that $\mathscr{T}$ has a unique fixed point in $\mathscr{P}$ for $|\beta|$ sufficiently small.

Let $y \in \mathscr{P}$ and consider

$$\begin{aligned}|\mathscr{T}y(a_2, t) - x(t)| \leqq {} & \left|\begin{pmatrix} 0 \\ x_2(t - \alpha, a_2) - x_2(t)\end{pmatrix}\right| \\ & + |\beta| \int_t^\infty |Q_1(t - s, y_1(a_2, s))g_1(s, y(a_2, s))|\,ds \\ & + |\beta| \int_\alpha^t |Q_2(t - s, y_2(a_2, s))g_2(s, y_2(a_2, s))|\,ds.\end{aligned}$$

Now, for fixed $a_2$, $y(a_2, t) \in \mathscr{S}_\alpha^+$. Therefore using $(H_1)$ (which implies $(H_1^*)$), we have

$$\begin{aligned}|\mathscr{T}y(a_2, t) - x(t)| &\leqq |x_2(t-\alpha, a_2) - x_2(t-\alpha, x_2(\alpha))| + 2|\beta|J \\ &\leqq B_2|a_2 - x_2(\alpha)| + 2|\beta|J \\ &\leqq d/2 + 2|\beta|J,\end{aligned}$$

where we have used the lemma of § 3 and the fact that $\bar{\delta} = d/B_2$. Thus $\bar{\rho}(\mathscr{T}y, x) \leqq d$, if $|\beta| \leqq d/(4J)$. Since $\mathscr{T}y$ is continuous on $E$ we see that $\mathscr{T}\mathscr{P} \subseteq \mathscr{P}$ for $|\beta|$ sufficiently small.

Now let $y, z \in \mathscr{P}$ and consider

$$\begin{aligned}|\mathscr{T}y(a_2, t) - \mathscr{T}z(a_2, t)| \leqq |\beta| \bigg| \int_t^\infty & Q_1(t-s, y_1(a_2, s))g_1(s, y(a_2, s)) \\ & - Q_1(t-s, z_1(a_2, s))g_1(s, z(a_2, s))\, ds \bigg| \\ + |\beta| \bigg| \int_\alpha^t & Q_2(t-s, y_2(a_2, s))g_2(s, y(a_2, s)) \\ & - Q_2(t-s, z_2(a_2, s))g_2(s, z(a_2, s))\, ds \bigg| .\end{aligned}$$

Consider the first integral above. It is less than or equal to

$$\begin{aligned}&\left| \int_t^\infty Q_1(t-s, y_1(a_2, s))[g_1(s, y(a_2, s)) - g_1(s, z(a_2, s))]\, ds \right| \\ &\qquad + \left| \int_t^\infty [Q_1(t-s, y_1(a_2, s)) - Q_1(t-s, z_1(a_2, s))]g_1(s, z(a_2, s))\, ds \right| \\ &\qquad \leqq NM\bar{\rho}(y, z) + M\bar{\rho}(y, z) \int_t^\infty \lambda(t-s)\, ds \\ &\qquad \leqq M(N + C)\bar{\rho}(y, z),\end{aligned} \tag{16}$$

where we use the hypotheses of this section. Similarly, the second integral is dominated by the same quantity. Thus

$$|\mathscr{T}y(a_2, t) - \mathscr{T}z(a_2, t)| \leqq 2|\beta|M(N + C)\bar{\rho}(y, z),$$

and therefore

$$\bar{\rho}(\mathscr{T}y, \mathscr{T}z) \leqq 2|\beta|M(N + C)\bar{\rho}(y, z).$$

Thus, for all $\beta$ with $|\beta|$ sufficiently small, we see that $\mathscr{T}$ is a contractual map of $\mathscr{P}$. Thus, by the Banach fixed-point theorem, (15) possesses a unique fixed point $\bar{y}(a_2, t) = \bar{y} \in \mathscr{P}$ for all small $|\beta|$. Clearly $\bar{y}(a_2, t)$ is continuous in $a_2$ (uniformly in $t$). Since $(H_1)$ and the conditions given in this section hold, then $(H_2)$ holds. By Theorem 9, for each $a_2$, $\bar{y}(a_2, t)$ is a solution of the differential equation (2), provided $|\beta|$ is sufficiently small (but *independent* of $a_2$). Furthermore $\bar{y}(a_2, t)$ is the unique solution of (2) satisfying $y \in \mathscr{S}_\alpha^+$ and $y_2(\alpha) = a_2$ (with $|a_2 - x_2(\alpha)| \leqq \bar{\delta}/2$). For, by an argument similar to that of Theorem 7, any solution of (2), satisfying these conditions, is a fixed point of (15) and is therefore unique for small $|\beta|$.

Let $Y_i = \bar{y}_i(\alpha, a_2)$ for $i = 1, 2$. Then (for all $|\beta|$ sufficiently small)

$$Y_1 = \psi(a_2) = x_1(\alpha) - \beta \int_\alpha^\infty Q_1(\alpha - s, \bar{y}_1(s, a_2)) g_1(s, \bar{y}(s, a_2))\, ds$$

and

$$Y_2 = a_2.$$

The equation $Y_1 = \psi(Y_2)$ then clearly defines, for $|Y_2 - x_2(\alpha)| \leqq \bar{\delta}/2$, a continuous $(n - k)$-dimensional initial manifold $\mathscr{M}^+$ such that if $y$ is a solution of (2), and $y(\alpha) \in \mathscr{M}^+$, then $y \in \mathscr{S}_\alpha^+$. Furthermore, any solution $y \in \mathscr{S}_\alpha^+$ of (2) with $|a_2 - x_2(\alpha)| \leqq \bar{\delta}/2$ is a fixed point of (15), where we take $a_2 = y_2(\alpha)$, and it follows that $y(\alpha) \in \mathscr{M}^+$.

We may similarly show the existence of an initial manifold $\mathscr{M}^-$ as stated in the theorem.

By Theorem 3, (5) and hence (2) possesses a solution $y \in \mathscr{S}$. Using the hypotheses of the above theorem we may apply the Banach fixed-point theorem to show that this solution is unique. Hence there is only one solution of (2) which is in both $\mathscr{S}_\alpha^+$ and $\mathscr{S}_\alpha^-$.

In the autonomous case we have the following theorem concerning almost periodic solutions over the entire real line.

THEOREM 11. *Let* (1) *be autonomous and suppose* $(\mathrm{H}_1)$ *and the hypotheses of this section hold. Suppose also that* $x(t)$ *and* $g(t, x)$ *are almost periodic* (*the latter uniformly so with respect to* $x$ *for* $x \in \bar{D}$). *Then for all* $\beta$ *with* $|\beta|$ *sufficiently small, the unique solution* $y \in \mathscr{S}$ *of* (2) *is almost periodic.*

*Proof.* For simplicity we assume that $f(x) = f_1(x)$ and we omit the subscript 1 in this proof. Let $\tau$ be any real number; we have

$$\begin{aligned}
|y(t + \tau) - y(t)| &\leqq |x(t + \tau) - x(t)| \\
&\quad + |\beta| \left| \int_{t+\tau}^\infty Q(t + \tau - s, y(s)) g(s, y(s))\, ds - \int_t^\infty Q(t - s, y(s)) g(s, y(s))\, ds \right| \\
&\leqq |x(t + \tau) - x(t)| \\
&\quad + |\beta| \left| \int_t^\infty Q(t - s, y(s + \tau)) g(s + \tau, y(s + \tau)) - Q(t - s, y(s)) g(s + \tau, y(s))\, ds \right| \\
&\quad + |\beta| \left| \int_t^\infty Q(t - s, y(s)) [g(s + \tau, y(s)) - g(s, y(s))]\, ds \right| \\
&\leqq \sup_{t \in R} |x(t + \tau) - x(t)| + |\beta| N(M + C) \sup_{t \in R} |y(t + \tau) - y(t)| \\
&\quad + |\beta| N \sup_{\substack{t \in R \\ y \in \mathscr{S}}} |g(t + \tau, y(t)) - g(t, y(t))|,
\end{aligned}$$

where we have used an inequality similar to (16) and have used the hypotheses of this section. We therefore have

$$[1 - |\beta| N(M + C)] \sup_{t \in R} |y(t + \tau) - y(t)| \leqq \sup_{t \in R} |x(t + \tau) - x(t)| + |\beta| \sup_{\substack{t \in R \\ y \in \mathscr{S}}} |g(t + \tau, y(t)) - g(t, y(t))| .$$

It follows that $y$ is almost periodic (since $x$ and $g$ are) provided $\beta$ is further restricted to satisfy $1 - |\beta| N(M + C) > 0$.

## REFERENCES

[1] V. M. Alekseev, *An estimate for the perturbations of the solutions of ordinary differential equations*, Vestnik Moskov. Univ. Ser. I Mat. Mek., 2 (1961), pp. 28–36.

[2] F. Brauer, *Perturbations of non-linear systems of differential equations. I*, J. Math. Anal. Appl., 14 (1966), pp. 198–206.

[3] ———, *Perturbations on non-linear systems of differential equations. II*, Ibid., 17 (1967), pp. 418–434.

[4] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[5] J. A. Marlin and R. A. Struble, *Asymptotic equivalence of non-linear systems*, J. Differential Equations, 6 (1969), pp. 578–596.

[6] V. V. Nemychkii, *The fixed point method in analysis*, Amer. Math. Soc. Transl. (2), 34 (1963), pp. 1–37.

[7] V. V. Nemyckii and V. V. Stepanov, *Qualitative Theory of Differential Equations*, Princeton University Press, Princeton, N.J., 1960.

[8] J. Schauder, *Der Fixpunktsatz in Functionalräumen*, Studia Math., 2 (1930), pp. 171–180.

SIAM J. MATH. ANAL.
Vol. 1, No. 3, August 1970

# THE ASYMPTOTIC APPROXIMATION OF CERTAIN INTEGRALS*

FRANK STENGER†

**1. Introduction and summary.** This paper investigates the use of quadratures for obtaining asymptotic approximations of certain definite integrals. The results of this paper were first announced in [7].

Let $f$ be a function bounded on the real line $R$ and let $f$ be smooth ("smooth" will be made more precise in later sections) in a neighborhood of the origin on $R$. Let $K \in L^1(R)$, i.e., $\int_R |K(t)|\,dt < \infty$; let $m$ and $n$ be positive integers and let $\lambda > 0$. The integrals have the form

$$I(f, \lambda) = \int_R \lambda K(\lambda t) f(t)\,dt \tag{1.1}$$

and the approximations take the form

$$Q_n(f, \lambda) = \sum_{j=1}^{m} W_j f(t_j/\lambda), \tag{1.2}$$

where

$$I(f, \lambda) = Q_n(f, \lambda) + \varepsilon_n(f, \lambda). \tag{1.3}$$

The expression (1.2) is obtained by applying a quadrature rule to the integral (1.1). It is assumed that $I(f, \lambda) = Q_n(f, \lambda)$ for $f(t) = 1, t, \cdots, t^{n-1}$.

Alternatively, let us consider

$$T_n(f, \lambda) = \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!\lambda^j} \mu_{j+1}, \tag{1.4}$$

where

$$\mu_{j+1} = \int_R t^j K(t)\,dt, \tag{1.5}$$

and let us define

$$\eta_n(f, \lambda) = I(f, \lambda) - T_n(f, \lambda). \tag{1.6}$$

The approximation (1.3) is often better than (1.4). Obviously, if one uses (1.3), one need not know derivatives of $f$ at $t = 0$; one need only be able to evaluate $f$ in a neighborhood of $t = 0$. Also, bounds on $\varepsilon_n$ are often easier to obtain than bounds on $\eta_n$. In addition, we present examples for which $|\varepsilon_n|$ is much smaller than $|\eta_n|$, although this is not true in general. For example, with $M = \sup_{t \in R} |f''(t)|$, it is shown that $I(f, \lambda) = \int_0^\infty \lambda e^{-\lambda t} f(t)\,dt = f(1/\lambda) + \varepsilon_2(f, \lambda)$, where $|\varepsilon_2(f, \lambda)| \leqq M/(2\lambda^2)$.

* Received by the editors March 20, 1969, and in revised form February 19, 1970.

† Department of Mathematics, University of Michigan, Ann Arbor, Michigan. Now at Department of Mathematics, University of Utah, Salt Lake City, Utah 84112. This work was supported by the National Science Foundation under Grant GP8687.

As a one-term approximation, $f(1/\lambda)$ is better (asymptotically, as $\lambda \to \infty$) than the well-known Laplace approximation $f(0)$ for which $I(f, \lambda) = f(0) + \eta$. Here $\eta \sim M_1/\lambda$ where $M_1$ is a constant. (Note that the approximation $f(0)$ is a special case of *either* (1.3) *or* (1.4).)

We establish the following result concerning the approximation of $I(f, \lambda)$ by $Q_n(f, \lambda)$:

Let $\alpha > 0$, let $n, r$ be positive integers such that $n \leqq r$ and let $K(t)t^n \in L^1(R)$. Let $\omega_r(f; N; s) = O(s^\alpha)$ as $s \to 0$ $(s > 0)$, where $\omega_r(f; N; s)$ (defined in (4.8)) denotes the $r$th modulus of continuity of $f$ in a neighborhood $N$ of the origin. Then $\varepsilon_n(f, \lambda)$ is $O(\lambda^{-n})$, $O(\lambda^{-\alpha} \log \lambda)$ or $O(\lambda^{-\alpha})$ as $\lambda \to \infty$ when $n < \alpha$, $n = \alpha$ or $n > \alpha$ respectively.

These bounds cannot be improved with regard to order.

Asymptotic approximations in terms of diminishing mesh length have been used before; see, e.g., Franklin and Friedman [6] where a form of a convergent expansion is given. Error bounds have also been obtained by some authors. A set of references concerning these is contained in a recent paper of Olver [3]. In [3] Olver finds error bounds for the Laplace approximation.

**2. Asymptotic approximation with error bounds.** At the outset we state two lemmas which we shall use in obtaining error bounds.

LEMMA 2.1. *Let $[c, d]$ be a closed and bounded interval on R, and let a polynomial $P(x)$ of degree n in x be bounded by L on $[c, d]$. If x is on R but not on $[c, d]$, then*

$$|P(x)| \leqq L\left[\frac{2|2x - c - d|}{d - c}\right]^n. \tag{2.1}$$

*Proof.* The proof is found in Timan [2, p. 88].

LEMMA 2.2. *Let $x > 0$, and for arbitrary real $\alpha$ let $I(\alpha)$ denote the integral*

$$I(\alpha) = \int_x^\infty t^{\alpha - 1} e^{-t}\, dt. \tag{2.2}$$

*If $\beta > 0$ and if $\rho = -[1 - \beta]$, then*

$$I(\beta) = (\beta - \rho)_\rho x^{\beta - \rho - 1} e^{-x}\left[\sum_{k=0}^{\rho} \frac{x^k}{(\beta - \rho)_k} + \varepsilon\right], \tag{2.3}$$

*where $\varepsilon \leqq 0$. Here*

$$(\alpha)_k = \begin{cases} \alpha(\alpha + 1) \cdots (\alpha + k - 1) & \text{if } k > 0, \\ 1 & \text{if } k = 0. \end{cases} \tag{2.4}$$

*Proof.* Substituting $\alpha = \beta$ in (2.2) and integrating by parts, we obtain

$$I(\beta) = x^{\beta - 1} e^{-x} + (\beta - 1) I(\beta - 1). \tag{2.5}$$

By repeated use of (2.5) we obtain (2.3) with

$$\varepsilon = (\beta - \rho - 1)_{\rho + 1} x^{\rho + 1 - \beta} e^x I(\beta - \rho - 1). \tag{2.6}$$

Now $\varepsilon \leqq 0$ since $x > 0$ implies that $I(\alpha) > 0$ for every real $\alpha$, and by hypothesis $0 \leqq \rho < \beta \leqq \rho + 1$.

**2.1. Quadrature rules.** Let us outline briefly some methods of constructing quadrature formulas (1.2). Let $t_1, t_2, \cdots, t_m$ be any $m$ distinct points on $R$, and let $\mu_1, \mu_2, \cdots, \mu_n$ be defined by (1.5). If $n \leqq m$, then the system of $n$ linear equations

$$\sum_{j=1}^{m} W_j t_j^{k-1} = \mu_k, \qquad k = 1, 2, \cdots, n, \tag{2.7}$$

can be used to determine the $m$ numbers $W_j$. If $K(t) \geqq 0$ on $R$ and if the points $t_j$ are chosen to be the $m$ zeros of a polynomial $P_m(t)$ of degree $m$ in $t$ which belongs to the sequence of polynomials orthogonal over $R$ with respect to the weight function $K(t)$, then the numbers $W_j$ determined from (2.7) with $n = m$ are nonnegative and satisfy $\sum_{j=1}^{m} W_j t_j^{k-1} = \mu_k$ for $k = 1, 2, \cdots, 2m$. This choice of points $t_j$ and weights $W_j$ is known as Gaussian quadrature; the reader should consult Davis and Rabinowitz [8] for further details.

**2.2. Integrals over** $[0, \omega]$. Let $\alpha, a > 0$ and let $C_*^{(2m)}[0, a]$ denote the class of functions with bounded $2m$th derivative on $[0, a]$. Let $\{L_k^\alpha\}_{k=0}^\infty$ denote the sequence of Laguerre polynomials orthogonal over $(0, \infty)$ with respect to the weight function $t^{\alpha-1}e^{-t}$. Here each $L_k^\alpha(t)$ is of degree $k$ in $t$, scaled so that it is monic. Let $\{t_j\}_{j=1}^m$ denote the $m$ zeros of $L_m^\alpha(t)$ and let $\{W_j\}_{j=1}^m$ denote the corresponding Gaussian integration weights. We prove the following theorem.

THEOREM 2.3. *If* $f \in C_*^{(2m)}[0, a]$ *and* $\lambda \geqq \max_j (t_j/a)$, *then*

$$\int_0^a \lambda^\alpha t^{\alpha-1} e^{-\lambda t} f(t)\, dt = \sum_{j=1}^{m} W_j f(t_j/\lambda) + \varepsilon_1 + \varepsilon_2, \tag{2.8}$$

*where*

$$\begin{aligned} |\varepsilon_1| &\leqq \frac{m!\Gamma(m+\alpha)}{(2m)!\lambda^{2m}} M_{2m}, \\ |\varepsilon_2| &\leqq (M + \omega_m) 2^{2m-1} \left(\frac{a\lambda}{2}\right)^{\alpha-2N-m} e^{-a\lambda} \\ &\quad \cdot \sum_{k=0}^{m} \frac{(-1)^k (1-\alpha)_k}{k!} (\alpha - N)_{N+2m-2-k} \left(\frac{a\lambda}{2}\right)^k \sum_{j=0}^{N+2m-2-k} \frac{(a\lambda/2)^j}{(\alpha - N)_j}, \end{aligned} \tag{2.9}$$

*where* $N = -[1 - \alpha]$,

$$\begin{aligned} M_{2m} &= \sup_{t \in (0,a)} |f^{(2m)}(t)|, \\ M &= \sup_{t \in (0,a)} |f(t)|, \\ \omega_m &= \frac{a^{2m}}{(2m)!} M_{2m}. \end{aligned} \tag{2.10}$$

*Furthermore, if* $a = \infty$ *then* $\varepsilon_2 = 0$.

*Proof.* The left side of (2.8) may be written in the form

$$F(\lambda) = \int_0^{a\lambda} t^{\alpha-1} e^{-t} f(t/\lambda)\, dt. \tag{2.11}$$

If $a = \infty$, the application of Gauss–Laguerre quadrature with error to (2.11) immediately yields (2.9) with $\varepsilon_2 = 0$.

For the remainder of the proof we shall assume that $a < \infty$.

It is known (see, e.g., Davis [1, p. 37]) that the Hermite interpolant polynomial of degree $2m - 1$ in $t$ (whose coefficients may be functions of $\lambda$) which is defined by

$$P_{2m-1}(t_k) = f\left(\frac{t_k}{\lambda}\right), \qquad P'_{2m-1}(t_k) = \frac{f'(t_k/\lambda)}{\lambda} \tag{2.12}$$

also satisfies

$$f\left(\frac{t}{\lambda}\right) - P_{2m-1}(t) = \frac{f^{(2m)}(\zeta)}{(2m)!\,\lambda^{2m}} L_m^\alpha(t)^2. \tag{2.13}$$

Here $\zeta \in (0, a)$ if $t \in (0, a\lambda)$. Since $m$-point Gaussian integration is exact for polynomials of degree $2m - 1$ and since (see [8, p. 96]) $\int_0^\infty t^{\alpha-1} e^{-t} L_m^\alpha(t)^2\, dt = m!\,\Gamma(m + \alpha)$,

$$F(\lambda) - \sum_{j=1}^{m} W_j f\left(\frac{t_j}{\lambda}\right) \leqq \frac{m!\,\Gamma(m+\alpha)}{(2m)!\,\lambda^{2m}} \sup_{\zeta\in(0,a)} |f^{(2m)}(\zeta)| + \int_{a\lambda}^\infty t^{\alpha-1}e^{-t}|P_{2m-1}(t)|\, dt. \tag{2.14}$$

If $\lambda \geqq \max_j (t_j/a)$, then in the interval $[0, a\lambda]$ we clearly have $|L_m^\alpha(t)| \leqq (a\lambda)^m$, since by the hypothesis of the theorem each zero of $L_m^\alpha(t)$ is in $[0, a\lambda]$. Hence in $[0, a\lambda]$, $|P_{2m-1}(t)| \leqq M + \omega_m$, where $\omega_m$ is given in (2.10). Upon applying Lemma 2.1 to the integral on the right of (2.14) we obtain

$$\int_{a\lambda}^\infty t^{\alpha-1}e^{-t}|P_{2m-1}(t)|\, dt \leqq (M + \omega_m) H\left(\frac{a\lambda}{2}, 2m, \alpha\right), \tag{2.15}$$

where

$$H(x, 2m, \alpha) = \int_{2x}^\infty t^{\alpha-1}e^{-t}(t - x)^{2m-1}\, dt. \tag{2.16}$$

Setting $t - x = y$, we have

$$H(x, 2m, \alpha) = \left(\frac{2}{x}\right)^{2m-1} e^{-x} \int_x^\infty \left(1 + \frac{x}{y}\right)^{\alpha-1} y^{2m+\alpha-2}e^{-y}\, dy. \tag{2.17}$$

Now if $t > 0$ and $N$ is defined by $N = -[1 - \alpha]$, then $(d/dt)^{N+1}(1 + t)^{\alpha-1} \leqq 0$, and therefore Taylor's formula with remainder yields

$$(1 + t)^{\alpha-1} \leqq \sum_{k=0}^{N} \frac{(-1)^k(1-\alpha)_k}{k!} t^k, \qquad t > 0. \tag{2.18}$$

By Lemma 2.2, we also have

$$\int_x^\infty y^{2m+\alpha-2-k}e^{-y}\, dy \leqq (\alpha - N)_{N+2m-2-k}\, x^{\alpha-N-1}e^{-x} \sum_{j=0}^{N+2m-2-k} \frac{x^j}{(\alpha - N)_j}. \tag{2.19}$$

Upon combining (2.18), (2.19) and substituting into (2.15) we get

$$\int_{a\lambda}^{\infty} t^{\alpha-1}e^{-t}|P_{2m-1}(t)|\,dt \leqq \delta_2, \tag{2.20}$$

where $\delta_2$ is the quantity on the right of (2.9).

This completes the proof.

For example, if $m = 1$, we have

$$\int_0^{\infty} \lambda^\alpha t^{\alpha-1}e^{-\lambda t}f(t)\,dt = \Gamma(\alpha)f(\alpha/\lambda) + \varepsilon, \qquad \varepsilon = \frac{\Gamma(\alpha+1)}{2\lambda^2}f''(\zeta)$$

for some $\zeta = \zeta(\lambda) \in (0, \infty)$. This should be compared with the well-known Laplace approximation

$$\int_0^{\infty} \lambda^\alpha t^{\alpha-1}e^{-\lambda t}f(t)\,dt = \Gamma(\alpha)f(0) + \eta, \qquad \eta \sim c/\lambda,$$

where $c$ is a constant.

In the case when $m \leqq 2$, the numbers $W_k$ and $t_k$ in (2.7) can be explicitly expressed:

For $m = 1$, $t_1 = \alpha$,

$$W_1 = \Gamma(\alpha).$$

For $m = 2$, $t_1, t_2 = \alpha + 1 \pm \sqrt{\alpha + 1}$,

$$W_1 = \Gamma(\alpha)(t_2 - 1)/(t_2 - t_1), \qquad W_2 = \Gamma(\alpha) - W_1.$$

These results can be obtained by application of the procedure described in § 2.1, together with $L_1^\alpha(t) = t - \alpha$, $L_2^\alpha(t) = t^2 - 2(\alpha+1)t + \alpha(\alpha+1)$ (see [1, p. 367]).

**2.3. Integrals over** $[-\infty, \infty]$. Let $a, b > 0$, and let $C_*^{(2m)}[-a, b]$ denote the class of functions whose $2m$th derivative is bounded on $[-a, b]$. For $k = 0, 1, 2, \cdots$, let $H_k(t)$ denote the Hermite polynomial of degree $k$ in $t$ normalized so that its coefficient of $t^k$ is unity. The polynomials $H_k(t)$ are orthogonal over $(-\infty, \infty)$ with respect to the weight function $e^{-t^2}$. Let $\{t_j\}_{j=1}^m$ denote the $m$ zeros of $H_m(t)$ and let $\{W_j\}_{j=1}^m$ denote the corresponding Gaussian integration weights.

THEOREM 2.4. *Let* $f \in C_*^{(2m)}[-a, b]$ *and let* $\lambda \geqq \max_j|t_j|/r$, *where* $r = \min(a, b)$. *Then*

$$\int_{-a}^{b} \lambda e^{-\lambda^2 t^2}f(t)\,dt = \sum_{j=1}^{m} W_j f(t_j/\lambda) + \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \tag{2.21}$$

*where*

$$\begin{aligned} |\varepsilon_1| &\leqq M(\lambda^2r^2)^{-1/2}e^{-\lambda^2r^2}, \\ |\varepsilon_2| &\leqq \frac{m!}{2^{2m}(2m)!}\frac{M_{2m}}{\lambda^{2m}}, \\ |\varepsilon_3| &\leqq \frac{2}{\lambda^{2m}}(M + \omega_m)e^{-\lambda^2r^2}(2/r)^{2m-1}(m-1)!\sum_{j=0}^{m-1}\frac{(\lambda^2r^2)^j}{j!}, \end{aligned} \tag{2.22}$$

*and where*

$$
\begin{aligned}
M &= \sup_{t\in(-a,b)} |f(t)|, \\
M_{2m} &= \sup_{t\in(-a,b)} |f^{(2m)}(t)|, \qquad (2.23)\\
W_m &= \frac{r^{2m}}{(2m)!} M_{2m}.
\end{aligned}
$$

*Furthermore, if* $a = b = \infty$, *then* $\varepsilon_1 = \varepsilon_3 = 0$.

*Proof.* The proof is similar to that of Theorem 2.3 and is omitted.

**3. Examples and comparison of quadrature and termwise integration.**

**3.1.** Let us start with the well-known expansion

$$\int_0^\infty \frac{\lambda e^{-\lambda t}}{1+t}\,dt = \sum_{k=0}^{2m-1} \frac{(-1)^k k!}{\lambda^k} + \eta_m, \qquad (3.1)$$

where it is readily shown that

$$|\eta_m| \leqq (2m)!/\lambda^{2m}. \qquad (3.2)$$

Asymptotically the bound on the right of (3.2) is the best bound possible. By applying Gaussian quadrature, we obtain

$$F(\lambda) = \sum_{k=1}^{m} \frac{W_k}{1 + t_k/\lambda} + \varepsilon_m, \qquad (3.3)$$

where

$$|\varepsilon_m| \leqq (m!)^2/\lambda^{2m}. \qquad (3.4)$$

Using Stirling's formula, we find that the bound on the right of (3.2) is roughly $2^{2m+1/2}/(\pi m)^{1/2}$ times as large as that on the right of (3.5). For example, with $m = 1$, (3.3) yields

$$F(\lambda) = \frac{\lambda}{1+\lambda} + \varepsilon_1, \qquad |\varepsilon_1| \leqq \frac{1}{\lambda^2}.$$

This result may also be interpreted as follows: If, in order to achieve an error $\leqq\delta$, it is necessary to take $\lambda \geqq \lambda_0(\delta)$ in (3.2), then (roughly, i.e., using an estimate based on Stirling's formula) (3.4) yields an error $\leqq\delta$ for all $\lambda$ such that $\lambda \geqq \frac{1}{2}\lambda_0(\delta)$.

**3.2.** An application of Gauss–Hermite quadrature with error to the integral form of the solution of the heat equation problem

$$
\begin{aligned}
u_t &= u_{xx}, &\quad x \in R,\quad t > 0, \\
u|_{t=0} &= g(x), &\quad x \in R,
\end{aligned} \qquad (3.5)
$$

yields the following corollary.

COROLLARY 3.1. *Let $g \in C_*^{(2m)}(R)$ and let $W_k$ and $t_k$ be defined as in Theorem 2.4. Then the function $u = u(x, t)$ which solves the problem (3.5) is given for all $x \in R$ and $t \geqq 0$ by*

$$u(x,t) = \frac{1}{\sqrt{\pi t}}\int_R e^{-(x-\xi)^2/(4t)} g(\xi)\, d\xi = \frac{1}{\sqrt{\pi}}\sum_{k=1}^{m} W_k g(x - t_k\sqrt{4t}) + \varepsilon_m, \tag{3.6}$$

$$\varepsilon_m = \frac{(4t)^m m!\, g^{(2m)}(\eta)}{2^m(2m)!}, \qquad \eta \in R.$$

The formula (3.6) thus provides a good method of computing the solution to (3.5) in the case when $t$ is small.

**3.3.** An asymptotic approximation of the integral

$$e^{\lambda}K_0(\lambda) = \frac{1}{2}\int_R e^{-\lambda(\cosh x - 1)}\, dx \tag{3.7}$$

together with an error bound for an approximation was obtained in [3]. It was shown in [3] that

$$e^{\lambda}K_0(\lambda) = \int_0^\infty e^{-\lambda p} p^{-1/2}(2 + p)^{-1/2}\, dp = \sum_{k=0}^{2m-1} \frac{[(1/2)_k]^2(-1)^k\sqrt{\pi}}{k!(2\lambda)^{k+1/2}} + \eta_m, \tag{3.8}$$

where

$$|\eta_m| \leqq \frac{[(1/2)_{2m}]^2\sqrt{\pi}}{(2m)!(2\lambda)^{2m+1/2}}. \tag{3.9}$$

If we substitute $\lambda p = t$ in the integral in (3.8) and then apply Gauss–Laguerre quadrature with weight $e^{-t}t^{-1/2}$ to the resulting integral, we obtain

$$e^{\lambda}K_0(\lambda) = \sum_{j=1}^{m} W_j \lambda^{-1/2}(2 + t_j/\lambda)^{-1/2} + \varepsilon_m, \tag{3.10}$$

where, by use of (2.14) with $(d/dt)^{2m}(1 + t)^{-1/2} = (1/2)_{2m}(1 + t)^{-2m-3/2}$, we get

$$|\varepsilon_m| \leqq \frac{\pi^{1/2} m!(1/2)_{2m}(1/2)_m}{(2m)!(2\lambda)^{2m+1/2}}. \tag{3.11}$$

Asymptotically, the bound on the right of (3.9) is the best bound possible. The ratio of the right of (3.9) over the right of (3.11) is equal to

$$\frac{2^{2m-1/2}}{(\pi m)^{1/2}[1 + O(1/m)]}$$

as $m \to \infty$. For example, with $m = 1$, (3.10) and (3.11) yield

$$K_0(\lambda) = \sqrt{\frac{\pi}{2\lambda}}e^{-\lambda}\left[\left(1 + \frac{1}{4\lambda}\right)^{-1/2} + \varepsilon_1\right], \tag{3.12}$$

where

$$|\varepsilon_1| \leqq \frac{3}{64\lambda^2}. \tag{3.13}$$

**3.4. Comparing quadrature and termwise integration.** Is the error of quadrature smaller than the error of termwise integration? If we compare (1.3) and (1.4), it is not always true that $|\varepsilon_n(f, \lambda)| \leqq |\eta_n(f, \lambda)|$, since it is easy to illustrate examples for which $\eta_n(f, \lambda) = 0$ ($\lambda$ fixed) and $\varepsilon_n(f, \lambda) \neq 0$. However, for $\lambda$ fixed, the question of comparing $\varepsilon_n$ and $\eta_n$ is the same as the following question: Is it more accurate to evaluate $I(g) = \int_R K(t)g(t)\,dt$ by termwise integration of the Taylor series expansion of $g$ about $t = 0$, i.e., to approximate $I(g)$ by $T_n(g) = \sum_{k=0}^n g^{(k)}(0)\int_R K(t)t^k\,dt$ or to approximate $I(g)$ by use of a quadrature formula, $I(g) \cong Q_n(g) = \sum_{j=1}^m W_j g(t_j)$? For arbitrary $K$ and arbitrary $g$ the question seems too difficult to answer and so we shall restrict $K$ and $g$ to have special properties.

If $K$ is nonnegative on a finite interval $[-a, b]$, where $a, b > 0$, and if $K = 0$ outside of $[-a, b]$, then we can take $K$ to be a weight function and construct Gaussian quadrature formulas. In this case, $\{Q_n(g)\}_{n=1}^\infty$ converges to $I(g)$ for every function $g$ that is continuous on $[-a, b]$. (Actually a more general result is valid; see, e.g., Davis [1, p. 353].) On the other hand there exist functions $g$ that are continuous and analytic on $[-a, b]$ for which $\{T_n(g)\}_{n=1}^\infty$ diverges. For example, if we take $0 < \varepsilon < \min(a, b)$, $K = 1$ on $[-a, b]$ and $g = (\varepsilon + x^2)^{-1}$, then

$$T_n(g) = \sum_{k=0}^n (-1)^k \frac{b^{2k+1} - (-a)^{2k+1}}{(2k+1)\varepsilon^{k+1}},$$

and therefore $T_{n+1}(g) - T_n(g)$ does not converge to zero as $n \to \infty$. Hence $T_n(g)$ does not converge to $I(g)$. Hence, for all $n$ sufficiently large, the error of Gaussian quadrature for this function $g$ is less than the error of termwise integration.

Similarly, in the case when $K(t) = t^{\alpha-1}e^{-t}$ if $t > 0$, $K(t) = 0$ if $t < 0$, $\alpha > 0$, a class of functions $g(t)$ for which Gaussian quadrature converges to $I(g)$ is the class of those functions $g$ that are infinitely differentiable on $[0, \infty)$ for which

$$\frac{m!\,\Gamma(m+\alpha)}{(2m)!} M_{2m} \to 0,$$

and where

$$M_{2m} = \sup_{t \in (0,\infty)} |g^{(2m)}(t)|;$$

this follows from (2.14). For example, $g = \sin at$, $0 \leqq a < 2$, is in this class, whereas $T_n(\sin ax)$ does not converge to $I(\sin ax)$ if $a > 1$. Conversely, if $g$ is infinitely differentiable on $[0, \infty)$ and if $T_n(g)$ converges, then we must have $\Gamma(m + \alpha)g^{(2m)}(0)/m! \to 0$ as $m \to \infty$. Additional sufficient conditions on the convergence of $Q_n(g)$ to $I(g)$ can be found in Davis and Rabinowitz [8, p. 97].

**4. A generalization.** Let $W$ denote the class of all functions $g(t)$ of bounded variation on $R$, i.e.,

$$V(g) = \int_R |dg(t)| < \infty, \tag{4.1}$$

and let $\hat{W}$ denote the isomorphic class of all transforms $\hat{g}$ of $dg$ defined by[1]

$$\hat{g}(x) = \int_R e^{ixt}\, dg(t), \tag{4.2}$$

where $g(t) \in W$.

Let us list some properties of $W$ and $\hat{W}$ which can be found in Bochner [4], and which we shall require.

The classes $W$ and $\hat{W}$ are rings with respect to ordinary multiplication of Fourier transforms, i.e., if $\hat{g}, \hat{h} \in \hat{W}$, then $\hat{g} + \hat{h} \in \hat{W}$, and $\hat{k} = \hat{g}\hat{h} \in \hat{W}$. The corresponding "product" in $W$ is given for all $f$ continuous and bounded on $R$ by

$$\int_R f(t)\, dk(t) = \int_R \int_R f(t+s)\, dg(t)\, dh(s). \tag{4.3}$$

If $\hat{g}$ is a function which is twice differentiable on $R$ and if $\hat{g} = 0$ on $R - E$, where $E \subset R$ is a closed and bounded interval, then $\hat{g} \in \hat{W}$. Finally, if $\hat{g}_k \in \hat{W}$ for $k = 1, 2, 3, \cdots$, and if $\hat{g}_k$ converges to $\hat{g} \in \hat{W}$ as $k \to \infty$ uniformly on each closed and bounded interval on $R$, then the corresponding element $g_k(t) - g_k(0) \in W$ defined by (4.2)′ converges to $g(t) - g(0) \in W$.

In what follows we let $dg_p$ and $dh$ be defined by

$$dg_p(t) = \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k}\, dH(t-k), \tag{4.4}$$

$$dh(t) = K(t)\, dt - \sum_{j=1}^{m} W_j\, dH(t - t_j), \tag{4.5}$$

where $p$ is a positive integer,

$$H(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geqq 0, \end{cases} \tag{4.6}$$

and where $t_j$ and $W_j$ are chosen in (4.5) so that

$$\int_R t^k\, dh(t) = \int_R t^k K(t)\, dt - \sum_{j=1}^{m} W_j t_j^k = 0 \tag{4.7}$$

for $k = 0, 1, \cdots, n-1$, $n \geqq 1$. Clearly $g_p \in W$, and since $K \in L^1(R)$, $h \in W$.

If $f$ is continuous on $R$, the $p$th modulus of continuity of $f$ on an interval $(a, b) \subset R$ is defined by

$$\omega_p(f; (a,b); s) = \sup_{t, t+rh \in (a,b), 0 < h \leqq s} |\Delta_h^p f(t)|, \tag{4.8}$$

where

$$\Delta_h^p f(t) = \sum_{k=0}^{p} (-1)^k \binom{p}{k} f(t + kh). \tag{4.9}$$

---

[1] Given $\hat{g} \in \hat{W}$ we can recover $g(t) - g(0) \in W$ by use of

$$g(t) - g(0) = \lim_{m \to \infty} \frac{1}{2\pi} \int_{-m}^{m} \hat{g}(x) \frac{1 - e^{-ixt}}{ix}\, dx. \tag{4.2′}$$

For example, if $f^{(p)}$ exists and is bounded on $[a,b]$, then $\omega_p(f;(a,b);s) = O(s^p)$ as $s \to 0$. Hence if $dg_p$ is defined by (4.4), then the quantity

$$Dg(f,s) = \sup_{t\in R}\left|\int_R f(t+su)\,dg_p(u)\right| = \sup_{t\in R}\left|\sum_{k=0}^{p}\binom{p}{k}(-1)^k f(t+ks)\right| \tag{4.10}$$

is related to $\omega_p$ by

$$\omega_p(f;R;s) \geqq Dg(f,s). \tag{4.11}$$

An easy computation using (4.2) shows that

$$\hat{g}_p(x) = (1-e^{ix})^p. \tag{4.12}$$

Also, it may be shown by use of Peano's theorem (for details, see Davis [1]) that there exists a function $k(t) \in W$ such that

$$\int_R f(t)\,dh(t) = \int_R K(t)f(t)\,dt - \sum_{j=1}^{m} W_j f(t_j) = \int_R f^{(n)}(t)\,dk(t) \tag{4.13}$$

for all functions $f$ that have a bounded $n$th derivative on $R$. Upon taking $f(t) = e^{ixt}$, we see that (4.13) yields

$$\hat{h}(x) = \int_R e^{ixt}\,dh(t) = (ix)^n \int_R e^{ixt}\,dk(t) = (ix)^n\hat{k}(x), \tag{4.14}$$

where the function $\hat{k} \in \hat{W}$ is given by

$$\hat{k}(x) = \int_R e^{ixt}\,dk(t). \tag{4.15}$$

Let us now prove the following theorem,

THEOREM 4.1. *Let $f$ be continuous and bounded on $R$, let $N$ be a neighborhood of $t = 0$, and let $K(t), t^nK(t) \in L^1(R)$. Then*

$$\left|\int_R \lambda K(\lambda t)f(t)\,dt - \sum_{j=1}^{m} W_j f\left(\frac{t_j}{\lambda}\right)\right| \leqq A\omega_n\left(f;N;\frac{1}{\lambda}\right) + B\sum_{k=0}^{\infty}\omega_n\left(f;N;\frac{2^{-k}}{\lambda}\right) + \frac{C}{\lambda^n}, \tag{4.16}$$

*where $A$, $B$ and $C$ are nonnegative numbers.*

*Proof.* (a) Let us first assume that $N = R$. Let us take $p = n$ in (4.12). The function $\hat{g}_n(x) = (1-e^{ix})^n$ is clearly not zero in the intervals $E = \{x : 1 \leqq |x| \leqq 4\}$. Let $\hat{q}$ be a twice differentiable function defined on $R$ such that $\hat{q} = 1$ if $|x| \leqq 1$ and $\hat{q} = 0$ if $|x| \geqq 2$. Then $\hat{q}(x/2) - \hat{q}(x) = 0$ for $x \in R - E$. Hence the function $\hat{\mu}$, defined by $\hat{\mu}(x) = [\hat{q}(x/2) - \hat{q}(x)]/\hat{g}_n(x)$ if $x \in E$, and $\hat{\mu}(x) = 0$ if $x \in R - E$, is a twice differentiable function on $R$ which vanishes on $R - E$. Therefore $\hat{\mu} \in \hat{W}$, and

$$\hat{q}(x/2) - \hat{q}(x) = \hat{g}_n(x)\hat{\mu}(x) \tag{4.17}$$

for all $x \in R$.

Now for all $x \in R$,

$$\sum_{k=0}^{s} [\hat{q}(x/2^{k+1}) - \hat{q}(x/2^k)] = \hat{q}(x/2^{s+1}) - \hat{q}(x) \to 1 - \hat{q}(x) \tag{4.18}$$

as $s \to \infty$. By (4.17) it follows therefore that

$$\hat{q}(x) + \sum_{k=0}^{s} \hat{g}_n(x/2^k)\hat{\mu}(x/2^k) \to 1 \tag{4.19}$$

as $s \to \infty$.

Now if $\hat{\sigma}_1(x)$ is defined by $\hat{\sigma}_1(x) = x^n\hat{q}(x)/\hat{g}_n(x)$, if $|x| \leqq 2$, and $\hat{\sigma}_1(x) = 0$, if $|x| > 2$, then $\hat{\sigma}_1(x)$ is a twice differentiable function on $R$ which vanishes if $|x| \geqq 2$. Therefore $\hat{\sigma}_1 \in \hat{W}$, and the function $\hat{\sigma}(x) = \hat{k}(x)\hat{\sigma}_1(x) \in \hat{W}$, where $\hat{k}(x)$ is defined in (4.14) and (4.15). Our construction shows that the identity

$$\hat{\sigma}(x)\hat{g}_n(x) = \hat{h}(x)\hat{q}(x) \tag{4.20}$$

is valid for all $x \in R$.

Upon multiplying (4.19) by $\hat{h}(x)$ and using (4.20), we get

$$\hat{\sigma}(x)\hat{g}_n(x) + \sum_{k=0}^{s} \hat{\mu}(x/2^k)\hat{g}_n(x/2^k) \to \hat{h}(x) \tag{4.21}$$

as $s \to \infty$.

We observe that the left side of (4.21) converges uniformly to $\hat{h}(x)$ on each closed and bounded interval on $R$. Thus if we combine (4.2)′, (4.3) and (4.10) in (4.21), we get

$$\left| \int_R f(t)\, dh(\lambda t) \right| \leqq V(\sigma)\omega_n(f; R; 1/\lambda) + V(\mu) \sum_{k=0}^{\infty} \omega_n(f; R; 2^{-k}/\lambda), \tag{4.22}$$

where $\sigma(t) - \sigma(0)$ and $\mu(t) - \mu(0)$ are defined by (4.2)′ using $\hat{\sigma}$ and $\hat{\mu}$ respectively on the right-hand side.

We now observe that the left side of (4.22) is equal to the left side of (4.16) and set $A = V(\sigma)$, $B = V(\mu)$ and $C = 0$. This completes the proof for the case when $N = R$.

(b) For the case when $N \neq R$, we set $N = (n_1, n_2)$ and $S = (s_1, s_2)$, where $n_1 < s_1 < 0 < s_2 < n_2$. Let $u \geqq 0$ be an infinitely differentiable function bounded by 1 on $R$ such that $u = 1$ on $S$, $u = 0$ on $R - N$. Then the function $f_1 = uf$ satisfies $\omega_n(f_1; R; 1/\lambda) = \omega_n(f; N; 1/\lambda)$, $f_1 = f$ on $S$, and $f_1 = 0$ on $R - N$. Hence

$$\begin{aligned} \left| \int_R f(t)\, dh(\lambda t) \right| &= \left| \int_R f_1(t)\, dh(\lambda t) + \int_{R-S} [f(t) - f_1(t)]\, dh(\lambda t) \right| \\ &\leqq \left| \int_R f_1(t)\, dh(\lambda t) \right| + C_1 \int_R |t|^n |\, dh(\lambda t)|, \end{aligned} \tag{4.23}$$

where

$$C_1 = \sup_{t \in R-S} |t^{-n} f(t)|. \tag{4.24}$$

Hence

$$\left|\int_R f(t)\,dh(\lambda t)\right| \leqq \left|\int_R f_1(t)\,dh(\lambda t)\right| + C/\lambda^n, \tag{4.25}$$

where

$$C = C_1 \int_R |t|^n|\,dh(t)|. \tag{4.26}$$

We now observe that the first term on the right of (4.25) is bounded by the right of (4.16), with $C = 0$. This completes the proof of Theorem 4.1.

*Remark* 4.2. The assumption $t^nK(t) \in L^1(R)$ is required only if $N \neq R$; if $N = R$, we merely require $K \in L^1(R)$ and the existence of $\int_R t^{k-1}K(t)\,dt$ for $k = 2, 3, \cdots, n$.

*Remark* 4.3. The term $\lambda K(\lambda t)\,dt$ in (1.1) may be replaced by $dk(\lambda t)$, provided that $k \in W$ and, also, provided that $\int_R t^{s-1}\,dk(t)$ exists for $s = 1, 2, \cdots, n$. The results would then be somewhat more general, although the proofs would remain essentially the same.

*Remark* 4.4. The technique of proof of Theorem 4.4 is in some ways similar (though more elementary) to that used by Shapiro [5] to establish a more general result.

COROLLARY 4.5. *Let the assumptions of Theorem* 4.1 *be satisfied. If* $r > n$ *and if* $\omega_r(f; N; 1/\lambda) = O(\lambda^{-\alpha})$ *as* $\lambda \to \infty$, $\alpha > 0$, *then*

$$\int_R \lambda K(\lambda t)f(t)\,dt - \sum_{j=1}^{m} W_j f(t_j/\lambda) = \begin{cases} O(\lambda^{-n}) & \text{if } n < \alpha, \\ O(\lambda^{-\alpha}\log\lambda) & \text{if } n = \alpha, \\ O(\lambda^{-\alpha}) & \text{if } n > \alpha. \end{cases} \tag{4.27}$$

*These bounds cannot be improved with regard to order.*

*Proof.* It is shown in [2, p. 107] that if $r > n$ and $\omega_r(f; N; 1/\lambda) = O(\lambda^{-\alpha})$, then

$$\omega_n(f; N; 1/\lambda) = \begin{cases} O(\lambda^{-n}) & \text{if } n < \alpha, \\ O(\lambda^{-\alpha}\log\lambda) & \text{if } n = \alpha, \\ O(\lambda^{-\alpha}) & \text{if } n > \alpha. \end{cases} \tag{4.28}$$

If we substitute $\omega_n(f; N; 1/\lambda) \leqq D/\lambda^n$ into (4.16), we find that the left of (4.26) is bounded by

$$AD/\lambda^n + BD\sum_{k=0}^{\infty} 2^{-kn}/\lambda^n + C/\lambda^n = O(\lambda^{-n}). \tag{4.29}$$

The proofs for the other two cases in (4.27) are similar and we omit them.

In order to show that the bounds in (4.27) cannot be improved with regard to order, we note first by the examples of the previous section that the first bound on the right of (4.27) can be achieved. Next, we take $K(t) = 1$ if $0 \leqq t \leqq 1$, $K(t) = 0$

otherwise, and $f(t) = t \log(1/t)$. It is easily verified then, that $\omega_1(f; R; 1/\lambda) = c(1/\lambda) \log \lambda$, where $c \neq 0$ is a constant and $\omega_r(f; R; 1/\lambda) = O(1/\lambda)$, $r \geqq 2$. If we take $t_1 = 2/3$, $W_1 = 1$, $m = 1$ in (4.5), then $\int_R t^{k-1}\, dh(t) = 0$ for $k = n = 1$. The resulting quadrature error $\varepsilon_1$ satisfies $\varepsilon_1 = (c_1/\lambda)\,[\log \lambda + O(1/\lambda)]$, where $c_1 \neq 0$ is a constant. However, any quadrature formula that is exact for 1 and $t$ has an error which is $O(1/\lambda)$ as $\lambda \to \infty$.

This completes the proof.

**Acknowledgments.** The author is very grateful to the referee for his valuable criticisms.

REFERENCES

[1] P. J. DAVIS, *Interpolation and Approximation*, Blaisdell, New York, 1963.

[2] A. F. TIMAN, *Theory of Approximation of Functions of a Real Variable*, Macmillan, New York, 1963.

[3] F. W. J. OLVER, *Error bounds for the Laplace approximation for definite integrals*, J. Approx. Theory, 1 (1968), pp. 293–313.

[4] S. BOCHNER, *Lectures on Fourier Integrals*, Princeton University Press, Princeton, N.J., 1959.

[5] H. S. SHAPIRO, *A Tauberian theorem related to approximation theory*, Acta Math., 120 (1968), pp. 279–292.

[6] J. FRANKLIN AND B. FRIEDMAN, *A convergent asymptotic representation for integrals*, Proc. Cambridge Philos. Soc., 53 (1957), pp. 612–619.

[7] F. STENGER, *The asymptotic approximation of certain integrals*, Notices Amer. Math. Soc., 16 (1969), p. 95.

[8] P. J. DAVIS AND P. RABINOWITZ, *Numerical Integration*, Blaisdell, Waltham, Mass., 1967.

SIAM J. MATH. ANAL.
Vol. 1, No. 4, November 1970

# LIE THEORY AND SOME SPECIAL SOLUTIONS OF THE HYPERGEOMETRIC EQUATIONS*

WILLARD MILLER, JR.†

**Abstract.** To obtain all solutions of the differential equations of hypergeometric type as basis functions corresponding to models of Lie algebra representations it is necessary to consider certain reducible representations. These representations are classified and models are constructed. A number of special hypergeometric functions arise naturally in the analysis, including the error and incomplete gamma functions, the incomplete beta functions, Legendre functions of the second kind, and some logarithmic solutions of the hypergeometric equation.

**Introduction.** In [1], the author studied the functions of hypergeometric type by obtaining them as basis functions in models of irreducible nonunitary representations of the 4-dimensional Lie algebras $\mathscr{G}(a, b)$. This approach is essentially equivalent to the factorization method of Infeld and Hull [2], [3]. However, not all solutions of the hypergeometric, confluent hypergeometric and parabolic cylinder equations were obtained by this analysis. To obtain these special solutions it is necessary to consider classes of reducible representations of $\mathscr{G}(a, b)$.

An explanation of the technique to be used is simplest in the language of the factorization method. Recall that a *factorization* of a sequence of second order differential operators $\{X_m\}$, $m \in S = \{m_0 + n : n = 0, \pm 1, \pm 2, \cdots\}$, consists of sequences of nonzero first order differential operators $\{L_m^+\}, \{L_m^-\}$ and constants $\{a_m\}, m \in S$, such that

$$L_m^+ L_m^- + a_m \equiv L_{m+1}^- L_{m+1}^- + a_{m+1} \equiv X_m, \quad m \in S. \tag{0.1}$$

Then if $\lambda$ is a complex constant and the function $y_l$ is a solution of

$$X_m y_m = \lambda y_m \tag{0.2}$$

for $m = l$, it follows from (0.1) that $y_{l+1} = L_{l+1}^+ y_l$ is a solution of (0.2) with $m = l + 1$. If $a_m \neq \lambda$ for all $m \in S$, then $y_{l+1}$ and $y_{l-1}$ are not identically zero if $y_l \neq 0$. Thus, if we start with one nonzero solution $y_l$ of (0.2), we can apply the operators $L_m^\pm$ to obtain nonzero solutions $y_m$ for all $m \in S$. Furthermore, if we choose two linearly independent solutions $y_l, y_l'$, then we can construct two ladders of solutions $\{y_m\}, \{y_m'\}$ such that $y_m$ and $y_m'$ are linearly independent for all $m$. This gives a basis for all solutions of (0.2) with $m \in S$.

However, if $a_l = \lambda$ for some $l \in S$, then the eigenvalue equation becomes

$$L_l^+ L_l^- y_l = 0, \tag{0.3}$$

and we can find a solution $y_l$ by solving the first order equation $L_l^- y_l = 0$. Applying the $\{L_m^+\}$ operators to $y_l$ we get a ladder of solutions $\{y_m\}, m = l, l + 1, l + 2, \cdots$ (bounded below). However, we do not get a basis of solutions for (0.2), $m = l + n$, $n \geqq 0$, nor do we get any solutions with $m = l - n$, $n > 0$. Similar remarks hold for ladders of solutions bounded above.

* Received by the editors October 24, 1969.

† Department of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455.

In this paper we extend our ladder of solutions by requiring $y_{l-1} = L_l^- y_l \not\equiv 0$ and interpreting (0.3) as the expression $L_l^+ y_{l-1} = 0$. Thus we solve one first order equation to get nonzero $y_{l-1}$ and another first order equation to get $y_l$. Then applying the operators $\{L_m^\pm\}$ we get a complete ladder of solutions $\{y_m\}$, $m \in S$.

To obtain a second ladder of solutions independent of the first we solve (0.3) by requiring $L_l^- y_l' = 0$. (We distinguish solutions on the second ladder with primes.) Applying the $\{L_m^+\}$ operators we then get solutions $y'_{l+1}, y'_{l+2}, \cdots$. Now require that $y'_{l-1}$ be a nonzero solution of the first order equation $L_l^+ y'_{l-1} = y'_l$. It then follows that $y'_{l-1}$ is a solution of (0.2) for $m = l - 1$. Applying the $\{L_m^-\}$ operators to $y'_{l-1}$ we get solutions $y'_{l-2}, y'_{l-3}, \cdots$. Since $y_l$, $y'_l$ are linearly independent solutions of (0.2) for $m = l$, it follows that $y_m, y'_m$ are linearly independent solutions for all $m \in S$.

*Note.* The above analysis is strictly correct only if there is exactly one $l \in S$ such that $a_l = \lambda$. If more than one solution of $a_m = \lambda$, $m \in S$, exists, then the analysis is slightly more complicated.

In this paper we shall use a Lie algebraic version of the above argument to embed all solutions of the differential equations of hypergeometric type in ladders. This method will automatically give recursion relations and addition theorems for these special functions. Among the functions which appear naturally in this algebraic analysis are the polynomials of Jacobi, Laguerre and Hermite, the error, incomplete gamma, and incomplete beta functions, Legendre functions of the second kind, and certain logarithmic solutions of the hypergeometric equation.

Since our method is completely algebraic, it can be used on factorizations other than those by first order differential operators. For example, it would work when applied to factorizations involving higher order differential operators [4], first order difference operators [3], or first order $q$-difference operators [5].

**1. Some reducible representations of $\mathscr{G}(0, 1)$.** The 4-dimensional complex Lie algebra $\mathscr{G}(0,1)$ with basis $\mathscr{J}^+, \mathscr{J}^-, \mathscr{J}^3, \mathscr{E}$ is defined by the commutation relations

$$[\mathscr{J}^3, \mathscr{J}^\pm] = \pm\mathscr{J}^\pm, \qquad [\mathscr{J}^+, \mathscr{J}^-] = -\mathscr{E}, \qquad [\mathscr{E}, \mathscr{J}^\pm] = [\mathscr{E}, \mathscr{J}^3] = \mathscr{O}, \tag{1.1}$$

where $\mathscr{O}$ is the zero element of the Lie algebra. $\mathscr{G}(0,1)$ is the Lie algebra of the simply connected Lie group consisting of all $4 \times 4$ matrices

$$g(a,b,c,\tau) = \begin{pmatrix} 1 & ce^\tau & a & \tau \\ 0 & e^\tau & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad a,b,c,\tau \in \mathbb{C}, \tag{1.2}$$

where the group operation is matrix multiplication. In fact,

$$g(a_1, b_1, c_1, \tau_1)g(a_2, b_2, c_2, \tau_2) = g(a_1 + a_2 + c_1 b_2 e^{\tau_1}, b_1 + e^{\tau_1} b_2, c_1 + e^{-\tau_1} c_2, \tau_1 + \tau_2) \tag{1.3}$$

and

$$\exp a\mathscr{E} = g(a, 0, 0, 0), \qquad \exp b\mathscr{J}^+ = g(0, b, 0, 0),$$
(1.4)
$$\exp c\mathscr{J}^- = g(0, 0, c, 0), \qquad \exp \tau\mathscr{J}^3 = g(0, 0, 0, \tau).$$

Let $\rho$ be a representation of $\mathscr{G}(0, 1)$ by linear operators on the complex vector space $V$ and set $J^\pm = \rho(\mathscr{J}^\pm)$, $J^3 = \rho(\mathscr{J}^3)$, $E = \rho(\mathscr{E})$. Then we have

$$[J^3, J^\pm] = \pm J^\pm, \qquad [J^+, J^-] = -E,$$
(1.5)
$$[E, J^\pm] = [E, J^3] = 0,$$

where 0 is the zero operator and $[A, B] = AB - BA$ for linear operators $A, B$ on $V$. The invariant operator

$$C_{0,1} = J^+J^- - J^3E$$

commutes with $J^\pm, J^3, E$ so that if $\rho$ is irreducible we would expect $C_{0,1}$ and $E$ to be multiples of the identity operator $I$ on $V$. The *spectrum* $S$ of $J^3$ is defined to be the set of eigenvalues of $J^3$ on $V$. The *multiplicity* of the eigenvalue $\lambda \in S$ is the dimension of the eigenspace $V^\lambda$,

$$V^\lambda = \{v \in V: J^3 v = \lambda v\}.$$

We shall classify all representations $\rho$ of $\mathscr{G}(0, 1)$ satisfying the following properties:

(A) There is a countable basis for $V$ consisting of eigenvectors of $J^3$. Each eigenvalue has multiplicity one.

(B) If $W_1, W_2$ are disjoint subspaces of $V$ such that $\rho(\alpha)W_i \subset W_i$ for all $\alpha \in \mathscr{G}(0, 1)$ and $W_1 \oplus W_2 = V$, then either $W_1 = V$, $W_2 = \varnothing$ or $W_2 = V$, $W_1 = \varnothing$.

Note that condition (B) requires only that $\rho$ be indecomposable, not necessarily irreducible.

An algebraic analysis similar to that given in [1, Chap. 2] yields the following possibilities for $\rho$.

THEOREM 1. *Every representation $\rho$ of $\mathscr{G}(0,1)$ satisfying conditions* (A), (B) *and for which $E \not\equiv 0$ is isomorphic to a representation in the following list:*

(i) *The representations $R(\omega, m_0, \mu)$ defined for all $\omega, m_0, \mu \in \mathbb{C}$ such that $\mu \neq 0, 0 \leqq \operatorname{Re} m_0 < 1$, and $\omega + m_0$ is not an integer, $S = \{m_0 + n : n = 0, \pm 1, \pm 2, \cdots\}$.*

(ii) *The representation $\uparrow_{\omega,\mu}$ defined for all $\omega, \mu \in \mathbb{C}$ such that $\mu \neq 0, S = \{-\omega + n : n = 0, 1, 2, \cdots\}$.*

(iii) *The representations $\uparrow'_{\omega,\mu}$ defined for all $\omega, \mu \in \mathbb{C}$ such that $\mu \neq 0, S = \{-\omega + n : n = 0, \pm 1, \pm 2, \cdots\}$.*

*For each of the cases* (i)–(iii) *there is a basis of $V$ consisting of vectors $f_m$, $m \in S$, such that*

$$J^3 f_m = m f_m, \qquad E f_m = \mu f_m,$$
(1.6)
$$J^+ f_m = \mu f_{m+1}, \qquad J^- f_m = (m + \omega) f_{m-1},$$
$$C_{0,1} f_m = (J^+J^- - EJ^3) f_m = \mu\omega f_m.$$

(On the right-hand side of these expressions we assume $f_m = 0$ if $m \notin S$.)

(iv) *The representation* $\downarrow_{\omega,\mu}$ *defined for all* $\omega, \mu \in \mathbb{C}$ *such that* $\mu \neq 0$, $S = \{-\omega - 1 - n : n = 0, 1, 2, \cdots\}$.

(v) *The representation* $\downarrow'_{\omega,\mu}$ *defined for all* $\omega, \mu \in \mathbb{C}$ *such that* $\mu \neq 0$, $S = \{-\omega + n : n = 0, \pm 1, \pm 2, \cdots\}$.

*For each of the cases* (iv) *and* (v) *there is a basis of* $V$ *consisting of vectors* $f_m, m \in S$, *such that*

$$J^3 f_m = m f_m, \qquad E f_m = \mu f_m,$$
$$J^+ f_m = (m + \omega + 1) f_{m+1}, \qquad J^- f_m = \mu f_{m+1}, \tag{1.7}$$
$$C_{0,1} f_m = (J^+ J^- - E J^3) f_m = \mu\omega f_m.$$

Classes (i), (ii) and (iv) contain irreducible representations which were studied in [1]. However, classes (iii) and (v) contain reducible representations. In particular, if we restrict $\uparrow'_{\omega,\mu}$ to the subspace generated by the basis vectors $f_m$ with $m = -\omega + n$, $n = 0, 1, 2, \cdots$, we get a representation isomorphic to $\uparrow_{\omega,\mu}$. Similarly, if we restrict $\downarrow'_{\omega,\mu}$ to the subspace generated by the $f_m$ with $m = -\omega - 1 - n$, $n = 0, 1, 2, \cdots$, we get a representation isomorphic to $\downarrow_{\omega,\mu}$. We shall construct models of the representations $\uparrow'_{\omega,\mu}$ and $\downarrow'_{\omega,\mu}$ in which the $J$-operators are differential operators in one and two complex variables.

First we construct a model of $\uparrow'_{\omega,\mu}$ in the form of differential operators acting on a space of functions of one complex variable $z$. Namely, we let $V$ be the space of all finite linear combinations of the functions $h_n(z) = z^n$, $n = 0, \pm 1, \pm 2, \cdots$, and define the $J$-operators on $V$ by

$$J^3 = -\omega + z\frac{d}{dz}, \quad J^+ = \mu z, \quad J^- = \frac{d}{dz}, \quad E = \mu. \tag{1.8}$$

The basis vectors $f_m$ are defined by $f_m(z) = h_n(z) = z^n$, $m = -\omega + n$, $n$ an integer. Then

$$J^3 f_m = \left(-\omega + z\frac{d}{dz}\right) z^n = (-\omega + n) z^n = m f_m,$$
$$J^+ f_m = \mu z^{n+1} = \mu f_{n+1}, \quad J^- f_m = \frac{d}{dz} z^n = n z^{n-1} = (\omega + m) f_{m-1} \tag{1.9}$$

for all $m \in S$ and we have a realization of $\uparrow'_{\omega,\mu}$. Just as in [1, Chap. 4] we can use our model to induce a local multiplier representation of the Lie group $G(0, 1)$ and can compute the matrix elements of the group representation. A simple computation shows that the local multiplier representation is defined by operators $B(g)$, $g \in G(0, 1)$, acting on the space of all functions $f(z)$ analytic in a deleted neighborhood of $z = 0$:

$$[B(g)f](z) = \exp[\mu(bz + a) - \omega\tau] f(e^\tau z + e^\tau c). \tag{1.10}$$

Here the group element $g$ is parametrized as in (1.2).

In the usual way [1], the matrix elements $B_{lk}(g)$ of the operators $B(g)$ are defined by

$$[B(g)h_k](z) = \sum_{l=-\infty}^{+\infty} B_{lk}(g) h_l(z), \qquad k = 0, \pm 1, \pm 2, \cdots, \tag{1.11}$$

where $h_n(z) = z^n$. Thus,

$$\exp[\mu(bz + a) + (k - \omega)\tau]z^k(1 + c/z)^k = \sum_{l=-\infty}^{\infty} B_{lk}(g)z^l, \qquad |c/z| < 1. \tag{1.12}$$

The representation property $B(g_1g_2) = B(g_1)B(g_2), g_1, g_2 \in G(0, 1)$, leads to the addition theorem

$$B_{lk}(g_1g_2) = \sum_{j=-\infty}^{+\infty} B_{lj}(g_1)B_{jk}(g_2), \qquad l, k = 0, \pm 1, \pm 2, \cdots. \tag{1.13}$$

Comparing coefficients of $z^l$ on both sides of (1.12) we find

$$B_{lk}(g) = \exp[\mu a + (k - \omega)\tau]c^{k-l}L_l^{(k-l)}(-\mu bc), \tag{1.14}$$

where the $L_\nu^{(\alpha)}(z)$ are generalized Laguerre functions [6]. Substituting (1.14) into (1.12) and simplifying we obtain the generating function

$$e^{-zt}(1 + t)^k = \sum_{j=-\infty}^{\infty} t^j L_j^{(k-j)}(z), \quad |t| < 1, \qquad k = 0, \pm 1, \pm 2, \cdots. \tag{1.15}$$

A careful examination of (1.15) shows that $L_l^{(k-l)}(z)$ in (1.14) is (a) a Laguerre polynomial if $k, l \geqq 0$, (b) identically zero if $k \geqq 0, l < 0$ and (c) a nonzero Laguerre function, not a polynomial, if $k < 0$. If we write $B_{lk}(g) = P_{lk}(g)$ for $k, l \geqq 0$, $B_{lk}(g) = F_{lk}(g)$ for $k < 0$, $l \geqq 0$, and $B_{lk}(g) = H_{lk}(g)$ for $k < 0$, $l < 0$, then we can write the matrix $(B_{lk}(g))$ in the form

$$(B_{lk}(g)) = \left[\begin{array}{c:c} P_{lk}(g) & F_{lk}(g) \\ \hdashline 0 & H_{lk}(g) \end{array}\right]. \tag{1.16}$$

The $P_{lk}(g)$ are the matrix elements of the irreducible representation $\uparrow_{\omega,\mu}$ (see [1]). In terms of the generalized Laguerre functions, the addition theorem (1.13) becomes

$$\begin{aligned} &e^{-c_1b_2}(c_1 + c_2)^n L_l^{(n)}[(b_1 + b_2)(c_1 + c_2)] \\ &\quad = \sum_{j=-\infty}^{\infty} (c_1)^{j+n} L_j^{(j+n)}[b_1c_1](c_2)^{-j} L_{l+j+n}^{(-j)}[b_2c_2], \qquad l, n = 0, \pm 1, \pm 2, \cdots. \end{aligned} \tag{1.17}$$

However, it follows easily from (1.16) that this addition theorem breaks up into the three identities:

$$\begin{aligned} \mathscr{I} &= \sum_{j=0}^{\infty} (c_1)^{j-l} L_l^{(j-l)}[b_1c_1](c_2)^{l+n-j} L_j^{(l+n-j)}[b_2c_2], \qquad l \geqq 0, \quad l + n \geqq 0, \\ \mathscr{I} &= \sum_{j=1}^{\infty} (c_1)^{-l-j} L_l^{(-l-j)}[b_1c_1](c_2)^{n+l+j} L_{-j}^{(n+l+j)}[b_2c_2], \qquad l < 0, \quad l + n < 0, \\ \mathscr{I} &= \sum_{j=-\infty}^{\infty} (c_1)^{j+n} L_l^{(j+n)}[b_1c_1](c_2)^{-j} L_{l+j+n}^{(-j)}[b_2c_2], \qquad l \geqq 0, \quad l + n < 0, \end{aligned} \tag{1.18}$$

where $\mathscr{I}$ is the left-hand side of (1.17) and all of the terms on the right-hand sides of expressions (1.18) are nonzero.

In a similar manner we can construct a model of $\downarrow'_{\omega,\mu}$. Let $V$ be the space of all finite linear combinations of the functions $h_n(z) = z^n$, $n = 0, \pm 1, \pm 2, \cdots$, and define the $J$-operators on $V$ by

$$J^3 = -\omega - 1 - z\frac{d}{dz}, \quad J^+ = -\frac{d}{dz}, \quad J^- = \mu z, \tag{1.19}$$
$$E = \mu.$$

The basis vectors $f_m$ are given by $f_m(z) = h_n(z) = z^n, m = -\omega - 1 - n, n = 0, \pm 1, \pm 2, \cdots$. Then

$$J^3 f_m = \left(-\omega - 1 - z\frac{d}{dz}\right)z^n = (-\omega - 1 - n)z^n = mf_m,$$

$$J^+ f_m = -\frac{d}{dz}z^n = -nz^{n-1} = (m + \omega + 1)f_{m+1},$$

$$J^- f_m = \mu z^{n+1} = \mu f_{m-1}, \quad m \in S,$$

so we indeed have a model of $\downarrow'_{\omega,\mu}$. The local multiplier representation of $G(0, 1)$ induced by this model is given by the operators

$$[C(g)f](z) = \exp[\mu(-cz - bc + a) - (\omega + 1)\tau]f(e^{-\tau}z + e^{-\tau}b).$$

The matrix elements $C_{lk}(g)$ are defined by

$$[C(g)h_k](z) = \sum_{l=-\infty}^{\infty} C_{lk}(g)h_l(z), \qquad k = 0, \pm 1, \pm 2, \cdots,$$

or

$$C_{lk}(g) = \exp[\mu(a - bc) - (\omega + k + 1)\tau](-b)^{k-l}L_l^{(k-l)}(\mu bc). \tag{1.20}$$

The addition theorem obtained from these matrix elements is equivalent to (1.17) so we omit it.

**2. Models of the representations by type $D'$ operators.** The type $D'$ operators

$$J^3 = t\frac{\partial}{\partial t}, \quad J^\pm = t^{\pm 1}\left(\pm\frac{\partial}{\partial z} - \frac{\mu}{2}z\right), \quad E = \mu, \tag{2.1}$$

satisfy the commutation relations (1.5) and can be used to construct models of representations of $\mathcal{G}(0, 1)$ in which $V$ is a space of analytic functions of two complex variables $t$ and $z$ (see [1, Chap. 4]). In particular, the basis vectors $f_m$ for a model of one of the representations classified in § 1 will take the form $f_m(z, t) = g_m(z)t^m$, $m \in S$. Furthermore, the equation $C_{0,1}f_m = \mu\omega f_m$ reduces to the second order differential equation

$$\left(\frac{d^2}{dz^2} - \frac{\mu^2 z^2}{4} + \frac{\mu}{2} + \mu(m + \omega)\right)g_m(z) = 0. \tag{2.2}$$

This is the parabolic cylinder equation, and its solutions $g_m(z)$ are parabolic cylinder functions. In fact, the parabolic cylinder functions $D_{m+\omega}(\sqrt{\mu}z)$ and $D_{-m-\omega-1}(i\sqrt{\mu}z)$ form a basis for the solutions of (2.2) (see [6]). (We see from this result that without loss of generality we can restrict ourselves to the case $\omega = 0$, $\mu = 0$, since this can always be achieved by an appropriate change of variables.) In [1] the functions $D_m(z)$ and $D_{-m-1}(iz)$, for $m$ not an integer, were obtained as basis vectors $f_m(z, t) = g_m(z)t^m$ of type $D'$ models of the representations $R(0, m_0, 1)$. We now proceed to give a group theoretic construction and interpretation of a basis of solutions for (2.2) when $\mu = 1$, $\omega = 0$ and $m$ is an integer.

First we use the type $D'$ operators to construct a model of $\uparrow_{0,1}$. The equation $J^- f_0 = 0$ implies $g_0(z) = c \exp(-z^2/4)$, $c \in \mathbb{C}$. Let us set $c = 1$. Then using the recurrence relation $J^+ f_m = f_{m+1}$ and mathematical induction we can verify the result

$$g_m(z) = \exp(z^2/4)\frac{d^m}{dz^m}\exp(-z^2/2), \qquad m = 0, 1, 2, \cdots, \tag{2.3}$$

or

$$g_m(z) = (-1)^m D_m(z) = (-1)^m \exp(-z^2/4)2^{-m/2}H_m(\sqrt{2}z), \quad m = 0, 1, 2, \cdots, \tag{2.4}$$

where the $H_m(z)$ are Hermite polynomials [6]. We still have to compute the $g_m(z)$ for $m$ a negative integer. The expression $J^+ f_{-1} = f_0$ becomes

$$g'_{-1}(z) - \frac{z}{2}g_{-1}(z) = \exp(-z^2/4)$$

with the general solution

$$\begin{aligned} g_{-1}(z) &= c\exp(z^2/4) + \exp(z^2/4)\int_z^\infty \exp(-w^2/2)\,dw \\ &= c\exp(z^2/4) - \sqrt{\frac{\pi}{2}}\exp(z^2/4)\operatorname{Erfc}\left(\frac{z}{\sqrt{2}}\right). \end{aligned}$$

Here,

$$\operatorname{Erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty \exp(-w^2)\,dt$$

is the error function [6]. Though the above integral definition makes sense only for $|\arg w| < \pi/4$ as $w \to \infty$, Erfc $(z)$ is actually an entire function of $z$. For definiteness we set $c = 0$. Then

$$g_{-1}(z) = -\sqrt{\frac{\pi}{2}}\exp(z^2/4)\operatorname{Erfc}(z(\sqrt{2})) = -D_{-1}(z). \tag{2.5}$$

Applying the recurrence formula $J^- f_m = m f_{m-1}$ and using mathematical induction we can easily derive the results

$$\begin{aligned} g_m(z) = (-1)^m D_m(z) &= -\sqrt{\frac{\pi}{2}}\frac{\exp(-z^2/4)}{n!}\frac{d^n}{dz^n}\left[\exp(z^2/2)\operatorname{Erfc}\left(\frac{z}{\sqrt{2}}\right)\right] \\ &= (-1)^{n+1}\exp(z^2)2^n\int_z^\infty \frac{(t-z)^n}{n!}\exp(-t^2/2)\,dt \\ & \qquad\qquad m = -n-1, \quad n = 0, 1, 2, \cdots. \end{aligned} \tag{2.6}$$

Thus the functions $g_m(z) = (-1)^m D_m(z)$, $m = 0, \pm 1, \pm 2, \cdots$, yield a model for the reducible representation $\uparrow'_{0,1}$. In terms of this model, relations (1.6) read

$$\begin{aligned}
&\left(\frac{d}{dz} - \frac{z}{2}\right) D_m(z) = -D_{m+1}(z),\\
(2.7)\qquad &\left(\frac{d}{dz} + \frac{z}{2}\right) D_m(z) = m D_{m-1}(z),\\
&\left(\frac{d^2}{dz^2} - \frac{z^2}{4} + \frac{1}{2} + m\right) D_m(z) = 0, \qquad m = 0, \pm 1, \pm 2, \cdots.
\end{aligned}$$

The multiplier representation induced by the type $D'$ operators is

$$\begin{aligned}
(2.8)\qquad [\underline{T}(g)f](z,t) = \exp\left[-\frac{t^2b^2}{4} - \frac{ztb}{2} - \frac{xt^{-1}c}{2} + \frac{t^{-2}c^2}{4} - \frac{bc}{2} + a\right]\\
\cdot f(z + bt - ct^{-1}, e^{\tau}t),
\end{aligned}$$

where $g \in G(0, 1)$ (see [1, Chap. 4]). Thus,

$$(2.9)\qquad [\underline{T}(g)f_k](z,t) = \sum_{l=-\infty}^{\infty} B_{lk}(g) f_l(z,t), \qquad k = 0, \pm 1, \pm 2, \cdots,$$

where the matrix elements $B_{lk}(g)$ are given by (1.14). In terms of special functions this relation is

$$\begin{aligned}
(2.10)\qquad &\exp\left(-\frac{b^2}{4} - \frac{zb}{2} - \frac{zc}{2} + \frac{c^2}{4} - \frac{bc}{2}\right) D_k(z + b - c)\\
&= \sum_{l=-\infty}^{\infty} (-c)^{-l} L_{m+l}^{(-l)}(-bc) D_{k+l}(z), \qquad k = 0, \pm 1, \pm 2, \cdots.
\end{aligned}$$

(Recall that $L_l^{(k-l)}(z) \equiv 0$ if $k \geqq 0$, $l < 0$.) Some special cases of (2.10) are

$$\begin{aligned}
&\exp\left(-\frac{b^2}{4} + \frac{zb}{2}\right) D_k(z - b) = \sum_{l=0}^{\infty} \frac{(b)^l}{l!} D_{k+l}(z),\\
(2.11)\qquad &\exp\left(\frac{c^2}{4} + \frac{zc}{2}\right) D_k(z + c) = \sum_{l=0}^{k} \binom{k}{l} c^l D_{k-l}(z) \quad \text{if } k = 0, 1, 2, \cdots,\\
&\exp\left(\frac{c^2}{4} + \frac{zc}{2}\right) D_k(z + c) = \sum_{l=0}^{\infty} \binom{k}{l} c^l D_{k-1}(z) \quad \text{if } k = -1, -2, \cdots.
\end{aligned}$$

The first of these equations in the case $k = 0$ yields the generating function

$$\exp\left[-\frac{z^2}{4} - \frac{b^2}{2} + zb\right] = \sum_{l=0}^{\infty} \frac{b^l}{l!} D_l(z).$$

Now we use the type $D'$ operators to construct a model of $\downarrow'_{0,1}$. Setting $f_m(z, t) = h_m(z)t^m$ we find $J^+ f_{-1} = 0$ or $h_{-1}(z) = c \exp(z^2/4)$. Setting $c = 1$ and using the recurrence relation $J^- f_m = f_{m-1}$ we obtain

$$h_{-n-1}(z) = \exp(-z^2/4)\frac{d^n}{dz^n}\exp(z^2/2)$$
$$= i^n D_n(iz), \qquad n = 0, 1, 2, \cdots.$$

The relation $J^- f_0 = f_{-1}$ leads to

$$h_0'(z) + \frac{z}{2}h_0(z) = -\exp(z^2/4)$$

with the solution

$$\begin{aligned} h_0(z) &= c\exp(-z^2/4) + \exp(-z^2/4)\int_z^\infty \exp(w^2/2)\,dw \\ &= c\exp(-z^2/4) + i\sqrt{\frac{\pi}{2}}\exp(-z^2/4)\,\mathrm{Erfc}\left(\frac{iz}{\sqrt{2}}\right). \end{aligned} \tag{2.12}$$

Again we choose $c = 0$ and obtain

$$h_0(z) = i\sqrt{\frac{\pi}{2}}\exp(-z^2/4)\,\mathrm{Erfc}\left(\frac{iz}{\sqrt{2}}\right) = -iD_{-1}(iz).$$

Finally the recurrence formula $J^+ f_m = (m+1)f_{m+1}$ yields

$$h_{-n-1}(z) = (i)^n D_n(iz), \qquad n = -1, -2, -3, \cdots.$$

Thus,

$$h_m(z) = (i)^{-m-1}D_{-m-1}(iz), \qquad m = 0, \pm 1, \pm 2, \cdots. \tag{2.13}$$

It follows from our analysis that $h_m(z) = (-1)^m D_m(z)$ and $h_m(z) = (i)^{-m-1}D_{-m-1}(iz)$ both satisfy the same differential equation, namely, the parabolic cylinder equation (2.2) with $\omega = 0$, $\mu = 1$. Furthermore, by the remarks following (0.2) in the Introduction, $D_m(z)$ and $D_{-m-1}(iz)$ must be linearly independent for all integers $m$. We have succeeded in embedding two linearly independent solutions of the parabolic cylinder equation for integer $m$ in models of representations of $\mathscr{G}(0, 1)$. The addition theorems and generating functions for the $h_m(z)$ are essentially the same as those for the $g_m(z)$, so we omit their derivation.

*Remark.* We already know from special function theory that $D_m(z)$ and $D_{-m-1}(iz)$ are linearly independent solutions of the parabolic cylinder equation. However, it is instructive to see that this result can be obtained directly from Lie algebraic techniques.

**3. Models of the representations by type $C'$ operators.** The type $C'$ operators

$$\begin{aligned} &J^3 = t\frac{\partial}{\partial t}, \quad J^+ = t\left(\frac{\partial}{\partial z} - 1\right), \quad J^- = t^{-1}\left(-z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t} + q\right), \\ &E = \mu, \qquad q, \mu \in \mathbb{C}, \end{aligned} \tag{3.1}$$

also satisfy the commutation relations (1.5) and can be used to construct models of representations of $\mathscr{G}(0, 1)$ in which $V$ is a space of analytic functions of $t$ and $z$

(see [1]). If we set $f_m(z,t) = g_m(z)t^m$, then the equation $C_{0,1}f_m = \mu\omega f_m$ reduces to

$$\left(z\frac{d^2}{dz^2} + (m+1-q-z)\frac{d}{dz} + q\right)g_m(z) = 0. \tag{3.2}$$

(It is easy to show that there is no loss of generality if we set $\omega = 0$, $\mu = 1$ and we have done this in (3.2).) This is the confluent hypergeometric equation with linearly independent solutions ${}_1F_1(-q; m-q+1; z)$ and $z^{q-m}{}_1F_1(-m; q-m+1; z)$ when $m-q$ is not an integer. In [1, Chap. 4] all solutions of (3.2) for which $m$ is not an integer were obtained as basis vectors of type $C'$ models of $R(0, m_0, 1)$. We now show that type $C'$ models of $\uparrow'_{0,1}$ and $\downarrow'_{0,1}$ will give a complete set of solutions for integer $m$.

To get a model for $\uparrow'_{0,1}$ we first examine the relation $J^-f_0 = 0$. This implies $g_0(z) = cz^q$. We set $c = 1$ and use the recurrence relation $J^+f_m = f_{m+1}$ to obtain

$$g_m(z) = e^z\left(\frac{d}{dz}\right)^m(z^qe^{-z}) = m!z^{q-m}L_m^{(q-m)}(z), \qquad m = 0,1,2,\cdots. \tag{3.3}$$

To compute $g_m(z)$ for negative $m$ consider the expression $J^+f_{-1} = f_0$ or

$$g'_{-1}(z) - g_{-1}(z) = z^q$$

with the general solution

$$g_{-1}(z) = -e^z\int_z^\infty e^{-w}w^q\,dw + ce^z, \qquad c \in \mathbb{C},$$

where

$$-e^z\int_z^\infty e^{-w}w^q\,dw = -e^z\Gamma(q+1,z) = -\Psi(-q,-q;z),$$

$\Gamma(a,z)$ is the incomplete gamma function and $\Psi(a,c;z)$ is a confluent hypergeometric function [6]. We choose the solution such that $c = 0$ and use the recurrence formula $J^-f_m = mf_{m-1}$ to obtain

$$\begin{aligned} g_m(z) &= \frac{(-1)^{n+1}}{n!}z^{q+n+1}\frac{d^n}{dz^n}[z^{-q-1}e^z\Gamma(q+1,z)] \\ &= (-1)^m\Psi(-q,m-q+1;z), \qquad m = -n-1, \quad n = 0,1,2,\cdots. \end{aligned} \tag{3.4}$$

It follows from the definition of the $\Psi$-functions that a basis for $\uparrow'_{0,1}$ is provided by the functions

$$g_m(z) = (-1)^m\Psi(-q,m-q+1;z), \qquad m = 0,\pm1,\pm2,\cdots. \tag{3.5}$$

Indeed, relations (1.6) become

$$\begin{aligned} &\left(\frac{d}{dz} - 1\right)\Psi(-q,m-q+1;z) = -\Psi(-q,m-q+2;z), \\ &\left(z\frac{d}{dz} + m - q\right)\Psi(-q,m-q+1;z) = m\Psi(-q,m-q;z), \end{aligned} \tag{3.6}$$

$$\left(z\frac{d^2}{dz^2} + (m + 1 - q - z)\frac{d}{dz} + q\right)\Psi(-q, m - q + 1; z) = 0,$$

$$m = 0, \pm 1, \pm 2, \cdots.$$

The multiplier representation induced by the type $C'$ operators is

$$[\underline{T}(g)f](z,t) = e^{a-bt}(1 - c/t)^{-q}f[(z + bt)(1 - c/t), e^{\tau}(t - c)],$$
$$|c/t| < 1, \quad g \in G(0,1)$$

(see [1, Chap. 4]). It follows that expression (2.9) holds and we obtain the identity

$$\begin{aligned} &e^{-bt}(1 + c/t)^{-q+k}\Psi(-q, k - q + 1; (z + bt)(1 + c/t)) \\ (3.7)\quad &= \sum_{j=-\infty}^{\infty} (-c)^{-j}L_{j+k}^{(-j)}(bc)\Psi(-q, j + k - q + 1; z)t^j, \qquad |c/t| < 1, \end{aligned}$$

where we recall that $L_{j+k}^{(-j)}(z) \equiv 0$ if $j + k < 0$, $k \geqq 0$. Various generating functions for the Laguerre polynomials and incomplete gamma functions are special cases of this identity.

We now use the type $C'$ operators to construct a model of $\downarrow'_{0,1}$. Set $f_m(z,t) = h_m(z)t^m$ and compute $h_{-1}(z)$ from $J^+f_{-1} = 0$. The result is $h_{-1}(z) = ce^z$ and we choose $c = 1$. From the recurrence relation $J^-f_m = f_{m-1}$ it follows that

$$\begin{aligned} h_m(z) &= z^{q+n+1}\frac{d^n}{dz^n}[e^z z^{-q-1}] \\ (3.8)\quad &= e^z\Psi(m + 1, m - q + 1; ze^{-i\pi}), \qquad m = -n - 1,\ n = 0, 1, 2, \cdots. \end{aligned}$$

To get $h_0(z)$ we solve $J^-f_0 = f_{-1}$ or $zh_0'(z) - qh_0(z) = -e^z$. The solution is

$$(3.9)\qquad h_0(z) = z^q\int_{-\infty}^{z} e^w w^{-q-1}\,dw + cz^q,$$

or $h_0(z) = (ze^{-i\pi})^q\Gamma(-q, ze^{-i\pi})$ if we set $c = 0$. The recurrence formula $J^+f_m = (m + 1)f_{m+1}$ then yields

$$\begin{aligned} h_m(z) &= \frac{e^z}{m!}\frac{d^m}{dz^m}[e^{-z}(-z)^q\Gamma(-q, -z)] \\ &= e^z\Psi(m + 1, m - q + 1; ze^{-i\pi}), \qquad m = 0, 1, 2, \cdots. \end{aligned}$$

Thus the functions

$$(3.10)\qquad h_m(z) = e^z\Psi(m + 1, m - q + 1; ze^{-i\pi}), \qquad m = 0, \pm 1, \pm 2, \cdots,$$

form a basis for our model. It follows from the remarks in the Introduction that $g_m(z) = (-1)^m\Psi(-q, m - q + 1; z)$ and $h_m(z)$ are linearly independent solutions of the same confluent hypergeometric equation for each integer value of $m$.

**4. Some reducible representations of** $sl(2)$. The 3-dimensional complex Lie algebra $sl(2)$ is defined by the relations

$$(4.1)\qquad [\mathscr{J}^3, \mathscr{J}^{\pm}] = \pm\mathscr{J}^{\pm}, \qquad [\mathscr{J}^+, \mathscr{J}^-] = 2\mathscr{J}^3.$$

Here, $sl(2)$ is the Lie algebra of the group $SL(2)$ of all $2 \times 2$ complex matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{4.2}$$

such that $\det g = 1$. The group operation is matrix multiplication. We can make the identifications

$$\begin{aligned} \exp a\mathscr{J}^3 &= \begin{pmatrix} e^{a/2} & 0 \\ 0 & e^{-a/2} \end{pmatrix}, \quad \exp b\mathscr{J}^+ = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}, \\ \exp c\mathscr{J}^- &= \begin{pmatrix} 1 & 0 \\ -c & 0 \end{pmatrix}, \quad a, b, c \in \mathbb{C}. \end{aligned} \tag{4.3}$$

Just as in § 1 we let $\rho$ be a representation of $sl(2)$ by linear operators on the vector space $V$ and set $\rho(\mathscr{J}^\pm) = J^\pm$, $\rho(\mathscr{J}^3) = J^3$. The invariant operator

$$C_{1,0} = J^+J^- + J^3J^3 - J^3 \tag{4.4}$$

commutes with $J^\pm$, $J^3$. Again we classify all representations $\rho$ of $sl(2)$ which satisfy conditions (A) and (B) of § 1. A straightforward analysis yields the following results.

THEOREM 2. *Every representation $\rho$ of $sl(2)$ satisfying conditions* (A), (B) *is isomorphic to a representation in the following list:*

(i) *The representations $D(u, m_0)$ defined for complex $u, m_0$ such that $m_0 \pm u$ are not integers and $0 \leqq \operatorname{Re} m_0 < 1$, $S = \{m_0 + n : n = 0, \pm 1, \pm 2, \cdots\}$, $D(u, m_0) \cong D(-u - 1, m_0)$.*

(ii) *The representations $\uparrow_u$, $u \in \mathbb{C}$, such that $2u \neq 0, 1, 2, \cdots$, $S = \{-u + n : n = 0, 1, 2, \cdots\}$.*

(iii) *The representations $\uparrow'_u$, $2u$ not an integer, $S = \{-u + n : n = 0, \pm 1, \pm 2, \cdots\}$.*

(iv) *The representations $\downarrow_u$, $u \in \mathbb{C}$, $2u \neq 0, 1, 2, \cdots$, $S = \{u - n : n = 0, 1, 2, \cdots\}$.*

(v) *The representations $\downarrow'_u$, $2u$ not an integer, $S = \{u - n : n = 0, \pm 1, \pm 2, \cdots\}$.*

(vi) *The representations $D(2u)$, $2u = 0, 1, 2, \cdots$, $S = \{-u, -u + 1, \cdots, u - 1, u\}$.*

(vii) *The representations $D'(2u)$, $2u = 0, 1, 2, \cdots$, $S = \{n : n = 0, \pm 1, \pm 2, \cdots\}$.*

*For each of these representations there is a basis $\{f_m\}$ for $V$ such that*

$$\begin{aligned} &J^3 f_m = m f_m, \quad J^+ f_m = (-u + m) f_{m+1}, \\ &J^- f_m = (-u - m) f_{m-1}, \\ &C_{1,0} f_m = (J^+J^- + J^3J^3 - J^3) f_m = u(u + 1) f_m, \qquad m \in S. \end{aligned} \tag{4.5}$$

(We make the convention that $f_m \equiv 0$ on the right-hand side of expressions (4.5) if $m \notin S$.)

(viii) *The representations $D^+(2u)$, $2u = -1, 0, 1, 2, \cdots$, $S = \{-u + n : n = 0, \pm 1, \pm 2, \cdots\}$. $D^+(2u)$ is defined by the relations*

$$\begin{aligned} &J^3 f_m = m f_m, \quad J^+ f_m = f_{m+1}, \\ &J^- f_m = (u + m)(u - m + 1) f_{m-1}, \quad C_{1,0} f_m = u(u + 1) f_m, \qquad m \in S. \end{aligned} \tag{4.6}$$

(ix) *The representations* $D^-(2u)$, $2u = -1, 0, 1, 2, \cdots$, $S = \{-u + n : n = 0, \pm 1, \pm 2, \cdots\}$, *defined by*

$$J^3 f_m = m f_m, \quad J^+ f_m = (u - m)(u + m + 1) f_{m+1},$$
$$J^- f_m = f_{m-1}, \quad C_{1,0} f_m = u(u + 1) f_m, \qquad m \in S. \tag{4.7}$$

(x) *The representation* $D^{+-}(2u)$, $2u = 0, 1, 2, \cdots$, $S = \{-u + n : n = 0, \pm 1, \pm 2, \cdots\}$, *defined by*

$$J^3 f_m = m f_m, \quad J^+ f_m = (u + m + 1) f_{m+1},$$
$$J^- f_m = (u - m + 1) f_{m-1}, \quad C_{1,0} f_m = u(u + 1) f_m, \qquad m \in S. \tag{4.8}$$

Classes (i), (ii), (iv) and (vi) contain irreducible representations and were analyzed in [1, Chap. 5]. The remaining representations are reducible and we shall construct models of them in this paper.

We start by constructing a model of $\uparrow'_u$ in which $V$ is a space of functions of one complex variable $z$. Let $V$ be the space of all finite linear combinations of $h_n(z) = z^n$, $n = 0, \pm 1, \pm 2$, and define the $J$ operators on $V$ by

$$J^3 = -u + z\frac{d}{dz}, \quad J^+ = -2uz + z^2\frac{d}{dz}, \quad J^- = -\frac{d}{dz}. \tag{4.9}$$

The choice $f_m(z) = z^n$, $m = -u + n$, is easily seen to yield a model of $\uparrow'_u$. Expressions (4.9) define a local multiplier representation of $SL(2)$ given by

$$[B(g)f](z) = (bz + d)^{2u} f\left(\frac{az + c}{bz + d}\right),$$

where $g \in SL(2)$ is defined by (4.2). The matrix elements $B_{lk}$ satisfy

$$[B(g)h_k](z) = \sum_{l=-\infty}^{\infty} B_{lk}(g) h_l(z), \qquad k = 0, \pm 1, \pm 2, \cdots,$$

or

$$(az)^k d^{2u-k}(1 + bz/d)^{2u-k}(1 + c/az)^k = \sum_{l=-\infty}^{\infty} B_{lk}(g) z^l,$$
$$|bz/d| < 1, \quad |c/az| < 1.$$

Thus,

$$B_{lk}(g) = a^k d^{2u-l} b^{l-k} \frac{\Gamma(2u - k + 1)}{\Gamma(2u - l + 1)} \frac{F(-k, -2u + l; l - k + 1; bc/(ad))}{\Gamma(l - k + 1)},$$
$$k, l = 0, \pm 1, \pm 2, \cdots. \tag{4.10}$$

(Recall that $F(a, b; c; z)/\Gamma(c)$ is an entire function of $a, b, c$ and is analytic and single-valued in the $z$-plane cut along the positive real axis from $+1$ to $\infty$.) Here we interpret the ratio $\Gamma(2u - k + 1)/\Gamma(2u - l + 1)$ as

$$\frac{\Gamma(2u - k + 1)}{\Gamma(2u - l + 1)} = \begin{cases} (2u - k)(2u - k - 1) \cdots (2u - l + 2)(2u - l + 1) & \text{if } l \geqq k + 1, \\ 1 & \text{if } l = k, \\ [(2u - l)(2u - l - 1) \cdots (2u - k + 2)(2u - k + 1)]^{-1} & \text{if } l \leqq k - 1. \end{cases} \tag{4.11}$$

The hypergeometric function defining $B_{lk}(g)$ is a nonzero polynomial if $k, l \geqq 0$, a nonzero function not a polynomial if $k < 0$, and is identically zero if $k \geqq 0$, $l < 0$. This result is similar to that for the matrix elements (1.14) in § 2. From (4.10) we have the generating function

$$z^k(1+z)^{2u-k}\left(1+\frac{b}{z}\right)^k = \sum_{l=-\infty}^{\infty} z^l \frac{\Gamma(2u-k+1)}{\Gamma(2u-l+1)} \frac{F(-k, -2u+l; l-k+1; b)}{\Gamma(l-k+1)},$$

$$|z| < 1, \quad \left|\frac{b}{z}\right| < 1.$$

The addition theorem for the matrix elements reads

$$B_{lk}(g_1 g_2) = \sum_{j=-\infty}^{\infty} B_{lj}(g_1) B_{jk}(g_2), \qquad l, k = 0, \pm 1, \pm 2, \cdots, \tag{4.12}$$

valid for $g_1, g_2$ in a suitably small neighborhood of the identity element in $SL(2)$. (See [1, Chap. 5] for a more detailed analysis of the domain of validity of expressions like (4.12).)

A similar computation will give us the matrix elements of $\downarrow'_u$. Namely, we choose

$$J^3 = u - z\frac{d}{dz}, \quad J^+ = -\frac{d}{dz}, \quad J^- = -2uz + z^2\frac{d}{dz} \tag{4.13}$$

and set $f_m(z) = h_n(z) = z^n$, $m = u - n$, $n = 0, \pm 1, \pm 2, \cdots$. The multiplier representation determined by these operators is

$$[C(g)f](z) = (cz + a)^{2u} f\left(\frac{dz + b}{cz + a}\right), \qquad g \in SL(2).$$

In the usual manner we find that the matrix elements $C_{lk}(g)$ are given by the generating function

$$(dz)^k a^{2u-k}(1 + cz/a)^{2u-k}(1 + b/dz)^k = \sum_{l=-\infty}^{\infty} C_{lk}(g) z^l,$$

$$|cz/a| < 1, \quad |b/dz| < 1.$$

It follows that the $C_{lk}(g)$ are obtained from the $B_{lk}(g)$ by making the interchanges $a \leftrightarrow d$, $b \leftrightarrow c$.

To get a model of $D'(2u)$ we use the differential operators (4.8) again and set $f_m(z) = z^n$, $m = -u + n$, $n = 0, \pm 1, \pm 2, \cdots$. It follows that the matrix elements $D_{lk}(g)$ of this representation are identical with the matrix elements $B_{lk}(g)$ in (4.10), except that now $2u$ is a nonnegative integer. The matrix elements are zero if $k$ is in the interval $[0, 2u]$ and $l$ is not in this interval; otherwise they are nonzero.

The representations $D^+(2u)$, $D^-(2u)$, $D^{+-}(2u)$ have no models in one complex variable; the simplest models require two complex variables. Nevertheless, it is not difficult to determine the matrix elements by a purely formal computation. The existence of an analytic basis for the two complex variable models will justify the formal computation [1, Chap. 2]. Thus, we formally exponentiate the Lie algebra representations in classes (viii), (ix) and (x) to local Lie group representations by operators $\underline{T}(g)$ and define matrix elements $T_{lk}(g)$ by

$$\underline{T}(g)f_{-u+k} = \sum_{l=-\infty}^{\infty} T_{lk}(g)f_{-u+l}, \qquad k = 0, \pm 1, \pm 2, \cdots. \tag{4.14}$$

Note that we have written $m = -u + k$ for $m \in S$. Comparing relations (4.6)–(4.8) with (4.5) we see that our desired matrix elements can be obtained from the $B_{lk}(g)$ of (4.10) by a simple formal change of basis. Thus, for the representation $D^+(2u)$ we have

$$T_{lk}(g) = a^k d^{2u-l} b^{l-k} \frac{F(-k, -2u + l; l - k + 1; bc/(ad))}{\Gamma(l - k + 1)}. \tag{4.15}$$

The matrix elements are zero if $k \geqq 0, l < 0$ or $k \geqq 2u + 1, 0 \leqq l \leqq 2u$; otherwise they are nonzero.

The matrix elements for $D^-(2u)$ are

$$T_{lk}(g) = a^l d^{2u-k}(-c)^{k-l} \frac{F(-l, -2u + k; k - l + 1; bc/(ad))}{\Gamma(k - l + 1)}. \tag{4.16}$$

They are zero if $k \leqq 2u, l > 2u$ or $k \leqq -1, 0 \leqq l \leqq 2u$; otherwise they are nonzero.

The matrix elements for $D^{+-}(2u)$ are

$$T_{lk}(g) = a^l d^{2u-k}(c)^{k-l} \frac{\Gamma(2u - k + 1)}{\Gamma(2u - l + 1)} \frac{F(-l, -2u + k; k - l + 1; bc/(ad))}{\Gamma(k - l + 1)}, \tag{4.17}$$

where the ratio of gamma functions is interpreted as in (4.11). The $T_{lk}(g)$ are zero if $k \geqq 2u + 1, l \leqq 2u$ or $k \leqq -1, l \geqq 0$; otherwise they are nonzero.

Each of the families of matrix elements (4.15)–(4.17) satisfies the addition theorem

$$T_{lk}(g_1 g_2) = \sum_{j=-\infty}^{\infty} T_{lj}(g_1)T_{jk}(g_2), \qquad l, k = 0, \pm 1, \pm 2, \cdots,$$

for all $g_1, g_2 \in SL(2)$.

**5. Models of the representations by type $B$ operators.** The type $B$ operators

$$\begin{aligned} J^+ &= t\left(z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t} - z\right), \\ J^- &= t^{-1}\left(z\frac{\partial}{\partial z} - t\frac{\partial}{\partial t}\right), \quad J^3 = t\frac{\partial}{\partial t} \end{aligned} \tag{5.1}$$

satisfy the commutation relations of $sl(2)$. We shall construct models of representations of $sl(2)$ such that the basis vectors take the form $f_m(z, t) = g_m(z)t^m$, $m \in S$. Then the equation $C_{1,0}f_m = u(u + 1)f_m$ becomes the equation

$$\left[z^2\frac{d^2}{dz^2} - z^2\frac{d}{dz} + zm\right]g_m(z) = u(u + 1)g_m(z). \tag{5.2}$$

Expression (5.2) is equivalent to the confluent hypergeometric equation. In fact, for $u$ not an integer, (5.2) has the two linearly independent solutions $z^{-u}{}_1F_1(-u-m;-2u;z)$ and $z^{1+u}{}_1F_1(u-m+1;2+2u;z)$. In [1] we were able to embed all solutions of (5.2), for which $u \pm m$ were not integers, as basis vectors in models of the representations $D(u, m_0)$. We shall now construct all solutions of (5.2) in the cases where at least one of $u \pm m$ is an integer.

We start by using the type $B$ operators to construct a model of $\uparrow'_u$. (Since this procedure should be familiar to the reader by now, we shall eliminate some of the details of the construction.) The relation $J^- f_{-u} = 0$ implies $g_{-u}(z) = cz^{-u}$. We set $c = 1$ and use the relation $J^+ f_m = (m-u)f_{m+1}$ to go up the ladder of solutions. The result is

$$g_{-u+n}(z) = z^{-u}{}_1F_1(-n;-2u;z), \qquad n = 0,1,2,\cdots.$$

The relation $J^+ f_{-u-1} = -(2u+1)f_{-u}$ yields, by choosing the integration constant appropriately,

$$g_{-u-1}(z) = -(2u+1)e^z z^{u+1}\gamma(-2u-1,z) = z^{-u}{}_1F_1(1;-2u;z),$$

where $\gamma(a, z)$ is the incomplete gamma function [6]. We can now use the relation $J^- f_m = -(m+u)f_{m-1}$ to move down the ladder and verify that in general

$$(5.3)\quad g_m(z) = z^{-u}{}_1F_1(-m-u;-2u;z), \qquad m = -u+n, \quad n = 0, \pm 1, \pm 2, \cdots.$$

The type $B$ operators define a local multiplier representation

$$[\underline{T}(g)f](z,t) = e^{bzt/(d+bt)} f\left[\frac{zt}{(at+c)(d+bt)}, \frac{at+c}{d+bt}\right],$$
$$|c/at| < 1, \quad |bt/d| < 1,$$

$g \in SL(2)$. It follows that the functions (5.3) satisfy the identities

$$(5.4)\quad \begin{aligned} &e^{bzt/(d+bt)}\left(1+\frac{c}{at}\right)^k\left(1+\frac{bt}{d}\right)^{2u-k} {}_1F_1\left(k;-2u;\frac{zt}{(at+c)(bt+d)}\right) \\ &= \sum_{l=-\infty}^{\infty} a^k d^{k-l}(bt)^{l-k}\frac{\Gamma(2u-k+1)}{\Gamma(2u-l+1)} \\ &\quad \cdot \frac{F(-k,-2u+l;l-k+1;bc/(ad))}{\Gamma(l-k+1)}\,{}_1F_1(l;-2u;z), \\ &\qquad\qquad k = 0, \pm 1, \pm 2, \cdots, \end{aligned}$$

where $d = (1+bc)/a$, $|c/at| < 1$, $|bt/d| < 1$.

An exactly similar computation applied to the representation $\downarrow'_u$ yields the basis functions

$$(5.5)\quad \begin{aligned} h_m(z) &= z_1^{-u}F_1(-m-u;-2u;z) \\ &= z^{-u}e^z_1F_1(m-u;-2u;-z), \qquad m = u-n, \quad n = 0,1,2,\cdots. \end{aligned}$$

Note that (5.2) depends on $u$ only in the form $u(u+1)$. Thus, it follows that the representations $\uparrow'_u$ and $\downarrow'_{-u-1}$ have the same spectra, and their eigenfunctions $g_m(z)$, $h_m(z)$ satisfy the same differential equations as long as $2u$ is not an integer. Thus, by the remarks in the Introduction,

$$z^{-u}{}_1F_1(-m-u;-2u;z),\quad z^{u+1}{}_1F_1(u-m+1;2u+2;z) \tag{5.6}$$

are linearly independent solutions of (5.2) if $m=-u+n$, $n=0,\pm1,\pm2,\cdots$, and $2u$ is not an integer. Similarly, comparing $\uparrow'_{-u-1}$ and $\downarrow'_u$ we find that the functions (5.6) are linearly independent solutions of (5.2) if $m=u-n$, $n=0$, $\pm1,\pm2,\cdots$, and $2u$ is not an integer. Due to the fact that (5.2) depends on $u(u+1)$ this leaves us with only the cases $2u=-1,0,1,\cdots$ to consider.

The reader can verify that the representation $D^+(2u)$ yields a model with basis functions

$$g_m(z)=(-1)^m z^{-u}\Psi(-m-u,-2u;z),\qquad m=-u+n,\quad n=0,\pm1,\pm2,\cdots. \tag{5.7}$$

In particular,

$$g_{-u-1}(z)=(-1)^{-u-1}e^z z^{u+1}\Gamma(-2u-1,z),\quad g_u(z)=(-1)^u z^{-u}e^z\Gamma(2u+1,z).$$

Similarly, $D^-(2u)$ yields a model with basis functions

$$g_m(z)=(-1)^m e^z\Psi(m-u,-2u;e^{-i\pi}z),\qquad m=-u+n,\quad n=0,\pm1,\pm2,\cdots. \tag{5.8}$$

According to the remarks in the Introduction the functions (5.7), (5.8) furnish linearly independent solutions of (5.2) when $2u=-1,0,1,2,\cdots$ and $u+m$ is an integer.

The representation $D^{+-}(2u)$ admits the basis functions

$$g_m(z)=z^{u+1}{}_1F_1(u-m+1;2+2u;z),\qquad m=-u+n,\quad n=0,\pm1,\pm2,\cdots,\quad 2u=0,1,2,\cdots. \tag{5.9}$$

However, there are no models of $D'(2u)$ and $D(2u)$ for type $B$ operators.

**6. Models of the representations by type $A$ operators.** The type $A$ operators

$$J^+=t\left(z\frac{\partial}{\partial z}+t\frac{\partial}{\partial t}\right),\quad J^3=t\frac{\partial}{\partial t},\qquad J^-=t^{-1}\left(z\left(1-z\right)\frac{\partial}{\partial z}-t\frac{\partial}{\partial t}+qz\right) \tag{6.1}$$

also satisfy the commutation relations of $sl(2)$. We construct models of representations of $sl(2)$ such that the basis vectors become $f_m(z,t)=g_m(z)t^m$ and the eigenvalue equation $C_{1,0}f_m=u(u+1)f_m$ becomes

$$\left[z^2(1-z)\frac{d^2}{dz^2}-z^2(m-q+1)\frac{d}{dz}+mqz\right]g_m(z)=u(u+1)g_m(z). \tag{6.2}$$

This equation is equivalent to the hypergeometric equation, and for $2u$ not an integer it has the linearly independent solutions $z^{-u}F(m-u, -u-q; -2u; z)$ and $z^{u+1}F(m+u+1, u-q+1; 2+2u; z)$. In [1, Chap. 5] models of $D(u, m_0)$ were used to construct all solutions of (5.2) such that $u \pm m$ were not integers. We shall now find all solutions when this condition is violated.

To construct a model of $\uparrow'_u$ using type $A$ operators we solve the relation $J^- f_{-u} = 0$ to get $g_{-u}(z) = cz^{-u}(1-z)^{q+u}$. Setting $c = 1$ and using $J^+ f_m = (m-u)f_{m+1}$ we obtain

$$
\begin{aligned}
g_m(z) &= z^{-u}F(m-u, -u-q; -2u; z) \\
&= z^{-u}(1-z)^{q-m}F(-u-m, -u+q; -2u; z)
\end{aligned}
\tag{6.3}
$$

for $m = -u + n$, $n = 0, 1, 2, \cdots$. The equation $J^+ f_{-u-1} = -(2u+1)f_{-u}$ yields

$$
\begin{aligned}
g_{-u-1}(z) &= -(2u+1)z^{u+1}\left[\int^z w^{-2u-2}(1-w)^{q+u}\,dw + c\right] \\
&= z^{-u}F(-2u-1, -u-q; -2u; z)
\end{aligned}
\tag{6.4}
$$

if the constant $c$ is chosen appropriately. (Note that for $0 \leqq z \leqq 1$,

$$
B(x, y, z) = \int_0^z w^{x-1}(1-w)^{y-1}\,dw, \qquad \operatorname{Re} x > 0, \quad \operatorname{Re} y > 0,
$$

is the incomplete beta function [6].) Finally, the relation $J^- f_m = -(m+u)f_{m-1}$ yields the basis functions (6.3) for all $m \in S$.

A similar computation applied to the representation $\downarrow'_u$ gives basis functions

$$
\begin{aligned}
h_m(z) &= z^{-u}F(m-u, -u-q; -2u; z), \\
&\qquad\qquad m = u - n, \quad n = 0, \pm 1, \pm 2, \cdots.
\end{aligned}
\tag{6.5}
$$

Since the differential equation (6.2) depends only on $u(u+1)$, it follows that $\uparrow'_u$ and $\downarrow'_{-u-1}$ have the same spectra and that the eigenfunctions $g_m(z)$, $h_m(z)$ satisfy the same equation (6.2) as long as $2u$ is not an integer. By the remarks in the Introduction,

$$
z^{-u}F(m-u, -u-q; -2u; z), \quad z^{u+1}F(m+u+1, u-q+1; 2u+2; z)
\tag{6.6}
$$

are linearly independent solutions of (6.2) if $m - u$ is an integer and $2u$ is not an integer. Also, comparing $\uparrow'_{-u-1}$ and $\downarrow'_u$ we can show that the functions (6.6) are linearly independent solutions of (6.2) if $m + u$ is an integer and $2u$ is not an integer. We are now left with only the cases $2u = -1, 0, 1, \cdots$ to consider.

To construct models of $D'(2u)$, $2u = 0, 1, 2, \cdots$, we must satisfy both $J^+ f_u = 0$ and $J^- f_{-u} = 0$. Thus, $g_u(z) = c_1 z^{-u}$, $g_{-u}(z) = c_2 z^{-u}(1-z)^{q+u}$. It is easy to show that this is possible only if $q$ takes one of the values $-u, -u+1, \cdots, u-1, u$, in which case a basis for the model is

$$g_m(z) = \sum_{n=0}^{-u-q} \frac{(m-u)_n(-u-q)_n}{(-2u)_n n!} z^n$$

(6.7)

$$= (1-z)^{q-m} \sum_{n=0}^{q-u} \frac{(-u-m)_n(q-u)_n}{(-2u)_n n!} z^n,$$

$$2u = 0, 1, 2, \cdots, \quad m + u \text{ an integer.}$$

To find a model of $D^{+-}(2u)$ we must construct a basis which satisfies $J^+ f_{-u-1} = 0$ and $J^- f_{u+1} = 0$. Thus, $g_{-u-1}(z) = c_1 z^{u+1}$, $g_{u+1}(z) = c_2 z^{u+1}(1-z)^{q-u-1}$. This time it turns out that a basis exists for all $q \in \mathbb{C}$:

(6.8)

$$g_m(z) = z^{u+1} F(m+u+1, u-q+1; 2+2u; z),$$

$$2u = -1, 0, 1, \cdots, \quad m + u \text{ an integer.}$$

It follows that for $q = -u, -u+1, \cdots, +u$, $2u = 0, 1, 2, \cdots$, the functions (6.7) and (6.8) are linearly independent solutions of (6.2).

The requirements $J^+ f_u = 0$, $J^+ f_{-u-1} = 0$ define a model of $D^-(2u)$. Then $g_u(z) = c_1 z^{-u}$, $g_{-u-1}(z) = c_2 z^{u+1}$, $c_1, c_2 \neq 0$. Setting $c_1 = 1$ and using $J^- f_{u+1} = f_u$ we find

(6.9)

$$g_{u+1}(z) = \frac{-1}{2u+1}(1-z)^{q-u-1} z^{-u} \left[ \sum_{\substack{n=0 \\ n \neq 2u+1}}^{\infty} \frac{(q-u)_n(-2u-1)_n}{(-2u)_n n!} z^n + \binom{u-q}{2u+1}(-1)^{2u}(2u+1) z^{u+1} \ln z \right].$$

Note that $q \neq -u, -u+1, \cdots, u-1, u$ for models of $D^-(2u)$. Otherwise we would obtain a model of $D'(2u)$. We could now use the relation $J^+ f_m = -(m+u+1)(m-u) f_{m+1}$ to compute $g_{u+n}(z)$ for $n = 2, 3, \cdots$, but as the expressions are somewhat complicated this will not be done here. The general theory of ladder operators guarantees that none of these functions will be identically zero. Now we apply the relation $J^- f_m = f_{m-1}$ to go down the ladder of solutions from $m = u$ to $m = -u$. The result is

$$(6.10) \qquad g_m(z) = (-1)^{m-u}\Gamma(u-m+1) z^{-u} \sum_{n=0}^{u-m} \frac{(m-u)_n(-u-q)_n z^n}{(-2u)_n n!},$$

$$m = u, u-1, \cdots, -u.$$

The relation $J^- f_{-u} = f_{-u-1}$ gives

$$g_{-u-1}(z) = -(2u)!(-u-q)_{2u+1} z^{u+1}$$

in agreement with our earlier result with $c_2 = -(2u)!(-u-q)_{2u+1}$. Finally, going down the ladder again we obtain

(6.11)

$$g_m(z) = -\frac{\Gamma(u-m+1)}{2u+1}(-u-q)_{2u+1} z^{u+1} F(m+u+1, u-q+1; 2+2u; z),$$

$$m = -u-n, \quad n = 1, 2, \cdots.$$

These basis functions and the basis functions for $D^{+-}(2u)$ are linearly independent solutions of (6.2) for $m \geqq -u$ but dependent for $m < -u$. In the special case $2u = -1$, the above expressions have to be slightly modified.

To construct a model for $D^+(2u)$ we make use of the relations $J^- f_{-u} = J^- f_{u+1} = 0$. Thus $h_{-u}(z) = c_1 z^{-u}(1-z)^{q+u}$, $h_{u+1}(z) = c_2 z^{u+1}(1-z)^{q-u-1}$. We set $c_1 = 1$ and apply $J^+ f_{-u-1} = f_{-u}$ to get

$$h_{-u-1}(z) = z^{-u} \sum_{\substack{n=0 \\ n \neq 2u+1}}^{\infty} \frac{(-q-u)_n z^n}{n!(n-2u-1)} + \binom{q+u}{2u+1}(-1)^{2u+1} z^{u+1} \ln z.$$

The functions $h_{-u-n}(z)$, $n = 1, 2, \cdots$, can then be computed by applying $J^-$ recursively. Applying $J^+$ successively to $f_{-u}$ we get

$$h_m(z) = \Gamma(u+m+1)(1-z)^{q-m} \sum_{n=0}^{u+m} \frac{(-u-m)_n(-u+q)_n}{(-2u)_n n!} z^n,$$

$$m = -u, -u+1, \cdots, u-1, u.$$

Then it follows that $J^+ f_u = f_{u+1}$, where $c_2 = -(2u)!(q-u)_{2u}$. Finally, we find that

$$h_m(z) = -\frac{(q-u)_{2u}}{2u+1}\Gamma(u+m+1) z^{u+1}(1-z)^{q-m}$$

$$\cdot F(u-m+1, u+q+1; 2+2u; z)$$

for $m = u+1, u+2, \cdots$. Again the results must be slightly modified for $2u = -1$. The basis vectors $g_m(z)$ of $D^-(2u)$ and $h_m(z)$ of $D^+(2u)$ are linearly independent solutions of (6.2) for all $m \in S$. This completes our construction of solutions of (6.2).

As a final example we consider a transformation of the type $A$ operators which leads directly to the Legendre and Gegenbauer functions:

$$J^{\pm} = t^{\pm 1}\left((z^2-1)\frac{\partial}{\partial z} \pm zt\frac{\partial}{\partial t} + \frac{z}{2}\right), \quad J^3 = t\frac{\partial}{\partial t}. \tag{6.12}$$

These operators were used in [1, Chap. 5] to construct addition theorems and recursion relations for Gegenbauer polynomials. Here, we use them to construct a model of $D^+(-1)$. The relation $J^- f_{1/2} = 0$ gives $g_{1/2}(z) = c$. Setting $c = 1$ and going up the ladder of solutions we find

$$g_{n+1/2}(z) = P_n(z), \qquad n = 0, 1, 2, \cdots, \tag{6.13}$$

where the $P_n(z)$ are Legendre polynomials. The relation $J^+ f_{-1/2} = f_{1/2}$ yields

$$g_{-1/2}(z) = \frac{1}{2}\ln\left(\frac{z-1}{z+1}\right) + c.$$

Setting $c = 0$ we find that $g_{-1/2}(z) = -Q_0(z)$, where $Q_n(z)$ is a Legendre function of the second kind. Now proceeding down the ladder of solutions we obtain

$$g_{-1/2-n}(z) = -Q_n(z), \qquad n = 0, 1, 2, \cdots. \tag{6.14}$$

The recurrence relations (4.6) become

$$\left[(z^2-1)\frac{d}{dz}+z(n+1)\right]g_{n+1/2}(z)=(n+1)g_{n+3/2}(z),$$

$$(6.15)\qquad \left[(z^2-1)\frac{d}{dz}-nz\right]g_{n+1/2}(z)=-ng_{n-1/2}(z),$$

$$\left[(z^2-1)\frac{d^2}{dz^2}+2z\frac{d}{dz}-n(n+1)\right]g_{n+1/2}(z)=0,$$

$$n=0,\pm 1,\pm 2,\cdots.$$

Since this last equation depends only on $n(n+1)$, it follows that $P_n(z)$ and $Q_n(z)$ satisfy the same differential equation. In fact, $P_n(z)$ and $Q_n(z)$ are linearly independent. We have not proved this here, but it could easily be shown by constructing a model of $D^-(-1)$ and comparing it with the present model. A number of important generating functions for the Legendre functions can be obtained from the multiplier representation associated with $D^+(-1)$, but this will be left to the reader. We note in conclusion that an analysis similar to the above suffices to construct all of the Legendre functions of the first and second kinds.

## REFERENCES


[1] W. Miller, *Lie Theory and Special Functions*, Academic Press, New York, 1968.

[2] L. Infeld and T. Hull, *The factorization method*, Rev. Modern Phys., 23 (1951), pp. 21–68.

[3] W. Miller, *Lie theory and difference equations. I*, J. Math. Anal. Appl., 28 (1969), pp. 383–399.

[4] ———, *Lie theory and special functions satisfying second order nonhomogeneous differential equations*, this Journal, 1 (1970), pp. 246–265.

[5] ———, *Lie theory and q-difference equations*, this Journal, 1 (1970), pp. 171–188.

[6] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, vols. I, II, McGraw-Hill, New York, 1953.

[7] N. Y. Vilenkin, *Special Functions and the Theory of Group Representations*, Amer. Math. Soc. Translations, no. 22, (1968), 613 pp.

[8] L. Weisner, *Group-theoretic origin of certain generating functions*, Pacific J. Math., 5 (1955), pp. 1033–1039.

SIAM J. MATH. ANAL.
Vol. 1, No. 4, November 1970

# A UNITARY TRANSFORM RELATED TO SOME INTEGRAL EQUATIONS*

KUSUM SONI†

**Summary.** The integral equations

$$F(x) = \frac{d}{dx}\int_0^x J_0[2\sqrt{k(x-t)}]f(t)\,dt$$

and

$$G(x) = \frac{d}{dx}\int_x^\infty J_0[2\sqrt{k(t-x)}]g(t)\,dt$$

are usually considered separately, and their solutions

$$f(x) = \frac{d}{dx}\int_0^x I_0[2\sqrt{k(x-t)}]F(t)\,dt$$

and

$$g(x) = \frac{d}{dx}\int_x^\infty I_0[2\sqrt{k(t-x)}]G(t)\,dt$$

respectively are regarded unique. These solutions in general are not square integrable. We prove that under certain conditions either one of these two integral equations gives the square integrable solution of the other. Moreover,

$$\frac{d}{dx}\int_{-\infty}^x J_0[2\sqrt{k(x-t)}]f(t)\,dt = 0 \quad \text{and} \quad \frac{d}{dx}\int_x^\infty J_0[2\sqrt{k(t-x)}]g(t) = 0, \qquad -\infty < x < \infty,$$

have nontrivial solutions. Therefore the assumption regarding the uniqueness of the solution of the second integral equation is not valid unless some additional conditions are specified.

It is well known that the homogeneous integral equation

$$\phi(x) = \int_{-\infty}^\infty k(x-y)\phi(y)\,dy$$

may have nontrivial solutions of the type $x^m e^{\beta x}$. The nontrivial solutions given here are not of this type.

Let $f(x)$ be a real-valued function of the Lebesgue class $L_2(-\infty, \infty)$ and let $\phi(\alpha)$ be its Fourier transform. It is well known that the simple transformation $\phi(\alpha) \to e^{ih\alpha}\phi(\alpha)$, $h$ real, corresponds to a translation of the function $f$ by $h$. Here we consider a similar bounded transformation $\phi(\alpha) \to e^{ih/\alpha}\phi(\alpha)$. It is of interest in the study of the following integral equations:

$$F(x) = \int_0^x (x-t)^{\nu/2} J_\nu[2\sqrt{k(x-t)}]f(t)\,dt \tag{1}$$

and

$$G(x) = \int_x^\infty (t-x)^{\nu/2} I_\nu[2\sqrt{k(t-x)}]f(t)\,dt. \tag{2}$$

* Received by the editors December 16, 1969.

† Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37916. This research was supported in part by the University of Tennessee Faculty Research Award—Summer 1969.

These were first studied by Burlak in connection with the solution of a pair of dual integral equations occurring in diffraction theory [2]. If $k$ is replaced by $0 - k$, the right-hand expression in (1) and (2), respectively, can be expressed in terms of Sonine operators $P$ and $Q$ defined and used earlier by Sneddon [5, p. 21]. Burlak gave formal solutions of these integral equations using Laplace transforms. Later Srivastav [7] obtained the solutions with the help of Sonine's first integral. We take a somewhat different approach. If $f$ is integrable in every finite interval, (1) can be written as

$$F(x) = [\Gamma(\nu + 1)]^{-1}k^{\nu/2}\int_0^x (x - t)^\nu \frac{d}{dt}\int_0^t J_0[2\sqrt{k(t - u)}]f(u)\,du\,dt.$$

Similarly, under suitable conditions regarding the convergence of the integral at infinity, we may write (2) as

$$G(x) = -[\Gamma(\nu + 1)]^{-1}k^{\nu/2}\int_x^\infty (t - x)^\nu \frac{d}{dt}\int_t^\infty I_0[2\sqrt{k(u - t)}]f(u)\,du\,dt.$$

Thus the solutions of (1) and (2) essentially depend upon the properties of fractional integrals and the corresponding solutions of the integral equations

(3) $$g(x) = \frac{d}{dx}\int_0^x J_0[2\sqrt{k(x - t)}]f(t)\,dt$$

and

(4) $$h(x) = \frac{d}{dx}\int_x^\infty I_0[2\sqrt{k(t - x)}]f(t)\,dt.$$

If we denote the Fourier transform of $f(x)$ and $g(x)$ by $\phi(\alpha)$ and $\psi(\alpha)$ respectively, we formally obtain, from (3),

(5) $$\psi(\alpha) = e^{-ik/\alpha}\phi(\alpha).$$

Obviously, $\psi(\alpha)$ and $\phi(\alpha)$ have the same $L_2$-norm. However, it is not true that the solution $f(x)$ of (3) belongs to $L_2(0, \infty)$ whenever $g(x)$ does [6, Theorem 1]. To resolve this apparent inconsistency, we obtain the unitary transformation which corresponds to (5) and determine the conditions under which it reduces to (3). This process yields some interesting results, particularly the relationship between integral equations (3), (4) and their solutions as given by Burlak and Srivastav.

**1. Notation.** $L_2$ denotes the class of functions $L_2(-\infty, \infty)$. A function belongs to $L_2(a, b)$, $-\infty < a \leqq b < \infty$, if it belongs to $L_2$ and is zero outside $(a, b)$. The Fourier transform of a function $f$ is defined and is denoted in the usual manner,

$$\hat{f}(\alpha) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} e^{i\alpha x}f(x)\,dx.$$

The integrals are convergent in the mean square sense. Two functions are equal if they differ at most on a set of measure zero. Finally $H(x)$ denotes Heavisides's unit function; it is 1 for $x > 0$ and zero otherwise.

**2. Main theorem.** Our main result is the following theorem.

THEOREM. *Let $f, g$ belong to $L_2$ and let $k > 0$. If*

$$Tf(x) = \frac{d}{dx}\int_{-\infty}^{\infty} \{J_0[2\sqrt{k(x-t)}]H(x-t) - J_0[2\sqrt{k(a-t)}]H(a-t)\} f(t)\, dt \tag{6}$$

*and*

$$T^*g(x) = \frac{d}{dx}\int_{-\infty}^{\infty} \{J_0[2\sqrt{k(t-b)}]H(t-b) - J_0[2\sqrt{k(t-x)}]H(t-x)\} g(t)\, dt \tag{7}$$

*for some $a, b$ real, then $T$ and $T^*$ are unitary transformations in $L_2$ and $T^* = T^{-1}$.*

*Proof.* Consider the transformations

$$S\hat{f}(\alpha) = e^{-ik/\alpha}\hat{f}(\alpha) \tag{8}$$

and

$$S^*\hat{g}(\alpha) = e^{ik/\alpha}\hat{g}(\alpha). \tag{9}$$

Since these transformations are unitary, there exist $L_2$ functions $\mu(x)$ and $\lambda(x)$ such that

$$\hat{\mu}(\alpha) = e^{-ik/\alpha}\hat{f}(\alpha) \tag{10}$$

and

$$\hat{\lambda}(\alpha) = e^{ik/\alpha}\hat{g}(\alpha). \tag{11}$$

From (10), using the inverse Fourier transform, we have

$$\begin{aligned} \int_a^x \mu(t)\, dt &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{e^{-i\alpha x} - e^{-i\alpha a}}{-i\alpha} e^{-ik/\alpha}\hat{f}(\alpha)\, d\alpha \\ &= \int_{-\infty}^{\infty} \hat{h}(\alpha)\hat{f}(\alpha)\, d\alpha \\ &= \int_{-\infty}^{\infty} h(-y) f(y)\, dy, \end{aligned} \tag{12}$$

where $\hat{h}(\alpha)$ is in $L_2$ and the Fourier transform of a function $h(y)$ is given by

$$\begin{aligned} h(y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\alpha x} - e^{-i\alpha a}}{-i\alpha} e^{-ik/\alpha} e^{-iy\alpha}\, d\alpha \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} z^{-1}\{e^{z(x+y)} - e^{z(a+y)}\} e^{-k/z}\, dz, \qquad z = e^{-i\pi/2}\alpha. \end{aligned}$$

The line of integration in the above integral can be shifted from $u = 0$ to $u = c$ ($c > 0$, $z = u + iv$) because the integrand is bounded and analytic and tends to zero uniformly as $|z| \to \infty$ in the strip $0 < u \leqq c$. Thus,

$$h(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \{(1/z)e^{z(x+y)}e^{-k/z} - (1/z)e^{z(a+y)}e^{-k/z}\}\, dz.$$

This integral can be evaluated easily by using contour integration. If $c, k > 0$,

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(1/z)e^{zs-k/z}\,dz = \begin{cases} J_0[2\sqrt{ks}], & s > 0, \\ 0, & s < 0. \end{cases}$$

For $s > 0$, this follows directly from the integral representation of Bessel functions [8, p. 177], and for $s < 0$, we complete the contour to the right. This gives

$$h(y) = J_0[2\sqrt{k(x+y)}]H(x+y) - J_0[2\sqrt{k(a+y)}]H(a+y).$$

Substituting for $h(y)$ in (12), we obtain

$$\begin{aligned}\mu(x) &= \frac{d}{dx}\int_{-\infty}^{\infty}\{J_0[2\sqrt{k(x-y)}]H(x-y) - J_0[2\sqrt{k(a-y)}]H(a-y)\}f(y)\,dy \\ &= Tf(x).\end{aligned} \tag{13}$$

Similarly from (11),

$$\int_b^x \lambda(t)\,dt = \int_{-\infty}^{\infty} h_1(-y)g(y)\,dy, \tag{14}$$

where

$$\begin{aligned} h_1(y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\alpha x} - e^{-i\alpha b}}{-i\alpha}e^{ik/\alpha}e^{-iy\alpha}\,d\alpha, \\ &= -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} z^{-1}(e^{-(x+y)z} - e^{-(b+y)z})e^{-k/z}\,dz \qquad (z = e^{i\pi/2}\alpha) \\ &= -J_0[2\sqrt{k(-x-y)}]H(-x-y) + J_0[2\sqrt{k(-b-y)}]H(-b-y). \end{aligned}$$

Hence substituting for $h_1(y)$ in (14), we have

$$\begin{aligned}\lambda(x) &= -\frac{d}{dx}\int_{-\infty}^{\infty}\{J_0[2\sqrt{k(y-x)}]H(y-x) - J_0[2\sqrt{k(y-b)}]H(y-b)\}g(y)\,dy \\ &= T^*g(x).\end{aligned} \tag{15}$$

Let $\mathfrak{F}$ and $\mathfrak{F}^{-1}$ denote the Fourier transform operator and its inverse respectively. From (8), (10) and (13), $\mathfrak{F}^{-1}S\mathfrak{F} = T$. Similarly from (9), (11) and (15), $\mathfrak{F}^{-1}S^*\mathfrak{F} = T^*$. From the fact that $S$ and $S^*$ are unitary and the inverse of each other, it follows that $T$ and $T^*$ also have these properties.

This proves the theorem. An obvious conclusion is that the integral equations $\phi(x) = Tf(x)$ and $\psi(x) = T^*g(x)$ have solutions in $L_2$ if and only if $\phi(x)$ and $\psi(x)$ are in $L_2$. These solutions are unique and are given by $f(x) = T^*\phi(x)$ and $g(x) = T\psi(x)$ respectively.

Now we consider integral equations which are of interest in applications:

$$\phi(x) = \frac{d}{dx}\int_{-\infty}^{x} J_0[2\sqrt{k(x-t)}]f(t)\,dt \tag{16}$$

and

$$\psi(x) = \frac{d}{dx}\int_x^{\infty} J_0[2\sqrt{k(t-x)}]g(t)\,dt. \tag{17}$$

For the sake of convenience, let

$$P(f,a) = \int_{-\infty}^{a} J_0[2\sqrt{k(a-t)}]f(t)\,dt \tag{18}$$

and

$$Q(g,b) = \int_b^{\infty} J_0[2\sqrt{k(t-b)}]g(t)\,dt. \tag{19}$$

If $P(f, a)$ and $Q(g, b)$ exist for some $a, b$, $Tf(x)$ and $-T^*g(x)$ reduce to the right side of (16) and (17) respectively. However, the square integrability of $f(x)$ and $g(x)$ does not guarantee the existence of $P(f, a)$ and $Q(g, b)$. For instance, let

$$h(x) = |x|^{-\alpha/2}J_\alpha(2\sqrt{k|x|}), \qquad -\infty < x < \infty.$$

By using the asymptotic expansion for the Bessel functions, it can be shown that although $h(x)$ is in $L_2$ for $\frac{1}{2} < \alpha < 1$, yet $P(h, a)$ and $Q(h, a)$ do not exist for any $a$. If the integral equation (16) has a solution $f(x)$ in $L_2$, then $P(f, a)$ exists for almost all $a$, $\phi(x) = Tf(x)$ and the solution is $f(x) = T^*\phi(x)$. In particular, it follows that (i) $\phi(x)$ is in $L_2$ and (ii) $P(T^*\phi, a)$ exists for some $a$. Conversely, if (i) and (ii) are satisfied, then $T^*\phi(x)$ satisfies (16). If, in addition, $Q(\phi, b)$ exists for some $b$, the solution takes a simpler form. This is our first corollary.

COROLLARY 1. *The integral equation* (16) *has a solution in $L_2$ if and only if $\phi(x)$ is in $L_2$ and $P(T^*\phi, a)$ exists for some $a$. If $Q(\phi, b)$ also exists for some $b$, then the solution is*

$$f(x) = T^*\phi(x) = -\frac{d}{dx}\int_x^{\infty} J_0[2\sqrt{k(t-x)}]\phi(t)\,dt. \tag{20}$$

In a similar manner, we can prove the following corollary.

COROLLARY 2. *The integral equation* (17) *has a solution in $L_2$ if and only if $\psi(x)$ is in $L_2$ and $Q(T\psi, b)$ exists for some $b$. If $P(\psi, a)$ also exists for some $a$, then the solution is*

$$g(x) = -T\psi(x) = -\frac{d}{dx}\int_{-\infty}^{x} J_0[2\sqrt{k(x-t)}]\psi(t)\,dt. \tag{21}$$

The $L_2$ solutions of (16) and (17), whenever they exist, are unique. In general, however, these integral equations also have solutions not belonging to $L_2$. Before we discuss this any further, we state some pertinent known results. In some of the statements below, the variables have been changed in an obvious manner.

The solution of

$$\phi(x) = \frac{d}{dx}\int_a^{x} J_0[2\sqrt{k(x-t)}]f(t)\,dt, \tag{22}$$

given by Burlak and Srivastav, can be expressed as

$$f(x) = \frac{d}{dx}\int_a^x I_0[2\sqrt{k(x-t)}]\phi(t)\,dt. \tag{23}$$

Srivastav gives the solution of

$$\psi(x) = \frac{d}{dx}\int_x^b J_0[2\sqrt{k(t-x)}]g(t)\,dt, \qquad b < \infty, \tag{24}$$

as

$$g(x) = \frac{d}{dx}\int_x^b I_0[2\sqrt{k(t-x)}]\psi(t)\,dt. \tag{25}$$

Finally, Burlak's solution of

$$\psi(x) = \frac{d}{dx}\int_x^b I_0[2\sqrt{k(t-x)}]g(t)\,dt, \qquad b \leqq \infty, \tag{26}$$

can be expressed as

$$g(x) = \frac{d}{dx}\int_x^b J_0[2\sqrt{k(t-x)}]\psi(t)\,dt. \tag{27}$$

A comparison of (16) and (22) shows that whenever $\phi(x)$ is in $L_2(a, \infty)$ and satisfies the conditions of Corollary 1, (16) has a solution $f_1(x)$ given by (20) and a solution $f_2(x)$ which is 0 for $x < a$ and is given by (23) for $x > a$. In general, these two solutions are different. For instance, let $\phi(x)$ be a step function,

$$\phi(x) = \begin{cases} 1, & a < \alpha \leqq x \leqq \beta < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

It can be shown that $\phi(x)$ satisfies all the conditions of Corollary 1. We omit this verification, which is straightforward. Below, we give the solutions $f_1(x)$ and $f_2(x)$ explicitly:

$$f_1(x) = \begin{cases} 0, & \beta < x, \\ J_0[2\sqrt{k(\beta - x)}], & \alpha < x < \beta, \\ J_0[2\sqrt{k(\beta - x)}] - J_0[2\sqrt{k(\alpha - x)}], & x < \alpha; \end{cases}$$

$$f_2(x) = \begin{cases} I_0[2\sqrt{k(x-\alpha)}] - I_0[2\sqrt{k(x-\beta)}], & \beta < x, \\ I_0[2\sqrt{k(x-\alpha)}], & \alpha < x < \beta, \\ 0, & x < \alpha. \end{cases}$$

Obviously $f_1(x)$ and $f_2(x)$ are linearly independent. It follows that $\xi(x) = f_1(x) - f_2(x)$ satisfies (16) for $\phi(x) \equiv 0$. With the help of asymptotic expansion for $I_0(z)$ (see [8, p. 203]), it can be easily established that, as $x \to \infty$,

$$f_2(x) > e^{2\sqrt{k'x}}, \qquad 0 < k' < k. \tag{28}$$

Hence we make the following conclusion.

COROLLARY 3. *If* $\phi \equiv 0$, *the integral equation* (16) *has nontrivial solutions which approach* 0 *as* $x \to -\infty$ *but are unbounded as* $x \to \infty$.

One basic difference between the solutions $f_1(x)$ and $f_2(x)$ is that if $\phi(x)$ is zero in $(-\infty, \alpha)$, then so is $f_2(x)$ but not necessarily $f_1(x)$. We investigate the conditions under which $f_1(x)$ also would vanish in this interval. It turns out that whenever this happens the two solutions are identical. We now prove the following corollary.

COROLLARY 4. *If* $\phi(x)$ *belongs to* $L_2(\alpha, \infty)$, *the following statements are equivalent*:

(i) *The integral equation* (16) *has a solution belonging to* $L_2(\alpha, \infty)$.

(ii) $\displaystyle\int_\alpha^\infty \{J_0[2\sqrt{k(t-x)}] - J_0[2\sqrt{kt}]\}\phi(t)\,dt = 0, \quad -\infty < x < \infty.$

(iii) $\displaystyle T^*\phi(x) = \frac{d}{dx}\int_\alpha^x I_0[2\sqrt{k(x-t)}]\phi(t)\,dt, \quad x > \alpha.$

*Proof*. The integral

$$\int_\alpha^\infty \{J_0[2\sqrt{k(t-a)}] - J_0[2\sqrt{kt}]\}\phi(t)\,dt \tag{29}$$

converges for all real $a$. Therefore the condition that the solution $f(x) = T^*\phi(x) = 0$ for $x < \alpha$ implies that

$$\frac{d}{dx}\int_\alpha^\infty \{J_0[2\sqrt{k(t-x)}] - J_0[2\sqrt{kt}]\}\phi(t)\,dt = 0 \tag{30}$$

for $x < \alpha$. Let $0 \leqq |\alpha| < R < \infty$ and let

$$F(z) = \int_\alpha^\infty \{J_0[2\sqrt{k(t-z)}] - J_0[2\sqrt{kt}]\}\phi(t)\,dt, \qquad z = x + iy. \tag{31}$$

We shall prove that $F(z)$ is analytic in $|z| < R$. By Taylor series expansion,

$$J_0[2\sqrt{k(t-z)}] - J_0[2\sqrt{kt}] = \sum_{n=1}^\infty \frac{z^n}{n!} k^{n/2} t^{-n/2} J_n[2\sqrt{kt}]. \tag{32}$$

Since $|J_n(x)| \leqq \min(1, |x/2|^n)$ and the above series converges absolutely and uniformly in $0 \leqq |z| \leqq R$, $\alpha \leqq t < \infty$, we have

$$F(z) = \sum_1^\infty A_n k^{n/2} \frac{z^n}{n!}, \tag{33}$$

where

$$A_n = \int_\alpha^\infty J_n[2\sqrt{kt}]t^{-n/2}\phi(t)\,dt,$$

and, for all $n \geqq 1$,

$$\begin{aligned} |A_n|^2 &\leqq \|\phi\|^2 \int_\alpha^\infty (J_n[2\sqrt{kt}])^2 t^{-n}\,dt \\ &\leqq \|\phi\|^2 (c_1 k^n + c_2), \end{aligned}$$

with $c_1, c_2$ depending upon $k$ and $\alpha$ only. Therefore $F(z)$ is analytic in $|z| < R$. By (30), $F'(x) = 0$ for $-R < x < \alpha$, so that $F'(z) \equiv 0$ and consequently $F(z)$ is a constant. But by (31), $F(0) = 0$. Hence $F(z) \equiv 0$ in $0 \leqq |z| < R$. In particular, $F(x) = 0$ for $-R < x < R$. Since $R$ is arbitrary, (ii) holds for all $x$. This proves that (i) implies (ii). To prove the converse, we have only to observe that if (ii) is satisfied for all $x$, then $T^*\phi(x) = 0$ for $x < \alpha$ and $P(T^*\phi, a)$ exists for all $a < \alpha$ so that by Corollary 1, $T^*\phi$ is the $L_2$ solution of (16). Thus (ii) implies (i).

Next, from (31), for $x > \alpha$,

$$F'(x) = \frac{d}{dx}\int_\alpha^x J_0[2\sqrt{k(t-x)}]\phi(t)\,dt$$

$$+\frac{d}{dx}\int_\alpha^\infty \{J_0[2\sqrt{k(t-x)}]H(t-x) - J_0[2\sqrt{kt}]\}\phi(t)\,dt$$

$$= \frac{d}{dx}\int_\alpha^x I_0[2\sqrt{k(x-t)}]\phi(t)\,dt - T^*\phi(x).$$

Since $F(z)$ is analytic, from the above relation it follows that (ii) and (iii) are equivalent. This proves Corollary 4. In applications usually $\phi(x)$ is 0 outside a bounded interval. Obviously, the solution $T^*\phi$ will also vanish outside that interval if and only if (ii) is satisfied. However in that case $\phi(x) = 0$ a.e.

Recently we proved [6] using a different technique that the integral equation

$$\phi(x) = \frac{d}{dx}\int_0^x J_0[2\sqrt{k(x-t)}]f(t)\,dt$$

has a solution in $L^2(0,\infty)$ if and only if $\phi(x)$ is in $L_2(0,\infty)$ and its Hankel transform $\mathfrak{H}_0\phi$,

$$\mathfrak{H}_0\phi(x) = \int_0^\infty J_0[2\sqrt{xt}]\phi(t)\,dt,$$

vanishes a.e. in $(0, \mathrm{k})$. We prove here that for $\alpha = 0$, Corollary 4 (ii) is in fact equivalent to this condition.

COROLLARY 5. *Let $\phi(x)$ belong to $L_2(0,\infty)$. Then*

$$G(x) = \int_0^\infty \{J_0[2\sqrt{k(t-x)}] - J_0[2\sqrt{kt}]\}\phi(t)\,dt$$

$$= 0, \qquad -\infty < x < \infty,$$

*if and only if $\mathfrak{H}_0\phi(x)$ is zero a.e. in* $(0, k)$.

*Proof.* Let

$$\lambda(t) = \begin{cases} J_0[2\sqrt{k(t+x)}] - J_0[2\sqrt{kt}], & t > 0, \\ 0, & t < 0. \end{cases}$$

Since $\lambda(t)$ belongs to $L_2(0,\infty)$, by Parseval's relation for the Hankel transform

[4, p. 213],

$$G(-x) = \int_0^\infty \mathfrak{H}_0\lambda(u)\mathfrak{H}_0\phi(u)\,du \tag{34}$$

$$= -\sqrt{x}\int_0^k (k-u)^{-1/2}J_1[2\sqrt{x(k-u)}]\mathfrak{H}_0\phi(u)\,du.$$

The Hankel transform of $\lambda(t)$ is obtained by first using the expansion (32). If $G(x) = 0$ for all $x$, since $G(z)$ is analytic, so is $G'(x)$. But from (34), for $x > 0$,

$$G'(-x) = \int_0^k J_0[2\sqrt{x(k-u)}]\mathfrak{H}_0\phi(u)\,du$$

$$= \int_0^\infty J_0[2\sqrt{xu}]\mathfrak{H}_0\phi(k-u)H(k-u)\,du.$$

Hence $G(x) = 0$ for all $x$ implies that $\mathfrak{H}_0\phi(u) = 0$ a.e. in $(0, k)$. Conversely if $\mathfrak{H}_0\phi(u) = 0$ a.e. in $(0, k)$, then, for all $n \geqq 1$,

$$B_n = [\Gamma(n)]^{-1}\int_0^k (k-u)^{n-1}\mathfrak{H}_0\phi(u)\,du$$

$$= 0.$$

By using Parseval's relation for the Hankel transform and the integral (see [8, p. 373]), the $B_n$ can be written as

$$B_n = \int_0^\infty k^{n/2}t^{-n/2}J_n[2\sqrt{kt}]\phi(t)\,dt$$

$$= 0, \qquad n \geqq 1.$$

Comparing this with (31) and (33), it follows that the analytic function $G(z) = 0$ in $|z| < \infty$ and consequently $G(x) = 0$ for $-\infty < x < \infty$. This completes the proof.

Now we consider integral equations (17) and (24). If $b = \infty$, (25) does not always give a solution of (24) even though the $L_2$ solution $-T\psi$ may still exist. Furthermore, the solution of (24), when it exists, is not unique.

If $g_1(x)$ and $g_2(x)$ denote the solutions given by (21) and (25) respectively, corresponding to the step function $\psi(x) = 1$ for $x$ in $(\alpha, \beta)$ and $\psi(x) = 0$ otherwise, then

$$g_1(x) = \begin{cases} J_0[2\sqrt{k(x-\beta)}] - J_0[2\sqrt{k(x-\alpha)}], & x > \beta, \\ -J_0[2\sqrt{k(x-\alpha)}], & \alpha < x < \beta, \\ 0, & x < \alpha; \end{cases}$$

and

$$g_2(x) = \begin{cases} 0, & x > \beta, \\ -I_0[2\sqrt{k(\beta-x)}], & \alpha < x < \beta, \\ I_0[2\sqrt{k(\alpha-x)}] - I_0[2\sqrt{k(\beta-x)}], & x < \alpha. \end{cases}$$

The behavior of these solutions as $x \to \pm\infty$ leads to the following conclusion.

COROLLARY 6. *If $\psi(x) \equiv 0$, the integral equation* (17) *has nontrivial solutions which tend to* 0 *as $x \to \infty$ but are unbounded as $x \to -\infty$.*

As usual, it is interesting to determine when the solutions of (17) vanish in some neighborhood of $\pm\infty$. Let $\psi(x)$ satisfy the conditions of Corollary 2. Obviously, the solution $-T\psi(x)$ must be 0 in $(-\infty, \alpha)$ whenever $\psi(x)$ is 0 there. However, $\psi(x)$ may vanish in some neighborhood of $+\infty$ but $T\psi(x)$ may not. In fact, the situation is no different from the one discussed earlier in the case of integral equation (16). The following corollary is similar to Corollary 4 and can be proved in essentially the same manner.

COROLLARY 7. *If $\psi(x)$ belongs to $L_2(-\infty, \beta)$, the following statements are equivalent*:

(i) *The integral equation* (17) *has a solution belonging to $L_2(-\infty, \beta)$.*

(ii) $$\int_{-\infty}^{\beta} \{J_0[2\sqrt{k(x-t)}] - J_0[2\sqrt{(-kt)}]\}\psi(t)\,dt = 0, \quad -\infty < x < \infty.$$

(iii) $$-T\psi(x) = \frac{d}{dx}\int_x^{\beta} I_0[2\sqrt{k(t-x)}]\psi(t)\,dt, \quad x < \beta.$$

In particular, if $\psi(x)$ belongs to $L_2(\alpha, \beta)$, the solution $-T\psi(x)$ vanishes outside this interval if and only if (ii) is satisfied.

Finally, we consider the integral equation (26). If $b = \infty$, $g(x)$ as determined by (27) is not necessarily its solution. We note that (27) can be written as $g(x) = -T^*\psi(x)$ provided that $Q(\psi, x)$ exists for some $x$. A necessary and sufficient condition that this be a solution of (26) is given below.

COROLLARY 8. *Let $\psi(x)$ belong to $L_2$ and let $Q(\psi, x)$ and $P(T^*\psi, x)$ exist for some $x$. Then* (27) *gives a solution of* (26), $b \leqq \infty$, *if and only if*

$$\frac{d}{dx}\int_{-\infty}^{\infty} J_0[2\sqrt{k(x-t)}]T^*\psi(t)\,dt = 0, \qquad -\infty < x < \infty. \tag{35}$$

*Proof.* The conditions on $\psi(x)$ are sufficient to ensure that

$$g(x) = -T^*\psi(x) = \frac{d}{dx}\int_x^{\infty} J_0[2\sqrt{k(t-x)}]\psi(t)\,dt$$

if and only if

$$\psi(x) = -Tg(x) = -\frac{d}{dx}\int_{-\infty}^{x} J_0[2\sqrt{k(x-t)}]g(t)\,dt.$$

The last relation, together with (26), proves (35).

## REFERENCES

[1] S. BOCHNER AND K. CHANDRASEKHARAN, *Fourier Transforms*, Ann. of Math. Studies, No. 19, Princeton Univ. Press, Princeton, 1953.

[2] J. BURLAK, *A pair of dual integral equations occurring in diffraction theory*, Proc. Edinburgh Math. Soc., 13 (1962), pp. 179–187.

[3] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. G. TRICOMI, *Tables of Integral Transforms*, vol. 2, McGraw-Hill, New York, 1954.

[4] A. Erdélyi and H. Kober, *Some remarks on Hankel transforms*, Quart. J. Math. Oxford Ser., 11 (1940), pp. 212–221.

[5] I. N. Sneddon, *Fractional integration and dual integral equations*, Appl. Math. Research Group, North Carolina State College, Raleigh, 1962.

[6] K. Soni, *An integral equation with Bessel function kernel*, to appear.

[7] R. P. Srivastav, *On certain integral equations of convolution type with Bessel function kernels*, Proc. Edinburgh Math. Soc. (2), 15 (1966–67), pp. 111–116.

[8] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, Cambridge, 1958.

SIAM J. MATH. ANAL.
Vol. 1, No. 4, November 1970

# NONLINEAR EIGENVALUE PROBLEMS FOR SOME FOURTH ORDER EQUATIONS. I: MAXIMAL SOLUTIONS*

SEYMOUR V. PARTER†

**Abstract.** A constructive, nonlinear iterative method is developed for the construction of a positive solution $(u(t), \theta(t))$ of nonlinear fourth order ordinary differential equations of the form $u'' = \lambda\theta H_1(t, u, \theta)$, $\theta'' = \lambda u H_2(t, u, \theta)$. A solution $(u(t), \theta(t))$ is positive if $u(t) \leqq 0 \leqq \theta(t)$. Under appropriate hypothesis, these solutions are maximal in the sense that if $(\omega, \Phi)$ is any other solution, then $u \leqq \omega$ and $\Phi \leqq \theta$. Thus, bounds on $(u, \theta)$ are a priori bounds on all solutions. Uniqueness is discussed. In special cases these positive solutions may be patched together to give other solutions.

**1. Introduction.** This work was motivated by the paper [8] of F. Odeh and I. Tadjbakhsh who discussed two specific nonlinear eigenvalue problems which arise in the study of the equilibrium states of a thin rotating rod. They consider the nonlinear system

$$\begin{aligned} u'' &= \lambda \sin\theta, \quad 0 < t < 1, \\ \theta'' &= \lambda u \cos\theta, \quad 0 < t < 1, \end{aligned} \tag{1.1}$$

and the two sets of boundary conditions

$$u'(0) = \theta(0) = u(1) = \theta'(1) = 0 \tag{A}$$

and

$$u'(0) = \theta'(0) = u(1) = \theta(1) = 0. \tag{B}$$

N. Bazley and B. Zwahlen [1] also studied equations (1.1) under the boundary conditions (A).

These interesting papers employ a variety of methods to obtain information about the existence of solutions when $\lambda > \lambda_0$, the smallest positive eigenvalue of the linearized problem (linearized about zero). In particular, Odeh and Tadjbakhsh prove that there always is a nontrivial solution (in both cases) when $\lambda_0 < \lambda$. Moreover, they make the following conjecture: Let $\lambda_j$ denote the $j$th positive eigenvalue of the linear problem at zero. If $\lambda_n < \lambda \leqq \lambda_{n+1}$, then there are (at least) $n + 1$ distinct nontrivial solutions $(u_j(t), \theta_j(t)), j = 0, 1, \cdots, n$.

We became interested in these problems because the physical solution $(u(t), \theta(t))$ must satisfy (see the discussion in [8, p. 83])

$$|\theta(t)| < \frac{\pi}{2}. \tag{1.2}$$

However, there is no discussion of the "size" of the solution obtained in [8] and [1].

In this report we formulate a general class of problems which includes equations (1.1) and study the existence and uniqueness of maximal solutions. While we are unable to prove that *all* solutions of (1.1) which satisfy the boundary

* Received by the editors January 29, 1970.

† Departments of Computer Sciences and Mathematics, University of Wisconsin, Madison, Wisconsin 53706. This research was supported by the Office of Naval Research under Contract N00014-67-A-0128-0004.

conditions (A) or (B) also satisfy (1.2), we are able to establish the existence of a maximal, positive solution which also satisfies condition (1.2).

The general problem is formulated in § 2. In § 3 we remind the reader of some basic facts about second order problems as developed in [2] and our previous work. Section 4 uses those results and an idea due to Picard [12, Chap. VII] (in the second order case) to establish the existence of positive solutions. Section 5 is devoted to the unicity of such positive solutions and their role as maximal solutions. Because these positive solutions are maximal solutions and provide bounds on all solutions, it is particularly relevant that our proof is a constructive proof. The basic existence proof is based on a nonlinear iteration which may be easily adapted to numerical computation. Finally in § 6 we establish the conjecture of Odeh and Tadjbakhsh for the boundary conditions (B).

In Part II of this work we turn to the application of fixed-point theorems to prove the existence of other solutions. In particular, the conjecture is established for the boundary conditions (A).

**2. The general problem.** Let

$$L_k[\varphi] \equiv (p_k(t)\varphi')' - c_k(t)\varphi(t), \qquad k = 1,2,$$

be two regular Sturm–Liouville operators; that is,

$$c_k(t) \in C[0,1], \quad c_k(t) \geqq 0, \quad 0 \leqq t \leqq 1,$$
$$p_k(t) \in C^1[0,1], \quad p_k(t) \geqq p_0 > 0, \quad 0 \leqq t \leqq 1,$$

for some positive constant $p_0$.

Consider the nonlinear systems of ordinary[1] differential equations

$$\begin{aligned} L_1[u] &= \lambda\theta H_1(t,u,\theta) = \lambda F_1(t,u,\theta), \quad 0 < t < 1, \\ L_2[\theta] &= \lambda u H_2(t,u,\theta) = \lambda F_2(t,u,\theta), \quad 0 < t < 1, \end{aligned} \tag{2.1}$$

where the functions $u(t)$, $\theta(t)$ are required to satisfy the homogeneous boundary conditions

$$\begin{aligned} A_0[u] &\equiv a_0u(0) - b_0u'(0) = 0, \\ A_1[u] &\equiv a_1u(1) + b_1u'(1) = 0, \\ B_0[\theta] &\equiv \alpha_0\theta(0) - \beta_0\theta'(0) = 0, \\ B_1[\theta] &\equiv \alpha_1\theta(1) + \beta_1\theta'(1) = 0, \end{aligned} \tag{2.2}$$

with

$$\begin{aligned} &\alpha_k, a_k, \beta_k, b_k \geqq 0, \qquad k = 1,2, \\ &a_k + b_k > 0, \quad \alpha_k + \beta_k > 0, \quad a_0 + a_1 > 0, \quad \alpha_0 + \alpha_1 > 0. \end{aligned} \tag{2.3}$$

For simplicity, we assume that the functions $H_k(t,u,\theta)$ are even, i.e.,

$$H_k(t,u,\theta) = H_k(t,|u|,|\theta|), \qquad k = 1,2. \tag{2.4}$$

[1] The operators $L_k[\varphi]$, $k = 1, 2$, could equally well be two uniformly elliptic second order operators on a smooth domain $\Omega \subset \Re^n$. However, the present treatment enables us to concentrate on the essential ideas and not get concerned with some technical "smoothness" questions.

With this convention we see that $-\lambda$ is an eigenvalue with eigenfunction $(-u, \theta)$ whenever $\lambda$ is an eigenvalue with eigenfunction $(u, \theta)$. Thus we may restrict our attention to the case where $\lambda > 0$.

DEFINITION 2.1. The problem described by (2.1), (2.2) is called *normal* if

$$H_k(t, u, \theta) > 0, \qquad k = 1, 2,$$

for all $t \in [0, 1]$ and all real $u, \theta$.

DEFINITION 2.2. The problem described by (2.1), (2.2) is called a *cutoff problem* if there is a finite positive constant $\Theta$ such that

(i) $H_k(t, u, \theta) = H_k(t, u, \Theta)$, $\Theta \leqq |\theta|$, $k = 1, 2$,

(ii) $H_k(t, u, \theta) > 0$, $|\theta| < \Theta$, $k = 1, 2$,

(iii) $H_2(t, u, \Theta) = 0$ for all $t \in [0, 1]$ and all $u$.

A pair of functions $u(t), \theta(t)$ is called a *solution to a cutoff problem* if and only if they satisfy equations (2.1), (2.2) and

$$|\theta(t)| < \Theta.$$

*Remark.* The problem of Odeh and Tadjbakhsh described by (1.1) is reduced to a cutoff problem by setting

$$H_1(t, u, \theta) = \begin{cases} \dfrac{\sin \theta}{\theta}, & |\theta| \leqq \dfrac{\pi}{2}, \\[2ex] \dfrac{2}{\pi}, & |\theta| > \dfrac{\pi}{2}, \end{cases}$$

$$H_2(t, u, \theta) = \begin{cases} \cos \theta, & |\theta| \leqq \dfrac{\pi}{2}, \\[2ex] 0, & |\theta| > \dfrac{\pi}{2}. \end{cases}$$

We assume that $F_k(t, u, \theta) \in C^1$ except possibly at $|\theta| = \Theta$ in the cutoff case. We shall consider the following hypotheses on the coefficients.

HYPOTHESIS 1. The function $F_1(t, u, \theta)$ is monotone nondecreasing in $\theta$ and $F_2(t, u, \theta)$ is monotone nondecreasing in $u$. We write

$$\frac{\partial}{\partial \theta} F_1(t, u, \theta) \geqq 0, \qquad \frac{\partial}{\partial u} F_2(t, u, \theta) \geqq 0, \tag{2.5}$$

even though this statement may not be true at $|\theta| = \Theta$. Observe that these conditions may be rewritten as

$$\begin{aligned} &H_1(t, u, \theta) + \theta \frac{\partial}{\partial \theta} H_1(t, u, \theta) \geqq 0, \\ &H_2(t, u, \theta) + u \frac{\partial}{\partial u} H_2(t, u, \theta) \geqq 0. \end{aligned} \tag{2.5a}$$

HYPOTHESIS 2. There are two functions $G_1(t, \theta)$, $G_2(t, u)$ such that

$$0 \leqq H_1(t, u, \theta) \leqq G_1(t, \theta),$$

$$0 \leqq H_2(t, u, \theta) \leqq G_2(t, u)$$

for all $t \in [0, 1]$ and all $u, \theta$.

HYPOTHESIS 3. The functions $H_k(t, u, \theta)$ are monotone nonincreasing in $|u|, |\theta|$; that is,

$$u\frac{\partial}{\partial u}H_k(t, u, \theta) \leqq 0, \quad \theta\frac{\partial}{\partial \theta}H_k(t, u, \theta) \leqq 0, \qquad k = 1, 2. \tag{2.6}$$

However, the system (2.1), (2.2) should be genuinely "nonlinear." Hence, in addition to (2.6) we assume if $\bar{u}, \bar{\theta}$ are positive and $C$ is a constant with $C > 1$, then

$$H_k(t, C\bar{u}, C\bar{\theta}) < H_k(t, \bar{u}, \bar{\theta}), \qquad k = 1, 2. \tag{2.6a}$$

**3. Second order problems, a review.** Let $L[\varphi]$ be a regular Sturm–Liouville operator and consider the nonlinear boundary value problem

$$\begin{aligned} L[\varphi] &= f(t, \varphi), \quad 0 < t < 1, \\ A_0[\varphi] &= A_1[\varphi] = 0, \end{aligned} \tag{3.1}$$

where the boundary operators $A_0[\varphi]$, $A_1[\varphi]$ are described by (2.2), (2.3). The function $f(t, \varphi)$ is continuous in $(t, \varphi)$ and satisfies a Lipschitz condition in $\varphi$ with Lipschitz constant $\gamma$.

DEFINITION 3.1. Let $\varphi_1(t), \varphi_2(t) \in C^1[0, 1]$. We say $\varphi_1$ *dominates* $\varphi_2$ if

$$\varphi_2(t) < \varphi_1(t), \qquad 0 < t < 1, \tag{3.2a}$$

$$\varphi_1(0) = \varphi_2(0) \Rightarrow \varphi_2'(0) < \varphi_1'(0), \tag{3.2b}$$

$$\varphi_1(1) = \varphi_2(1) \Rightarrow \varphi_1'(1) < \varphi_2'(1). \tag{3.2c}$$

If $\varphi_1(t)$ dominates $\varphi_2(t)$ we write

$$\varphi_2 \prec \varphi_1. \tag{3.3}$$

The concept of domination[2] arises in the study of second order equations through the strong form of the maximum principle and Hopf's lemma [2]. Together these principles give the following assertion: If

$$L[\varphi] \leqq 0, \tag{3.4a}$$

then

$$\varphi(t) \geqq \min\{0, \varphi(0), \varphi(1)\}. \tag{3.4b}$$

Moreover, if equality (in (3.4b)) occurs at any interior point, then

$$\varphi(t) \equiv \text{const.} \tag{3.4c}$$

Furthermore, if $\varphi(0) \geqq 0$ and

$$\varphi(0) = \min \varphi(t), \qquad 0 \leqq t \leqq 1, \tag{3.5a}$$

then either (3.4c) holds or

$$\varphi'(0) > 0. \tag{3.5b}$$

[2] We shall make essential use of this concept only in § 5 where we are concerned with uniqueness.

Similarly, if $\varphi(1) \geqq 0$ and $\varphi(t)$ assumes its minimum at $t = 1$, then either (3.4c) holds or

$$\varphi'(1) < 0. \tag{3.5c}$$

These facts lead to the following basic lemma.

LEMMA 3.1. *If* $L[\varphi] \leqq 0$, $A_0[\varphi] = A_1[\varphi] = 0$, *then either* $\varphi(t) \equiv 0$ *or*

$$0 \prec \varphi(t).$$

The next lemmas collect some basic facts about solutions of the problem (3.1) as developed in [2] and [11]. In [11] we developed the basic ideas for the special case where

$$L \equiv \left(\frac{d}{dt}\right)^2, \qquad b_0 = b_1 = 0.$$

However, using Lemma 3.1 one may easily adapt the proofs to the general case.

LEMMA 3.2. *Let* $f(t, \varphi)$ *be bounded for all* $(t, \varphi)$. *Suppose* $a(t) \in C^2[0, 1]$ *satisfies*

$$\begin{aligned} L[a] &\leqq f(t, a), \qquad L[a] \not\equiv f(t, a), \\ A_0[a] &\geqq 0, \qquad A_1[a] \geqq 0. \end{aligned} \tag{3.6}$$

*Then there is a function* $u(t)$ *which is a solution of* (3.1) *which satisfies*

$$u(t) \prec a(t). \tag{3.7}$$

*Moreover, if* $z(t)$ *is any other solution of* (3.1) *which satisfies*

$$z(t) \leqq a(t), \tag{3.8a}$$

*then*

$$z(t) \leqq u(t). \tag{3.8b}$$

*This solution is uniquely determined by an iterative process. Finally, if* $f_1(t, u) \geqq f(t, u), f_1(t, u) \not\equiv f(t, u)$ *for all* $u$, *then the corresponding solution* $u_1$ *of*

$$L[u_1] = f_1(t, u_1), \qquad A_0[u_1] = A_1[u_1] = 0,$$

*which is determined by this process, satisfies*

$$u_1 \prec u. \tag{3.8c}$$

*Similarly, let* $b(t) \in C^2[0, 1]$ *satisfy*

$$\begin{aligned} L[b] &\geqq F(t, b), \qquad L[b] \not\equiv f(t, b), \\ A_0[b] &\leqq 0, \qquad A_1[b] \leqq 0. \end{aligned} \tag{3.9}$$

*Then, there is a function* $v(t)$ *which is a solution of* (3.1) *which satisfies*

$$b(t) \prec v(t). \tag{3.10}$$

*Moreover, if* $z(t)$ *is any solution of* (5.1) *which satisfies*

$$b(t) \leqq z(t), \tag{3.11a}$$

*then*

$$v(t) \leqq z(t). \tag{3.11b}$$

*This solution is uniquely determined by an iterative process. Finally, if* $f_2(t,u) \leqq f(t,u), f_2(t,u) \not\equiv f(t,u)$ *for all u, then the corresponding solution* $v_2(t)$ *of*

$$L[v_2] = f_2(t, v_2), \qquad A_0[v_2] = A_1[v_2] = 0,$$

*which is determined by this process, satisfies*

$$v \prec v_2. \tag{3.11c}$$

*Proof.* Consider the iteration

$$L[z_{n+1}] - \gamma z_{n+1} = f(t, z_n) - \gamma z_n$$

with $z_0(t) = a(t)$ or $z_0(t) = b(t)$. The argument proceeds by induction as in [11].

On the basis of this lemma we define two operations $U(a)$, $V(b)$ by

$$U(a) = u(t), \qquad V(b) = v(t). \tag{3.12}$$

LEMMA 3.3. *If* $f(t, \varphi)$ *is monotone nondecreasing in* $\varphi$, *then* (3.1) *has a unique solution.*

COROLLARY 3.1. *Suppose* $f(t,\varphi) \leqq 0$ *and is monotone nondecreasing in* $\varphi$ *for* $\varphi \geqq 0$. *Then there exists a unique nonnegative solution* $\varphi(t)$. *Similarly, suppose* $f(t,\varphi) \geqq 0$ *and is monotone nondecreasing in* $\varphi$ *for* $\varphi \leqq 0$. *Then there exists a unique nonpositive solution* $\varphi(t)$.

*Proof.* We consider only the first case. We observe that if there is a solution $\varphi(t)$ of (3.1) it is nonnegative. Let

$$f_0(t,\varphi) = \begin{cases} f(t,\varphi), & \varphi \geqq 0, \\ f(t,0), & \varphi \leqq 0. \end{cases}$$

Then, $\varphi(t)$ is a solution of (3.1) if and only if $\varphi(t)$ is a solution of

$$L[\varphi] = f_0(t, \varphi(t)).$$

But, this equation has a unique solution because $f_0(t,\varphi)$ is nondecreasing in $\varphi$.

LEMMA 3.4. *Suppose* $f(t,\varphi) \leqq 0$. *Suppose there is a constant* $k > 0$ *such that*

$$f(t,\varphi) = 0 \quad \textit{for } k \leqq \varphi.$$

*Let* $\varphi(t)$ *be a solution of* (3.1). *Then*

$$0 \leqq \varphi(t) \leqq k.$$

*Proof.* Suppose there is a point $t_0 \in (0,1)$ such that

$$\varphi(t_0) > k.$$

Then there is an interval $[\rho, \delta]$ about $t_0$ such that

$$\varphi(t) \geqq k \quad \text{for } t \in [\rho, \delta]. \tag{3.13}$$

Naturally, we take $[\rho,\delta]$ as large as possible.

*Case* 1. $\rho = 0$, $\delta = 1$. Then $L[\varphi] \equiv 0$ and the maximum principle asserts that $\varphi(t) \equiv 0 < k$.

*Case* 2. $\rho = 0, \delta < 1$. Then $\varphi(\delta) = k$ and $\varphi(0)$ is a maximum of $\varphi(t)$ for $t \in [0, \delta]$. If $\varphi(t)$ is not constant on this interval, we have

$$\varphi'(0) < 0.$$

However, the boundary condition $A_0[\varphi] = 0$ implies that $\varphi(0) = 0$ or $\varphi'(0) \cdot \varphi(0) > 0$. Since $\varphi(0) \geqq k$, we have a contradiction.

*Case* 3. $\rho > 0, \delta = 1$. In this case we see that

$$\varphi'(1) > 0, \qquad \varphi(1) \geqq k.$$

But the boundary condition $A_1[\varphi] = 0$ implies that

$$\varphi(1) = 0 \quad \text{or} \quad \varphi'(1)\varphi(1) < 0.$$

*Case* 4. $0 < \rho < \delta < 1$. Then $\varphi(\rho) = \varphi(\delta) = k$ and

$$L[\varphi] = 0, \qquad \rho < t < \delta.$$

The maximum principle asserts that

$$\varphi(t) = k, \qquad \rho \leqq t \leqq \delta.$$

This completes the proof of the lemma.

COROLLARY 3.2. *Suppose the functions $H_k(t, u, \theta)$ satisfy the conditions* (i), (ii), (iii) *of Definition* 2.2 *for a cutoff problem. Suppose* $(u(t), \theta(t))$ *is a solution of* (2.1), (2.2). *Then*

$$|\theta(t)| \leqq \Theta.$$

**4. Positive solutions.** We now return to the general problem (2.1).

DEFINITION 4.1. A pair of functions $(u(t), \theta(t))$ will be called a *positive solution* of (2.1) if it is a solution and also satisfies

$$u \prec 0 \prec \theta. \tag{4.1}$$

*Note.* If $(u(t), \theta(t))$ is a solution, so is $(-u(t), -\theta(t))$. Moreover, if either function, $u(t)$ or $\theta(t)$, is nonpositive (but not identically zero), the other function dominates the zero function.

DEFINITION 4.2. A positive solution $(u(t), \theta(t))$ will be called a *maximal* solution if whenever $(w, \Phi)$ is another nontrivial solution of (2.1), (2.2) (not necessarily positive) then

$$|\Phi(t)| \leqq \theta(t), \qquad |w(t)| \leqq |u(t)| = -u(t). \tag{4.2}$$

*Note.* By the remarks above, (4.2) is equivalent to

$$\Phi(t) \leqq \theta(t), \qquad u(t) \leqq w(t). \tag{4.2'}$$

LEMMA 4.1. *Suppose* $(w(t), \Phi(t))$ *is a nontrivial solution of* (2.1), (2.2) *and* $\Phi(t) \geqq 0$. *Then* $(w, \Phi)$ *is a positive solution.*

*Proof*. Apply Lemma 3.1 and the remarks above.

Consider now the problem obtained by "linearizing" (2.1) about $(u, \theta) = (0, 0)$.

We obtain

$$L_1[u] = \lambda\theta H_1(t,0,0), \qquad A_0[u] = A_1[u] = 0. \tag{4.3}$$
$$L_2[\theta] = \lambda u H_2(t,0,0), \qquad B_0[\theta] = B_1[\theta] = 0.$$

Let $K_1(s,t)$, $K_2(s,t)$ be the Green's functions associated with the operators $-L_1[u]$ and $-L_2[\theta]$ respectively, subject to the appropriate homogeneous boundary conditions ($A_j[u] = B_j[\theta] = 0, j = 1, 2$). Then the equations (4.3) are equivalent to

$$u(t) = -\lambda \int_0^1 K_1(t,x)H_1(x,0,0)\theta(x)\,dx,$$

$$\theta(x) = -\lambda \int_0^1 K_2(x,y)H_2(y,0,0)u(y)\,dy.$$

On substitution, we obtain

$$\theta(t) = \lambda^2 \int_0^1 G(t,s)\theta(s)\,ds \tag{4.4}$$

with

$$G(t,s) = \int_0^1 K_2(t,x)K_1(x,s)H_1(s,0,0)H_2(x,0,0)\,dx. \tag{4.4a}$$

The kernel[3] $G(t,s)$ is a positive (nonnegative) kernel. Hence, the smallest eigenvalue $\lambda_0^2$ corresponds to an eigenfunction of constant sign (see [4], [5], [6], [7]). Thus, we may normalize the eigenfunction $(u_0(t), \theta_0(t))$ associated with the smallest positive eigenvalue $\lambda_0 > 0$ so that

$$u_0 \prec 0 \prec \theta_0. \tag{4.5}$$

Moreover, if $\lambda_0 < \lambda$, we may scale $(u_0(t), \theta_0(t))$ so that (4.5) holds and

$$R_k \equiv \left[1 - \frac{\lambda H_k(t,u_0,\theta_0)}{\lambda_0 H_k(t,0,0)}\right] < 0, \qquad k = 1, 2. \tag{4.6}$$

A straightforward calculation now shows that

$$L_1[u_0] \leqq \lambda F_1(t,u_0,\theta_0), \qquad A_0[u_0] = A_1[u_0] = 0, \tag{4.7}$$
$$L_2[\theta_0] \geqq \lambda F_2(t,u_0,\theta_0), \qquad B_0[\theta_0] = B_1[\theta_0] = 0.$$

These inequalities, together with the mappings of Lemma 3.2, enable us to construct an "increasing" sequence $(u_n(t), \theta_n(t))$.

LEMMA 4.2. *Let Hypotheses* 1 *and* 2 *hold. Suppose* $(u_{n-1}(t), \theta_{n-1}(t))$ *satisfies*

$$L_1[u_{n-1}] \leqq \lambda F_1(t, u_{n-1}(t), \theta_{n-1}(t)), \tag{4.8a}$$
$$L_2[\theta_{n-1}] \geqq \lambda F_2(t, u_{n-1}(t), \theta_{n-1}(t)),$$

[3] In fact, $G(t, s)$ is an oscillation kernel in the sense of Gantmacher-Krein [3]. However, we shall not make use of this fact in this report.

*and*

$$A_0[u_{n-1}] \geqq 0, \qquad A_1[u_{n-1}] \geqq 0. \tag{4.8b}$$
$$B_0[\theta_{n-1}] \leqq 0, \qquad B_1[\theta_{n-1}] \leqq 0.$$

*Let $u_n(t)$ be the solution of the nonlinear equation*

$$L_1[u_n] = \lambda F_1(t, u_n, \theta_{n-1}(t)), \qquad A_0[u_n] = A_1[u_n] = 0 \tag{4.9}$$

*determined by Lemma* 3.2; *that is,*

$$u_n = U(u_{n-1}). \tag{4.10}$$

*Then, unless $u_{n-1}(t)$ satisfies* (4.9) *and $u_n(t) \equiv u_{n-1}(t)$, we have*

$$u_n \prec u_{n-1} \tag{4.11a}$$

*and*

$$L_2[\theta_{n-1}] \geqq \lambda F_2(t, u_{n-1}, \theta_{n-1}) \geqq \lambda F_2(t, u_n(t), \theta_{n-1}). \tag{4.11b}$$

*Then, we choose $\theta_n(t)$ as the solution of*

$$L_2[\theta_n] = \lambda F_2(t, u_n(t), \theta_n), \qquad B_0[\theta_n] = B_1[\theta_n] = 0, \tag{4.12}$$

*determined by Lemma* 3.2; *that is,*

$$\theta_n = V(\theta_{n-1}). \tag{4.13}$$

*Then, unless $\theta_{n-1}$ satisfies* (4.12) *and $\theta_{n-1}(t) \equiv \theta_n(t)$, we have*

$$\theta_{n-1} \prec \theta_n \tag{4.14a}$$

*and*

$$L_1[u_n] = \lambda F_1(t, u_n, \theta_{n-1}) \leqq \lambda F_1(t, u_n, \theta_n). \tag{4.14b}$$

*In either case,*

$$u_n \leqq u_{n-1}, \qquad \theta_{n-1} \leqq \theta_n, \tag{4.15}$$

*and* (4.8a) *and* (4.8b) *hold with $n-1$ replaced by $n$.*

*Proof*. Hypothesis 2 permits us to apply Lemma 3.2, while (4.11b) and (4.14b) follow from Hypothesis 1.

COROLLARY 4.1. *Suppose $0 < \lambda_0 < \lambda$. Then we choose $(u_0(t), \theta_0(t))$ as the solution of the linear eigenvalue problem* (4.3) *associated with $\lambda_0$ which also satisfies* (4.5), (4.6) *and* (4.7). *Thus we generate a sequence $(u_n(t), \theta_n(t))$ with*

$$u_n(t) \leqq u_{n-1}(t) \prec 0 \prec \theta_{n-1}(t) \leqq \theta_n(t). \tag{4.16}$$

*The functions $u_n(t)$, $\theta_n(t)$ satisfy* (4.9) *and* (4.12) *respectively. Moreover, either $u_n(t) \equiv u_{n-1}(t)$ or*

$$u_n \prec u_{n-1}. \tag{4.17a}$$

*And, either $\theta_n(t) \equiv \theta_{n-1}(t)$ or*

$$\theta_{n-1} \prec \theta_n. \tag{4.17b}$$

COROLLARY 4.2. *Let Hypothesis* 3 *hold also. Then each of the equations* (4.9) *and* (4.12) *has a unique solution. These are* $u_n(t)$ *and* $\theta_n(t)$ *respectively.*

*Proof.* Apply Corollary 3.1.

COROLLARY 4.3. *If we are dealing with a cutoff problem and we further "scale"* $(u_0(t), \theta_0(t))$ *so that* $\theta_0(t) < \Theta$, *then*

$$\theta_n(t) \leqq \Theta, \qquad n = 1, 2, \cdots.$$

*Proof.* Apply Lemma 3.4.

We obtain our next result from the same argument.

LEMMA 4.3. *Let Hypotheses* 1 *and* 2 *hold. Suppose* $(w_{n-1}(t), \Phi_{n-1}(t))$ *satisfies*

$$\begin{aligned} L_1[w_{n-1}] &\geqq \lambda F_1(t, w_{n-1}(t), \Phi_{n-1}(t)), \\ L_2[\Phi_{n-1}] &\leqq \lambda F_2(t, w_{n-1}(t), \Phi_{n-1}(t)), \end{aligned} \tag{4.18a}$$

*and*

$$\begin{aligned} A_0[w_{n-1}] &\leqq 0, \qquad A_1[w_{n-1}] \leqq 0, \\ B_0[\Phi_{n-1}] &\geqq 0, \qquad B_1[\Phi_{n-1}] \geqq 0. \end{aligned} \tag{4.18b}$$

*Let* $w_n(t)$ *be the solution of the nonlinear equation*

$$L_1[w_n] = \lambda F_1(t, w_n, \Phi_{n-1}(t)), \qquad A_0[w_n] = A_1[w_n] = 0 \tag{4.19}$$

*determined by Lemma* 3.2; *that is,*

$$w_n = V(w_{n-1}).$$

*Then, unless* $w_n(t) \equiv w_{n-1}(t)$ *and* $w_{n-1}(t)$ *satisfies* (4.19),

$$w_{n-1} \prec w_n \tag{4.20a}$$

*and*

$$L_2[\Phi_{n-1}] \leqq \lambda F_2(t, w_n, \Phi_{n-1}). \tag{4.20b}$$

*Thus, we may choose* $\Phi_n(t)$ *as the solution of*

$$L_2[\Phi_n] = \lambda F_2(t, w_n(t), \Phi_n), \qquad B_0[\Phi_n] = B_1[\Phi_n] = 0, \tag{4.21}$$

*determined by Lemma* 3.2; *that is,*

$$\Phi_n = U(\Phi_{n-1}).$$

*Then, unless* $\Phi_n(t) \equiv \Phi_{n-1}(t)$ *and* $\Phi_{n-1}$ *satisfies* (4.21),

$$\Phi_n \prec \Phi_{n-1} \tag{4.22a}$$

*and*

$$L_1[w_n] = \lambda F_1(t, w_n, \Phi_{n-1}) \geqq \lambda F_1(t, w_n, \Phi_n). \tag{4.22b}$$

*In either case,*

$$w_{n-1} \leqq w_n, \qquad \Phi_n \leqq \Phi_{n-1},$$

*and* (4.18a) *and* (4.18b) *hold with* $n - 1$ *replaced by n.*

THEOREM 4.1. *Suppose* $\lambda_0 < \lambda$ *and Hypotheses* 1 *and* 2 *hold. Suppose* $(u_0(t), \theta_0(t))$ *is the eigenfunction pair of the linear eigenvalue problem* (4.3) *which also satisfies* (4.5), (4.6), (4.7). *Suppose there exists a pair of functions* $(w, \Phi)$ *such that*

$$w \prec u_0 \prec 0 \prec \theta_0 \prec \Phi, \tag{4.23a}$$

$$L_1[w] \geqq \lambda F_1(t, w, \Phi), \quad A_0[w] \leqq 0, \quad A_1[w] \leqq 0, \tag{4.23b}$$

$$L_2[\Phi] \leqq \lambda F_2(t, w, \Phi), \quad B_0[\Phi] \geqq 0, \quad B_1[\Phi] \geqq 0. \tag{4.23c}$$

*Then, there exists a positive solution* $(u(t), \theta(t))$ *of* (2.1), (2.2). *Moreover, either* $(w, \Phi)$ *is a solution or*

$$w \prec u \prec u_0 \prec 0 \prec \theta_0 \prec \theta \prec \Phi. \tag{4.24}$$

*Proof.* Let $(u_n(t), \theta_n(t))$ be the monotone sequence generated by Lemma 4.2 with $(u_0, \theta_0)$ chosen as above. Let $(w_n, \Phi_n)$ be the monotone sequence generated by Lemma 4.3 with $w_0 \equiv w$, $\Phi_0 \equiv \Phi$. We shall prove

$$w_0 \leqq u_n, \qquad \theta_n \leqq \Phi_0, \tag{4.25a}$$

$$w_n \leqq u_0, \qquad \theta_0 \leqq \Phi_n. \tag{4.25b}$$

Then the theorem follows from standard estimates and the Ascoli–Arzela lemma. Indeed, each pair of sequences $(u_n, \theta_n)$, $(w_n, \Phi_n)$ will converge to a solution pair $(u, \theta)$ and $(\hat{w}, \hat{\Phi})$ respectively. Thus, there may be two solutions.

The proof follows by induction. By (4.23a) we have (4.25a), (4.25b) for $n = 0$. Suppose

$$w_0 \leqq u_{n-1}, \qquad \theta_{n-1} \leqq \Phi_0.$$

Then

$$L_1[u_{n-1}] \leqq \lambda F_1(t, u_{n-1}, \theta_{n-1}) \leqq \lambda F_1(t, u_{n-1}, \Phi_0).$$

Using Lemma 4.2 we construct $u_n(t)$ which satisfies (4.9), and using Lemma 3.2 we construct a function $\hat{v}(t)$ which satisfies

$$L_1[\hat{v}] = \lambda F_1(t, \hat{v}, \Phi_0), \qquad A_0[\hat{v}] = A_1[\hat{v}] = 0$$

and

$$w_0 = w \leqq \hat{v} \leqq u_n.$$

Thus, we establish (4.25a) for all $n$. A similar argument establishes (4.25b) and completes the proof.

THEOREM 4.2. *Let* $\lambda_0 < \lambda$ *and suppose Hypotheses* 1 *and* 2 *hold. Suppose we have a cutoff problem. Then there is a maximal solution* $(u(t), \theta(t))$.

*Proof.* Let

$$W = \max\{\lambda F_1(t, w, \Theta); 0 \leqq t \leqq 1, |w| < \infty\},$$

and let $w(t)$ be the solution of

$$L_1[w] = W \geqq \lambda F_1(t, v, \Theta) \quad \text{for all } v(t),$$

$$A_0[w] = 0, \qquad A_1[w] = 0.$$

Let $\Phi(t) \equiv \Theta$. Then

$$L_2[\Phi] = -c_2(t)\Theta \leqq \lambda F_2(t, w, \Theta) = 0,$$
$$B_0[\Phi] \geqq 0, \qquad B_1[\Phi] \geqq 0.$$

Thus, the pair $(w, \Phi)$ satisfies the conditions of Theorem 4.1 and there is a positive solution $(\hat{w}(t), \hat{\Phi}(t))$ which is the limit of $(w_n(t), \Phi_n(t))$.

Let $(v(t), \Psi(t))$ be any other solution. Then, because $(v, \Psi)$ is a solution to the cutoff problem, we have

$$|\Psi(t)| \leqq \Theta. \tag{4.26}$$

Hence,

$$\lambda F_1(t, v(t), \Psi(t)) \leqq \lambda F_1(t, v(t), \Theta) \leqq W.$$

Therefore,

$$w(t) \leqq v(t).$$

And, of course, $(-v(t), -\Psi(t))$ is also a solution so that

$$|v(t)| \leqq -w(t) = |w(t)|. \tag{4.27}$$

An induction based on Lemma 3.2 and Lemma 4.3 shows that

$$w_n \leqq v(t), \qquad \Psi(t) \leqq \Phi_n(t).$$

The theorem follows at once.

Returning to the normal (noncutoff) problems, we seek conditions which will guarantee the existence of a pair $(w(t), \Phi(t))$ satisfying (4.23a), (4.23b) and (4.23c). Clearly, Hypotheses 1 and 2 are not sufficient because these conditions include the linear case.

THEOREM 4.3. *Let $\lambda_0 < \lambda$. Let Hypotheses* 1 *and* 2 *hold. Let $K_1(s,t)$, $K_2(s,t)$ be the Green's functions of $-L_1[u]$, $-L_2[\theta]$ respectively which were discussed earlier. Suppose there are four positive constants $M$, $U_0$, $\Theta_0$, $\alpha$ with $0 < \alpha < 1$ such that*

$$\begin{aligned} &H_k(t,0,0) \geqq H_k(t, u(t), \theta(t)), && k = 1,2, \\ &K_j(t,s)H_j(t, u(s), \theta(s)) \leqq M, && j = 1,2, \end{aligned} \tag{4.28a}$$

$$\lambda^2 \int_0^1 K_1(s,t)K_2(x,s)H_1(t, u(t), \theta(t))H_2(s, \hat{u}(s), \hat{\theta}(s))\, ds \leqq \alpha \tag{4.28b}$$

*for all functions $u(x)$, $\theta(x)$, $\hat{u}(x)$, $\hat{\theta}(x)$ which satisfy*

$$\begin{aligned} &U_0 \leqq |u(x)|, |\hat{u}(x)|, && 0 \leqq x \leqq 1, \\ &\Theta_0 \leqq |\theta(x)|, |\hat{\theta}(x)|, && 0 \leqq x \leqq 1. \end{aligned} \tag{4.28c}$$

*Then there exists a pair $(w, \Phi)$ with $w(t) \leqq -U_0 < \Theta_0 \leqq \Phi(t)$ which satisfies* (4.23a), (4.23b) *and* (4.23c). *Finally, there exists a positive solution $(u(t), \theta(t))$.*

*Proof.* Consider the inhomogeneous, nonlinear equation

$$\begin{aligned} L_1[v] &= \lambda \Psi H_1(t, v - U_0, \Psi + \Theta_0) + \lambda \Theta_0 H_1(t,0,0), \\ L_2[\Psi] &= \lambda v H_2(t, v - U_0, \Psi + \Theta_0) - \lambda U_0 H_2(t,0,0), \\ A_0[v] &= A_1[v] = B_0[\Psi] = B_1[\Psi] = 0. \end{aligned} \tag{4.29}$$

We shall show that there exists a positive solution, i.e., a solution $(v, \Psi)$ with

$$v(t) \leqq 0 \leqq \Psi(t). \tag{4.30}$$

Let

$$\begin{aligned} g_1(s) &= \lambda\Theta_0 \int_0^1 K_1(s,t)H_1(t,0,0)\,dt, \\ g_2(s) &= \lambda U_0 \int_0^1 K_2(s,t)H_2(t,0,0)\,dt, \\ K_0 &= (\lambda M\|g_1\|_\infty + \|g_2\|_\infty)/(1-\alpha), \\ K_1 &= \lambda K_0 \cdot M + \|g_1\|_\infty. \end{aligned} \tag{4.31}$$

Let $S$ be the convex set

$$S = \{(\bar{v}(t), \overline{\Psi}(t)) \in C[0,1];\ -K_1 \leqq \bar{v}(t) \leqq 0 \leqq \overline{\Psi}(t) \leqq K_0\}. \tag{4.32}$$

Let $(\bar{v}(t), \overline{\Psi}(t)) \in S$ and let $V(t)$, $\Psi(t)$ be the solution of the linear equations

$$\begin{aligned} L_1[V] &= \lambda\overline{\Psi}H_1(t, \bar{v} - U_0, \overline{\Psi} + \Theta_0) + \lambda\Theta_0 H_1(t,0,0), \\ L_2[\Psi] &= \lambda V H_2(t, V - U_0, \overline{\Psi} + \Theta_0) - \lambda U_0 H_2(t,0,0), \\ A_0[V] &= A_1[V] = B_0[\Psi] = B_1[\Psi] = 0. \end{aligned} \tag{4.33}$$

Using the integral representations of the solution, we have

$$V(s) = -\lambda \int_0^1 K_1(s,t)\overline{\Psi}(t)H_1(t, v(t) - U_0, \overline{\Psi}(t) + \Theta_0)\,dt - g_1(s) \tag{4.34a}$$

and

$$\begin{aligned} \Psi(x) &= \int_0^1 G(x,t)\overline{\Psi}(t)\,dt \\ &\quad + \lambda \int_0^1 K_2(x,s)H_2(s, V - U_0, \overline{\Psi} + \Theta_0)g_1(s)\,ds + g_2(x), \end{aligned} \tag{4.34b}$$

where

$$\begin{aligned} G(x,t) = \lambda^2 \int_0^1 & K_1(s,t)K_2(x,s)H_1(t, \bar{v}(t) - V_0, \overline{\Psi}(t) + \Theta_0) \\ & \cdot H_2(s, V(s) - V_0, \overline{\Psi}(s) + \Theta_0)\,ds. \end{aligned} \tag{4.34c}$$

From (4.34a) and (4.28a) we see that

$$-K_1 \leqq V(t) \leqq 0. \tag{4.35a}$$

From (4.34b), (4.28a) and (4.28b) we see that

$$0 \leqq \psi \leqq \alpha K_0 + \lambda M \cdot \|g_1\|_\infty + \|g_2\|_\infty = K_0. \tag{4.35b}$$

Thus, (4.33) provides a mapping of $S$ into $S$. Standard estimates show that this is a compact continuous mapping. Thus, there is a fixed point, i.e., a solution of (4.29) which satisfies (4.30).

Let

$$w = v - U_0 \leqq -U_0, \quad \Phi = \Psi + \Theta_0 \geqq \Theta_0. \tag{4.36}$$

Then,

$$L_1[w] = \lambda\Phi H_1(t, w, \Phi) + \lambda\Theta_0[H_1(t,0,0) - H_1(t, w, \Phi)] + C_1 U_0,$$
$$L_2[\Phi] = \lambda w H_2(t, w, \Phi) - \lambda U_0[H_2(t,0,0) - H_2(t, w, \Phi)] - C_2\Theta_0.$$

Thus,

$$\begin{aligned} &L_1[w] \geqq \lambda F_1(t, w, \Phi), \\ &A_0[w] = -a_0 U_1 \leqq 0, \qquad A_1[w] = -a_1 U_1 \leqq 0 \end{aligned} \tag{4.37a}$$

and

$$\begin{aligned} &L_2[\Phi] \leqq \lambda F_2(t, w, \Phi), \\ &B_0[\Phi] = \alpha_0\Theta_0 \geqq 0, \qquad B_1[\Phi] = \alpha_1\Theta_0 \geqq 0. \end{aligned} \tag{4.37b}$$

The theorem now follows from Theorem 4.1.

**5. Uniqueness of positive solutions, existence of maximal solutions.** In this section we strengthen the hypotheses on the functions $H_k(t, u, \theta)$, $k = 1, 2$, and study the unicity of the positive solution.

LEMMA 5.1. *Let Hypotheses* 1 *and* 3 *hold. Then, of course, Hypothesis* 2 *holds as well. Assume that*

$$\lambda_0 < \lambda$$

*and assume there are two distinct positive solutions* $(v_1, \Psi_1)$, $(v_2, \Psi_2)$.

*Then, there are two positive solutions* $(u, \theta)$ *and* $(v, \Psi)$ *which satisfy*

$$u \prec v \prec 0 \prec \Psi \prec \theta. \tag{5.1}$$

*Proof.* Let $(u_0(t), \theta_0(t))$ be an eigenfunction pair of the linear eigenvalue problem (4.3) which also satisfies (4.5), (4.6), (4.7) and

$$v_k \prec u_0 \prec 0 \prec \theta_0 \prec \Psi_k, \qquad k = 1, 2. \tag{5.2}$$

Since both pairs $(v_k, \Psi_k)$ satisfy the conditions (4.23a), (4.23b), (4.23c), we may apply Theorem 4.1 to obtain a positive solution $(v, \Psi)$ which satisfies

$$v_k \leqq v \prec 0 \prec \Psi \leqq \Psi_k, \qquad k = 1, 2.$$

Suppose

$$v_1(t) \not\equiv v(t), \qquad \Psi_1(t) \not\equiv \Psi(t). \tag{5.3}$$

Then,

$$\begin{aligned} &L_2[\Psi] = \lambda F_2(t, v, \Psi) \geqq \lambda F_2(t, v_1, \Psi), \\ &B_0[\Psi] = B_1[\Psi] = 0. \end{aligned} \tag{5.4}$$

By Lemma 3.2, there is a function $a(t)$ which satisfies

$$L_2[a] = \lambda F_2(t, v_1, a), \qquad B_0[a] = B_1[a] = 0, \tag{5.5a}$$

and either $\Psi(t) \equiv a(t)$ or

$$\Psi \prec a. \tag{5.5b}$$

But, since $v_1(t) \leqq 0$ and Hypothesis 3 holds, Corollary 3.1 asserts that the solution of (5.5a) is unique. Hence,

$$a(t) \equiv \Psi_1(t).$$

Thus, using (5.3), we have

$$0 \prec \Psi \prec \Psi_1.$$

A similar argument shows that

$$v_1 \prec v \prec 0.$$

On the other hand, if (5.3) does not hold, we apply the same argument to $(v_2, \Psi_2)$.

LEMMA 5.2. *Suppose Hypotheses* 1 *and* 3 *hold and there are two positive solutions of* (2.1), (2.2) *which satisfy* (5.1). *Let* $\alpha$ *be any constant such that*

$$0 < \alpha < 1, \tag{5.6a}$$

$$\alpha\theta \leqq \Psi. \tag{5.6b}$$

*Then*

$$v \prec \alpha u. \tag{5.7}$$

*Similarly, if*

$$v \leqq \alpha u, \tag{5.8}$$

*then*

$$\alpha\theta \prec \Psi. \tag{5.9}$$

*Proof.* Using (2.6a) we see that

$$L_1[\alpha u] = \lambda\alpha\theta H_1(t, u, \theta) < \lambda\alpha\theta H_1(t, \alpha u, \alpha\theta) = \lambda F_1(t, \alpha u, \alpha\theta).$$

Using (5.6b) we have

$$L_1[\alpha u] < \lambda F_1(t, \alpha u, \Psi), \qquad A_0[\alpha u] = A_1[\alpha u] = 0.$$

By Lemma 3.2, there is a function $w(t)$ which satisfies

$$\begin{aligned} &L_1[w] = \lambda F_1(t, w, \Psi), \qquad A_0[w] = A_1[w] = 0, \\ &w \prec \alpha u. \end{aligned} \tag{5.10}$$

However, because Hypothesis 3 holds, Corollary 3.1 implies that $w(t) \equiv v(t)$ and the lemma is proved in the first case. The other case follows by a similar argument.

THEOREM 5.1. *Let Hypotheses* 1 *and* 3 *hold. Let*

$$\lambda_0 < \lambda.$$

*Then there is at most one positive solution of* (2.1), (2.2).

*Proof.* Suppose there are two positive solutions. By Lemma 5.1, we may assume that there are two positive solutions $(u, \theta)$, $(v, \Psi)$ which satisfy (5.1).

There is a positive number $\alpha < 1$ such that

$$\alpha\theta \leqq \Psi, \tag{5.11}$$

but

$$\alpha\theta \not\prec \Psi. \tag{5.12}$$

To see this we merely observe that for $\beta$ small enough, $\beta\theta \prec \Psi$. We may let $\beta$ increase until either $\beta\theta(t_0) = \Psi(t_0)$ for some interior point $t_0$, or $\beta\theta'(0) = \Psi'(0)$ or $\beta\theta'(1) = \Psi'(1)$.

Then, using Lemma 5.2 and (5.11), we have

$$v \prec \alpha u.$$

In particular, $v \leqq \alpha u$. By Lemma 5.2 again,

$$\alpha\theta \prec \Psi,$$

which contradicts (5.12).

*Remark.* The above uniqueness theorem applies to the cutoff case as well as the normal case. The fact that $0 < \alpha < 1$ implies that we have been in the region

$$|\theta| \leqq \Theta.$$

THEOREM 5.2. *Suppose Hypotheses* 1 *and* 3 *hold and*

$$\lambda_0 < \lambda.$$

*Suppose also that the hypotheses of Theorem* 4.3 *hold. Then the positive solution constructed in Theorem* 4.3 *is also a maximal solution.*

*Proof.* Let $(v, \Psi)$ be any solution. Let

$$v_1 = \max|v(t)|, \qquad \Psi_1 = \max|\Psi(t)|.$$

Let

$$U_2 = U_0 + v_1, \qquad \Theta_2 = \Theta_0 + \Psi_1.$$

Then, following the construction of Theorem 4.3 we may construct a pair $(w, \Phi)$ such that

$$w(t) \leqq v(t), \qquad \Psi(t) \leqq \Phi(t)$$

and (4.23b), (4.23c) hold. A simple induction similar to the basic proof of Theorem 4.2 shows that the iterates $(w_n, \Phi_n)$ constructed in the proof of Theorem 4.3 satisfy

$$w_n(t) \leqq v(t), \qquad \Psi(t) \leqq \Phi_n(t).$$

Thus the functions $(w_n(t), \Phi_n(t))$ converge to a positive solution $(\hat{u}(t), \hat{\theta}(t))$ which also satisfies

$$\hat{u}(t) \leqq v(t), \qquad \Psi(t) \leqq \hat{\theta}(t).$$

However, there is only one positive solution and the theorem follows at once.

A very similar argument shows that the maximal solution is monotone in $\lambda$.

THEOREM 5.3. *Assume Hypotheses* 1 *and* 3 *hold,* $\lambda_0 < \lambda$ *and the hypotheses of Theorem* 4.3 *hold. Let* $(u(t, \lambda), \theta(t, \lambda))$ *denote the maximal solution of* (2.1), (2.2). *Then*

$$u(t, \lambda + \delta) \leqq u(t, \lambda) \leqq \theta(t, \lambda) \leqq \theta(t, \lambda + \delta).$$

*Proof.* Let

$$v_1 = \max |u(t, \lambda)|, \qquad \Psi_1 = \max |\theta(t, \lambda)|$$

and

$$U_2 = U_0 + v_1, \qquad \Theta_2 = \Theta_0 + \Psi_1.$$

As in Theorem 4.3 we construct a pair $(w, \Phi)$ so that (4.23b), (4.23c) hold and

$$w(t) \leqq u(t, \lambda), \qquad \theta(t, \lambda) \leqq \Phi(t).$$

Consider (2.1) and (2.2) with $\lambda$ replaced by $\lambda + \delta$. Since

$$L_1[u(t, \lambda)] = \lambda F_1(t, u, \theta) \leqq (\lambda + \delta) F_1(t, u, \theta),$$
$$L[\theta(t, \lambda)] = \lambda F(t, u, \theta) \geqq (\lambda + \delta) F(t, u, \theta),$$

we may use the induction of Lemma 4.2 to produce a sequence which increases, and as in the proof of Theorem 4.1, we have

$$w(t) \leqq u_n(t) \leqq u(t, \lambda) \leqq \theta(t, \lambda) \leqq \theta_n(t) \leqq \Phi(t).$$

Thus the sequence $(u_n(t), \theta_n(t))$ will converge to the unique positive solution, and the theorem is proved.

**6. Other solutions, special cases.** Let us now consider the very special case where (2.1) takes the form

$$u'' = \lambda \theta H_1(u, \theta), \qquad \theta'' = \lambda u H_2(u, \theta) \tag{6.1}$$

subject to the boundary conditions (B) (of Odeh and Tadjbakhsh) or the boundary condition

$$u(0) = u(1) = 0, \qquad \theta(0) = \theta(1) = 0. \tag{S}$$

Let

$$P = H_1(0,0) H_2(0,0). \tag{6.2a}$$

$$J = H_2(0,0)/H_1(0,0). \tag{6.2b}$$

Consider the linear eigenvalue problem

$$u'' = \lambda \theta H_1(0,0), \qquad \theta'' = \lambda u H(0,0). \tag{6.3}$$

In the case of the boundary conditions (S) the eigenvalues are

$$\lambda_j = \pm \frac{(\pi j)^2}{\sqrt{P}}, \tag{6.4S}$$

while the eigenfunctions are given by $(\lambda_j > 0)$

$$u_j(t) = A \sin \pi j t, \qquad \theta_j(t) = -\sqrt{J} u_j(t) = -\sqrt{J}(A \sin \pi j t). \tag{6.5S}$$

In the case of the boundary conditions (B) we have

$$\lambda_j = \pm \frac{((2_j + 1)\pi/2)^2}{\sqrt{P}} \tag{6.4B}$$

and, for $\lambda_j > 0$,

$$\begin{aligned} u_j(t) &= A \cos \frac{2j+1}{2}\pi t, \\ \theta_j(t) &= -\sqrt{J} u_j(t) = -\sqrt{J}\left(A \cos \frac{2j+1}{2}\pi t\right). \end{aligned} \tag{6.5B}$$

We must also consider the differential equation (6.3) on other intervals. For this reason we introduce the following notation. Let

$$\lambda_k(m, S) \tag{6.6}$$

be the $k$th positive eigenvalue of the differential equations (6.3) on an interval of length $m$ subject to the boundary conditions (S). For example, consider (6.3) on the interval $(a, a + m)$ subject to the boundary conditions

$$u(a) = u(a + m) = \theta(a) = \theta(a + m) = 0.$$

Then, the $k$th positive eigenvalue is denoted by (6.6). Similarly, let

$$\lambda_k(m, B) \tag{6.7}$$

be the $k$th eigenvalue of the differential equations (6.3) on an interval of length $m$ subject to the boundary conditions (B). For example, $\lambda_k(m, B)$ denotes the $k$th eigenvalue of (6.3) on the interval $(a, a + m)$ subject to the boundary conditions

$$u'(a) = \theta'(a) = u(a + m) = \theta(a + m) = 0.$$

A straightforward calculation shows that

$$\begin{aligned} \lambda_0\left(\frac{1}{k}, S\right) &= \lambda_{k-1}(1, S), \qquad k = 1, \cdots, \\ \lambda_0(2m, S) &= \lambda_0(m, B), \\ \lambda_0\left(\frac{2}{2k+1}, S\right) &= \lambda_0\left(\frac{1}{2k+1}, B\right) = \lambda_k(1, B), \qquad k = 0, 1, \cdots. \end{aligned} \tag{6.8}$$

These facts lead immediately to the following results.

LEMMA 6.1. *Let Hypotheses* 1 *and* 2 *be satisfied. Let*

$$\lambda_0(m, B) < \lambda$$

*and suppose that* $H_1(u, \theta)H_2(u, \theta)$ *gets small enough for large* $(u, \theta)$ *that one knows there is a positive solution* $(u(t, m), \theta(t, m))$ *of* (6.1) *subject to the boundary conditions* (B) *on an interval of length* $m$, *say* $(a, a + m)$.

*Then there is a positive solution* $(U(t, m), \Theta(t, m))$ *of* (6.1) *subject to the boundary condition* (S) *on the interval* $(a, a + 2m)$. *Moreover,*

$$U'(a) = -U'(a + 2m) < 0,$$
$$\Theta'(a) = -\Theta'(a + 2m) > 0, \tag{6.9}$$
$$u'(a + m) = \Theta'(a + m) = 0.$$

*Proof.* Let

$$U(t) = \begin{cases} u(2a + m - t), & a \leqq t \leqq a + m, \\ u(t - m), & a + m \leqq t \leqq a + 2m, \end{cases} \tag{6.10a}$$

$$\Theta(t) = \begin{cases} \theta(2a + m - t), & a \leqq t \leqq a + m, \\ \theta(t - m), & a + m \leqq t \leqq a + 2m. \end{cases} \tag{6.10b}$$

A direct computation verifies that these functions have the desired properties.

THEOREM 6.1. *Let Hypotheses* 1 *and* 2 *be satisfied. Let* $k \geqq 0$ *and assume that*

$$\lambda_0\left(\frac{1}{k + 1}, S\right) = \lambda_k(1, S) < \lambda.$$

*Suppose* $H_1(u, \theta)H_2(u, \theta)$ *gets small enough for large* $(u, \theta)$ *that one may apply Lemma* 6.1 *to assert the existence of the functions* $U(t, 1/(k + 1))$, $\Theta(y, 1/(k + 1))$ *of the previous lemma.*

*Then there is a solution* $(U_k(t), \Theta_k(t))$ *of* (6.1) *which satisfies the boundary conditions* (S). *Moreover,*

$$U_k\left(\frac{l}{k + 1}\right) = \Theta_n\left(\frac{l}{k + 1}\right) = 0, \tag{6.11}$$

*and these are the only zeros of* $U_k(t)\Theta_k(t)$.

*Proof.* Let $a = 0$ and $U(t)$, $\Theta(t)$ be the functions whose existence is assumed by Lemma 6.1. Let

$$U_k(t) \equiv (-1)^l U\left(t - \frac{l}{k + 1}\right), \quad \frac{l}{k + 1} \leqq t \leqq \frac{l + 1}{k + 1}, \qquad l = 0, 1, \cdots, k,$$
$$\Theta_k(t) \equiv (-1)^l \Theta\left(t - \frac{l}{k + 1}\right), \quad \frac{l}{k + 1} \leqq t \leqq \frac{l + 1}{k + 1}, \qquad l = 0, 1, \cdots, k. \tag{6.12}$$

A direct computation verifies that $(U_k(t), \Theta_k(t))$ is the desired solution.

THEOREM 6.2. *Let Hypotheses* 1 *and* 2 *be satisfied. Let*

$$\lambda_0\left(\frac{2}{2k + 1}, S\right) = \lambda_0\left(\frac{1}{2k + 1}, B\right) = \lambda_k(1, B) < \lambda. \tag{6.13}$$

*Suppose that* $H_1(u, \theta)H_2(u, \theta)$ *gets small enough for large* $(u, \theta)$ *that we may apply Lemma* 6.1 *to assert the existence of the functions* $U(t, 2/(2k + 1))$, $\Theta(t, 2/(2k + 1))$ *of Lemma* 6.1.

*Then there is a solution* $(u_k(t),\ \theta_k(t))$ *of* (6.1) *which satisfies the boundary conditions* (B). *Moreover,*

$$u_k\left(\frac{2l+1}{2k+1}\right) = \theta_k\left(\frac{2l+1}{2k+1}\right) = 0, \qquad l = 0, 1, \cdots, k,$$

*and these are the only zeros of* $u_k(t)\theta_k(t)$ *in* $[0, 1]$.

*Proof.* Let $a = -1/(2k+1)$. Let

$$(6.14)\qquad \begin{aligned} u_k(t) &= (-1)^l U\left(t - \frac{2(l+1)}{2k+1}\right), \quad \frac{2l+1}{2k+1} \leqq t \leqq \frac{2l+3}{2k+1}, \\ &\qquad\qquad l = -1, 0, 1, 2, \cdots, k-1, \\ \theta_k(t) &= (-1)^l \Theta\left(t - \frac{2(l+1)}{2k+1}\right), \quad \frac{2l+1}{2k+1} \leqq t \leqq \frac{2l+3}{2k+1}, \\ &\qquad\qquad l = -1, 0, 1, \cdots, k-1. \end{aligned}$$

Once more, a direct computation verifies that these functions have the desired features.

*Remark.* In the cutoff case, we are assured of the existence of the necessary positive solutions. Thus, in particular, in the case of (1.1) subject to the boundary condition (B), if

$$\lambda_k < \lambda \leqq \lambda_{k+1},$$

there are at least $k + 1$ distinct nontrivial solutions $(u_j(t), \theta_j(t)), j = 0, 1, \cdots, k$. The pair $(u_j(t),\ \theta_j(t))$ is characterized by the fact that each function has exactly $j$ interior nodal zeros and no other zeros.

*Remark.* This method of "patching together" positive solutions is clearly of limited applicability. Nevertheless, it is an interesting direct consequence of this theory of positive solutions.

**7. Remarks on another iterative method.** While the basic nonlinear iterations described by (4.9), (4.12), or by (4.19), (4.21) have certain advantages from a theoretical point of view, there are linear iterative methods which converge also. These linear iterative methods have advantages from the point of view of computation. We now describe one such method without proof.

Suppose $(u_{n-1}(t), \theta_{n-1}(t))$ satisfy (4.8a) and (4.8b). Let $u_n(t)$ be the solution of the linear equation

$$(7.1)\qquad \begin{aligned} L_1[u_n] - \alpha_1 u_n &= \lambda F_1(t, u_{n-1}, \theta_{n-1}) - \alpha_1 u_{n-1}, \\ A_0[u_n] &= A_1[u_n], \end{aligned}$$

where $\alpha_1$ is a Lipschitz constant for the function $\lambda F_1(t, u, \theta_{n-1})$; that is,

$$(7.1\text{a})\qquad |\lambda F_1(t, x, \theta_{n-1}) - \lambda F_1(t, y, \theta_{n-1})| \leqq \alpha_1 |x - y|.$$

Let $\theta_n(t)$ be the solution of the linear equation

$$(7.2)\qquad L_2[\theta_n] - \alpha_2 \theta_n = \lambda F_2(t, u_n, \theta_{n-1}) - \alpha_2 \theta_{n-1},$$

where $\alpha_2$ is a Lipschitz constant for the function $\lambda F_2(t, u_n, \theta)$; that is,

(7.2a) $$|\lambda F_2(t, u_n, x) - \lambda F_2(t, u_n, y)| \leqq \alpha_2|x - y|.$$

It is now an easy matter to show that either

$$u_{n-1}(t) \equiv u_n(t)$$

or

$$u_n \prec u_{n-1};$$

and, either

$$\theta_{n-1}(t) \equiv \theta_n(t)$$

or

$$\theta_{n-1}(t) \prec \theta_n(t).$$

A similar linear system may be used to replace the nonlinear iteration of Lemma 4.3.

## REFERENCES


[1] N. BAZLEY AND B. ZWAHLEN, *Remarks on the bifurcation of solutions of a nonlinear eigenvalue problem*, Arch. Rational Mech. Anal., 28 (1968), pp. 51–58.

[2] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics*, vol. II, Interscience, New York, 1953.

[3] F. R. GANTMACHER AND M. G. KREIN, *Oscillation matrices and kernels, and small vibrations of mechanical systems*, A. E. C. Transl. 4481, Office of Technical Services, Dept. of Commerce, Washington, D.C., 1961.

[4] E. HOPF, *An inequality for positive linear integral operators*, J. Math. Mech., 12 (1963), pp. 683–692.

[5] R. JENTZSCH, *Über Integralgleichungen mit positivem Kern*, Crelle's J., 141 (1912), pp. 235–249.

[6] S. KARLIN, *Positive operators*, J. Math. Mech., 8 (1959), pp. 907–932.

[7] M. G. KREIN AND M. A. RUTMAN, *Linear operators leaving invariant a cone in a Banach space*, Uspekhi Mat. Nauk, 3 (1948), no. 1 (23), pp. 3–95; English transl., Amer. Math. Soc. Translations, no. 26, 1950, 128 pp.

[8] F. ODEH AND I. TADJBAKHSH, *A nonlinear eigenvalue problem for rotating rods*, Arch. Rational Mech. Anal., 20 (1965), pp. 81–94.

[9] S. V. PARTER, *Mildly nonlinear elliptic partial differential equations and their numerical solution. I*, Numer. Math., 7 (1965), pp. 113–128.

[10] ———, *Remarks on the numerical computation of solutions of $\Delta u = f(p, u)$*, Numerical Solution of Partial Differential Equations, J. Bramble, ed., Academic Press, New York, 1966, pp. 73–82.

[11] ———, *Maximal solutions of mildly nonlinear elliptic equations*, Numerical Solutions of Nonlinear Differential Equations, D. Greenspan, ed., John Wiley, New York, 1966, pp. 213–238.

[12] E. PICARD, *Traite d'analyse*, vol. III, 2nd ed., Gauthier-Villars, Paris, 1908.

SIAM J. MATH. ANAL.
Vol. 1, No. 4, November 1970

# NONLINEAR EIGENVALUE PROBLEMS FOR SOME FOURTH ORDER EQUATIONS. II: FIXED-POINT METHODS*

SEYMOUR V. PARTER†

## 1. Introduction.

Let

$$L_k[\varphi] \equiv (p_k(t)\varphi')' - C_k(t)\varphi(t), \qquad k = 1, 2, \tag{1.1}$$

be two regular Sturm-Liouville operators defined on $[0, 1]$; that is,

$$\begin{aligned} &p_k(t) \in C^1[0, 1], \qquad C_k(t) \in C[0, 1], \\ &p_k(t) \geqq p_0 > 0, \qquad C_k(t) \geqq 0. \end{aligned} \tag{1.2}$$

Consider the nonlinear system of ordinary differential equations

$$\begin{aligned} L_1[u] &= \lambda\theta H_1(t, u, \theta), \qquad 0 < t < 1, \\ L_2[\theta] &= \lambda u H_2(t, u, \theta), \qquad 0 < t < 1, \end{aligned} \tag{1.3}$$

where the functions $u(t)$, $\theta(t)$ are required to satisfy the boundary conditions

$$\begin{aligned} A_0[u] &\equiv a_0 u(0) - b_0 u'(0) = 0, \qquad A_1[u] \equiv a_1 u(1) + b_1 u'(1) = 0, \\ B_0[\theta] &\equiv \alpha_0\theta(0) - \beta_0\theta'(0) = 0, \qquad B_1[\theta] \equiv \alpha_1\theta(1) + \beta_1\theta'(1) = 0, \end{aligned} \tag{1.3a}$$

with

$$\begin{aligned} &a_k, \alpha_k, b_k, \beta_k \geqq 0, \qquad k = 1, 2, \\ &a_k + b_k > 0, \quad \alpha_k + \beta_k > 0, \quad a_0 + a_1 > 0, \quad \alpha_0 + \alpha_1 > 0. \end{aligned} \tag{1.3b}$$

The functions $H_k(t, u, \theta)$ are even and positive, i.e.,

$$H_k(t, u, \theta) = H_k(t, |u|, |\theta|) > 0, \qquad k = 1, 2. \tag{1.3c}$$

In a companion paper [12] we studied such problems under a set of assumptions which allowed the iterative construction of a maximal, positive solution. In this report we apply the Schauder fixed-point theorem to obtain (under appropriate hypotheses) the existence of solutions having a specified number of zeros.

This work and the work described in [12] were motivated by a problem studied by F. Odeh and I. Tadjbakhsh [10] and N. Bazley and B. Zwahlen [1]. These authors consider the nonlinear system

$$\begin{aligned} u'' &= \lambda \sin\theta, \qquad 0 < t < 1, \\ \theta'' &= \lambda u \cos\theta, \qquad 0 < t < 1, \end{aligned} \tag{1.4}$$

subject to the boundary conditions

$$u'(0) = u(1) = 0, \qquad \theta(0) = \theta'(1) = 0, \tag{A}$$

* Received by the editors January 29, 1970.

† Departments of Computer Sciences and Mathematics, University of Wisconsin, Madison, Wisconsin 53706. This research was sponsored by the Office of Naval Research under Contract N00014-67-A-0128-0004.

or the boundary conditions

(B) $$u'(0) = \theta'(0) = \theta(1) = u(1) = 0.$$

Because of the physical interpretation of the function $(u(t), \theta(t))$ an important condition which was not imposed by these earlier authors is

(1.4a) $$|\theta(t)| < \pi/2.$$

The case of the boundary conditions (B) has been discussed in [12]. Thus one of our major aims is to obtain "physical" solutions of (1.4) subject to the boundary conditions (A). This result will follow from the general results obtained here together with a simple construction based on the a priori estimates of [12].

The approach we use here is closely related to our work [11] on sublinear Hammerstein equations and is related to the work of Pimbley [14] and Wolkowisky [16]. Indeed, using the method of [11, Lemma 7] these results go over to problems involving pairs of integral equations with oscillation kernels. Nevertheless, at this time, we limit ourselves to the case of differential equations. The interested reader would be well advised to look at the book [15] of Pimbley.

In § 2 we discuss some preliminary ideas relating these problems to the theory of oscillation kernels [4] and variational problems. In § 3 we develop some basic facts of oscillation theory for fourth order problems. Section 4 is devoted to the basic existence theorem. In § 5 we show how the results of [12] may be used to obtain additional existence theorems. In particular, we obtain results which apply to Problem A of Odeh and Tadjbakhsh.

**2. Preliminary notions.** In addition to the assumptions (1.3c), we make the following assumptions about $H_k(t, u, \theta)$.

ASSUMPTION 1. $H_k(t, u, \theta) \in C[0, 1] \times C[-\infty, \infty] \times C[-\infty, \infty], \quad k = 1, 2.$

ASSUMPTION 2. $0 < a \leqq H_k(t, u, \theta) \leqq b, \ k = 1, 2$, where $a$ and $b$ are positive constants.

Let

(2.1) $$A \equiv \{(q_1(t), q_2(t)) \in C[0, 1] \times C[0, 1]; a \leqq q_k(t) \leqq b, k = 1, 2\},$$

(2.2) $$B \equiv \{(q_1(t), q_2(t)) \in L^2(0, 1) \times L^2(0, 1); a \leqq q_k(t) \leqq b \text{ a.e.}, k = 1, 2\}.$$

For any pair $(q_1(t), q_2(t)) \in B$ we consider the linear eigenvalue problem

(2.3) $$\begin{aligned} L_1[u] &= \lambda\theta q_1(t), \qquad A_0[u] = A_1[u] = 0, \\ L_2[\theta] &= \lambda u q_2(t), \qquad B_0[\theta] = B_1[\theta] = 0. \end{aligned}$$

Let $K_1(s, t), K_2(s, t)$ be the Green's functions associated with the operators $-L_1[u]$ and $-L_2[\theta]$ subject to the appropriate homogeneous boundary conditions $(A_j[u] = 0, B_j[\theta] = 0)$. Then the equations (2.3) are equivalent to

(2.4a) $$u(t) = -\lambda \int_0^1 K_1(t, x) q_1(x)\theta(x)\, dx,$$

(2.4b) $$\theta(x) = -\lambda \int_0^1 K_2(x, y) q_2(y) u(y)\, dy.$$

Upon substitution, we see that this pair of integral equations is equivalent to either

$$u(t) = \lambda^2 \int_0^1 G_1(t, s)u(s)\, dx,$$

(2.5a)

$$G_1(t, s) = \int_0^1 K_1(t, x)q_1(x)K_2(x, s)q_2(s)\, dx$$

or

$$\theta(t) = \lambda^2 \int_0^1 G_2(t, s)\theta(s)\, ds,$$

(2.5b)

$$G_2(t, s) = \int_0^1 K_2(t, x)K_1(x, s)q_1(s)q_2(x)\, dx.$$

The kernels $K_j(s, t)$, and therefore the kernels $G_j(s, t)$,[1] are "oscillation kernels" in the sense of Gantmacher–Krein [4], and hence a great deal is known about their spectrum. In particular, consider equations (2.5a) or (2.5b). The spectrum consists of positive, simple eigenvalues

(2.6) $$0 < \lambda_0^2 < \lambda_1^2 < \cdots < \lambda_k^2 < \cdots.$$

Moreover, the associated eigenfunctions $\varphi_k(t)$, $k = 0, 1, \cdots$, satisfy the oscillation condition; that is, in the open interval $(0, 1)$, $\varphi_k(t)$ has exactly $k$ nodal zeros and no other zeros.

Thus, returning to our original problem, we see that the eigenvalues are real and occur in pairs $(\lambda_k, -\lambda_k)$. Indeed, if $\lambda$ is an eigenvalue with associated eigenfunction $(\mu(t), \theta(t))$, then $-\lambda$ is an eigenvalue associated with $(-u(t), \theta(t))$. Thus we may restrict ourselves to a consideration of the positive eigenvalues

(2.7) $$0 < \lambda_1 < \lambda_2 \cdots.$$

If $(u_k(t), \theta_k(t))$ is the eigenfunction associated with $\lambda_k$, then each function $u_k(t)$ or $\theta_k(t)$ has exactly $k$ interior nodal zeros and no other zeros.

Another useful fact about oscillation kernels which is clearly related to the above remarks is the *variation diminishing property*: For $f(t) \in C[0, 1]$ let $Z(f)$ denote the number of interior nodal zeros of $f(t)$, let $K(s, t)$ be an oscillation kernel and

(2.8a) $$\varphi(s) = \int_0^1 K(s, t)\psi(t)\, dt;$$

then

(2.8b) $$Z(\varphi) \leqq Z(\psi).$$

[1] The theory developed in [4] is restricted to the symmetric case $G(s, t) = G(t, s)$. However the results required here are valid in the general case. Gantmacher and Krein assert the validity of their results in the general case and cite references to the Russian literature. A discussion of the general case was given by S. Karlin in classroom lectures and will appear in his book [6].

The representations (2.5a), (2.5b) show that $\lambda_j$ is a continuous function of $(q_1(t), q_2(t)) \in B$ (see [7, p. 213]).

In the special case where $L_1 \equiv L_2$, $A_j \equiv B_j$ the linear eigenvalue problem (2.3) is essentially self-adjoint and we know even more about the spectrum. The eigenvalues $\lambda_k$ are given by the variational characterization of Courant [2], Weyl, Ritz, etc.; that is,

$$\lambda_0^2 = \min_{u \not\equiv 0} \frac{\int_0^1 [q_1(t)]^{-1}(L_1[u])^2\,dt}{\int_0^1 q_2(t)(u(t))^2\,dt} \tag{2.9a}$$

and, for $j \geqq 1$,

$$\lambda_j^2 = \max_{S_{j-1}} \min_{u \in S_{j-1}^{\perp}} \frac{\int_0^1 [q_1(t)]^{-1}(L_1[u])^2\,dt}{\int_0^1 q_2(t)(u(t))^2\,dt}, \tag{2.9b}$$

where $S_k$ denotes an arbitrary $k$-dimensional subspace of $W_2^2(0, 1)$ whose elements satisfy the boundary conditions

$$A_0[u] = A_1[u] = 0 \tag{2.9c}$$

and $S_k^{\perp}$ denotes the orthogonal complement of $S_k$ in $L_2[(0, 1), q_2\,dt]$, i.e., $\varphi(t) \in S_k^{\perp}$ if (2.9c) holds and

$$\int_0^1 q_2(t)\varphi(t)u(t)\,dt = 0 \tag{2.9d}$$

for every $u(t) \in S_k$.

From this basic fact we obtain the following lemma.

LEMMA 2.1. *Let $(q_1(t;\sigma), q_2(t;\sigma)) \in A$, $0 \leqq \sigma \leqq \infty$, be a one-parameter family of pairs of functions which is continuous in B as a function of $\sigma$. Let $\lambda_j(\sigma)$ denote the j-th positive eigenvalue of*

$$\begin{aligned} L_1[u] &= \lambda\theta q_1(t;\sigma), \qquad A_0[u] = A_1[u] = 0, \\ L_1[\theta] &= \lambda u q_2(t;\sigma), \qquad A_0[\theta] = A_1[\theta] = 0. \end{aligned} \tag{2.10}$$

*Suppose $\sigma_1 < \sigma_2$ implies*

$$q_j(t, \sigma_1) \leqq q_j(t, \sigma_2), \qquad q_1(t, \sigma_1)q_2(t, \sigma_1) \not\equiv q_1(t, \sigma_2)q_2(t, \sigma_2). \tag{2.11}$$

*Then the eigenvalue $\lambda_j(\sigma)$ is a continuous function of $\sigma$ and*

$$\lambda_j(\sigma_1) > \lambda_j(\sigma_2). \tag{2.12}$$

*Moreover, for each j, there exist two positive constants $\Lambda_j$ and $M_j$ such that*

$$0 < \Lambda_j \leqq \lambda_j(q_1, q_2) \leqq M_j \tag{2.13}$$

*for all $(q_1, q_2) \in B$.*

**3. Linear problems, oscillation theory.** In this section we develop some further properties of (2.3). Our fundamental tool is an extension of some basic results of W. Leighton and Z. Nehari [9].

Let $\lambda > 0$ be a fixed constant, let $(q_1(t), q_2(t)) \in A$ and consider the linear differential equation

$$L_1[u] = \lambda\theta q_1(t), \qquad 0 < t < 1, \tag{3.1}$$
$$L_2[\theta] = \lambda u q_2(t), \qquad 0 < t < 1.$$

LEMMA 3.1. *Let* $(u(t), \theta(t))$ *be a solution of* (3.1) *and let* $a \in [0, 1)$. *If* $u(a)$, $u'(a)$, $\theta(a)$, $\theta'(a)$ *are nonnegative (but not all zero), then the functions* $u(x)$, $u'(x)$, $\theta(x)$, $\theta'(x)$ *are all positive for* $a < x \leqq 1$.

*Proof.* In the case where

$$L_1[u] = L_2[u] \equiv u''$$

this result is Lemma 2.1 of [9]. In the general case we use the representations (Volterra integral equations)

$$\theta(s) = \theta(a) + p_2(a)\theta'(a)\int_a^s \frac{dx}{p_2(x)} + \lambda\int_a^s \frac{dx}{p_2(x)}\int_a^x u(t)q_2(t)\,dt + \int_a^s \frac{dx}{p_2(x)}\int_a^x C_2(t)\theta(t)\,dt, \tag{3.2a}$$

$$u(s) = u(a) + p_1(a)u'(a)\int_a^s \frac{dx}{p_1(x)} + \lambda\int_a^s \frac{dx}{p_1(x)}\int_a^x \theta(t)q_1(t)\,dt + \int_a^s \frac{dx}{p_1(x)}\int_a^x C_1(t)u(t)\,dt. \tag{3.2b}$$

*Case* 1. $\theta(a) + \theta'(a) > 0$. There is an interval $(a, a + \delta)$ in which $\theta(t)$ is positive. Let us assume $\theta(t)$ is known and use (3.2b) to obtain $u(t)$ in this interval. Since we are dealing with a Volterra integral equation we may use Picard iterations with $u_0(t) \equiv u(a)$. A straightforward induction shows that $u_n(t)$ is positive on $(a, a + \delta)$ and hence

$$u(t) \geqq 0, \qquad a < t < a + \delta.$$

Using this result in (3.2a) we see that

$$\theta(a + \delta) > 0.$$

Hence $\theta(t)$ and $u(t)$ are (strictly) positive for $t \in (a, 1]$. Using the representations (3.1a) and (3.1b) we see that $\theta'(t)$ and $u'(t)$ are also positive for $t \in (a, 1]$.

*Case* 2. $u(0) + u'(0) > 0$. A similar argument (reversing the roles of $u$ and $\theta$) completes the proof in this case.

LEMMA 3.2. *Let* $(u(t), \theta(t))$ *be a solution of* (3.1) *and let* $a \in (0, 1]$. *Suppose* $u(a) \geqq 0$, $\theta(a) \geqq 0$ *while* $u'(a) \leqq 0$, $\theta'(a) \leqq 0$ *(but not all zero). Then for* $t \in [0, a)$ *we have*

$$u(t) > 0, \quad \theta(t) > 0, \quad u'(t) < 0, \quad \theta'(t) < 0.$$

*Proof*. As in the proof of Lemma 2.2 of [9], we let $s = 1 - t$ and apply Lemma 3.1.

LEMMA 3.3. *Let* $(u(t), \theta(t))$ *be a nontrivial solution of* (3.1) *and let* $a \in (0, 1)$. *Suppose either*

$$u(a) = u'(a) = 0$$

*or*

$$\theta(a) = \theta'(a) = 0.$$

*Then, in* (*at least*) *one of the two intervals* $[0, a)$, $(a, 1]$ *all four functions* $u(t)$, $u'(t)$, $\theta(t)$, $\theta'(t)$ *are different from zero.*

*Proof*. This result follows from the two preceding lemmas exactly as in [9].

These rather elementary results are the basis of some interesting theorems on the "continuity" of the spectrum of (2.3) which are stronger than the results mentioned earlier [7, p. 213].

LEMMA 3.4. *Suppose there is a sequence* $(q_1^{(k)}(t), q_2^{(k)}(t)) \in A$, $k = 1, 2, \cdots$, *and functions* $(\bar{q}_1(t), \bar{q}_2(t)) \in B$ *such that*

$$q_j^{(k)} \rightharpoonup \bar{q}_j(t) \quad \text{weakly in } L_2[0, 1] \text{ as } k \to \infty, \qquad j = 1, 2. \tag{3.3}$$

*Let* $(u_n^{(k)}(t), \theta_n^{(k)}(t))$ *and* $\lambda\sigma_n^k$ *be the n-th eigenfunction and n-th positive eigenvalue of*

$$\begin{aligned} L_1[u_n^{(k)}] &= \lambda\sigma_n^{(k)}\theta_n^{(k)}q_1^{(k)}(t), \qquad A_0[u_n^{(k)}] = A_1[u_n^{(k)}] = 0, \\ L_2[\theta_n^{(k)}] &= \lambda\sigma_n^{(k)}u_n^{(k)}q_2^{(k)}(t), \qquad B_0[\theta_n^{(k)}] = B_1[\theta_n^{(k)}] = 0. \end{aligned} \tag{3.4}$$

*Suppose there is a positive constant* $\bar{\sigma}$ *such that*

$$\sigma_n^{(k)} \to \bar{\sigma} \quad \text{as } k \to \infty. \tag{3.4a}$$

*Finally, suppose there are functions* $\bar{u}(t), \bar{\theta}(t) \in C^1(0, 1]$ *such that*

$$\begin{aligned} u_n^{(k)}(t) &\to \bar{u}(t) \quad \text{in } C^1[0, 1] \text{ as } k \to \infty, \\ \theta_n^{(k)}(t) &\to \bar{\theta}(t) \quad \text{in } C^1[0, 1] \text{ as } k \to \infty. \end{aligned} \tag{3.4b}$$

*Then the functions* $(\bar{u}(t), \bar{\theta}(t))$ *are the n-th eigenfunction associated with the n-th* (*positive*) *eigenvalue* $\lambda\bar{\sigma}$ *of*

$$\begin{aligned} L_1[\bar{u}] &= \lambda\bar{\sigma}\bar{\theta}q_1 \text{ a.e.}, \qquad A_0[\bar{u}] = A_1[\bar{u}] = 0, \\ L_2[\theta] &= \lambda\bar{\sigma}\bar{u}\bar{q}_2 \text{ a.e.}, \qquad B_0[\theta] = B_0[\bar{\theta}] = 0. \end{aligned} \tag{3.5}$$

*Proof*. While the functions $(\bar{q}_1(t), \bar{q}_2(t))$ need not be continuous, they belong to $B$. Moreover, the functions $\bar{u}(t)$, $\bar{\theta}(t)$ are weak solutions of (3.5), hence, strong solutions. Thus, as in the development of (2.5a), (2.5b) we see that $\bar{u}(t)$ and $\bar{\theta}(t)$ are separately eigenfunctions of a linear integral equation whose kernel is an oscillation kernel. Thus each has only a finite number of interior zeros in $(0, 1)$ and each such interior zero is a nodal zero. Let $N$ be the number of interior zeros.

Because of the $C^1[0, 1]$ convergence there is a $k_0$ such that $k \geqq k_0$ implies that $u_n^{(k)}(t)$ has at least $N$ interior nodal zeros. Since each $u_n^{(k)}(t)$ has exactly $n$ interior nodal zeros, we have

$$N \leqq n. \tag{3.6}$$

Let $n \geqq 1$ and let

$$0 < \xi_1^{(k)} < \xi_2^{(k)} < \cdots < \xi_n^{(k)} < 1$$

be the $n$ interior zeros of $u_n^{(k)}(t)$. There is a subsequence $(k')$ and a set of values

$$0 \leqq \bar{\xi}_1 \leqq \bar{\xi}_2 \leqq \cdots \leqq \bar{\xi}_n \leqq 1$$

such that

$$\xi_j^{(k')} \to \bar{\xi}_j \quad \text{as } k' \to \infty. \tag{3.7}$$

If $N < n$, then either there is a pair

$$0 < \bar{\xi}_j = \bar{\xi}_{j+1} < 1, \tag{3.8a}$$

or

$$\bar{\xi}_1 = 0 \tag{3.8b}$$

or

$$\bar{\xi}_n = 1. \tag{3.8c}$$

However, if (3.8a) occurs, then $\bar{u}(t)$ has a double zero at $\bar{\xi}_j$. If

$$\bar{\theta}(\bar{\xi}_j)\bar{\theta}'(\bar{\xi}_j) \geqq 0,$$

then (because of the linearity) we may take

$$\bar{\theta}(\bar{\xi}_j) \geqq 0, \qquad \bar{\theta}'(\bar{\xi}_j) \geqq 0;$$

and then Lemma 3.1 contradicts the boundary conditions at $t = 1$. On the other hand, if

$$\bar{\theta}(\bar{\xi}_j) \geqq 0, \qquad \bar{\theta}'(\bar{\xi}_j) \leqq 0,$$

then the boundary conditions at $t = 0$ and Lemma 3.2 lead to a contradiction.

If (3.8b) occurs as a result of $\xi_1^{(k')} \to 0$, then (because of the boundary condition at $t = 0$) $\bar{u}(t)$ has a double zero at $t = 0$. However, we also have

$$\bar{\theta}(0)\bar{\theta}'(0) \geqq 0.$$

Hence, the boundary conditions at $t = 1$ and Lemma 3.1 lead to a contradiction.

A similar argument disposes of the case (3.8c).

This result leads us to consider another basic assumption.

ASSUMPTION 3. For every fixed $n$ there are constants $A_n$, $B_n$ such that $\lambda_n$, the $n$th positive eigenvalue of the linear eigenvalue problem (2.3), satisfies

$$0 < A_n \leqq \lambda_n \leqq B_n \tag{3.9}$$

for all $(q_1(t), q_2(t)) \in A$.

THEOREM 3.1. *Suppose Assumption 3 holds. Let $(q_1^{(k)}(t),\ q_2^{(k)}(t)) \in A$ for $k = 1, 2, \cdots$. Suppose there are two functions $(\bar{q}_1(t), \bar{q}_2(t))$ such that*

$$q_j^{(k)}(t) \rightharpoonup \bar{q}(t) \quad \textit{weakly in } L_2(0,1) \textit{ as } k \to \infty, \qquad j = 1, 2. \tag{3.10}$$

*The convexity of $A$ implies that $(\bar{q}_1, \bar{q}_2) \in B$. Let $\lambda_n^{(k)}$ and $(u_n^{(k)}(t),\ \theta_n^{(k)}(t))$ be the n-th (positive) eigenvalue and the corresponding eigenfunction of* (2.3) *with $q_j(t)$ replaced by $q_j^{(k)}(t)$. Let $(u_n^{(k)}(t),\ \theta_n^{(k)}(t))$ be normalized so that*

$$\max\left(\|\theta_n^{(k)}\|_\infty,\ \|u_n^{(k)}\|_\infty\right) = 1,$$

*where*

$$\|f\|_\infty = \max\{|f(t)|, 0 \leqq t \leqq 1\}.$$

*Let $\bar{\lambda}_n$ and $(\bar{u}(t), \bar{\theta}(t))$ be the n-th (positive) eigenvalue and corresponding eigenfunction of* (2.3) *with $q_j(t)$ replaced by $\bar{q}_j(t)$. Then*

$$\lambda_n^{(k)} \to \bar{\lambda},$$

$$u_n^{(k)}(t) \to \bar{u}(t) \quad in\ C'[0,1],$$

$$\theta_n^{(k)}(t) \to \bar{\theta}(t) \quad in\ C'[0,1].$$

*Proof.* There is a subsequence $(k')$ and a constant $\mu$ and two functions $U(t)$, $\Theta(t)$ such that

$$\lambda_n^{(k')} \to \mu,$$

$$u_n^{(k)}(t) \to U(t) \quad \text{in } C'[0,1],$$

$$\theta_n^{(k)}(t) \to \Theta(t) \quad \text{in } C'[0,1].$$

On applying Lemma 3.3 we see that

$$\mu = \bar{\lambda},$$

$$U(t) = \bar{u}(t), \qquad \Theta(t) = \bar{\theta}(t).$$

A straightforward argument based on the uniqueness of the quantities $\bar{\lambda}$, $\bar{u}(t)$, $\bar{\theta}(t)$ shows that the entire sequence converges.

Having established this result, one is naturally led to the question: When does Assumption 3 hold? Clearly, Lemma 2.1, the variational characterization of $\bar{\lambda}_n$ given by (2.9b), asserts that Assumption 3 holds in the symmetrizable case. It seems reasonable to conjecture that Assumption 3 always holds. However, we have not established this assertion. On the other hand, the methods of this section may be used to establish this fact for certain cases. These results are presented in the Appendix. We note that the case (A) of Odeh and Tadjbakhsh is included. The case (B) of Odeh and Tadjbakhsh is a symmetrizable case.

**4. The basic existence theorem.** In this section we return to our original nonlinear problem (1.3), (1.3a). We assume that Assumptions 1–3 hold.

Let

$$h_k(t) = H_k(t,0,0), \qquad k = 1,2, \tag{4.1}$$
$$g_k(t) = H_k(t,\infty,\infty), \quad k = 1,2.$$

Let $\lambda_j$, $j = 0, 1, \cdots$, denote the positive eigenvalues of the linear eigenvalue problem

$$L_1[v] = \lambda\phi h_1(t), \qquad A_0[v] = A_1[v] = 0, \tag{4.2a}$$
$$L_2[\phi] = \lambda v h_2(t), \qquad B_0[\phi] = B_1[\phi] = 0.$$

Let $\mu_j$, $j = 0, 1, \cdots$, denote the positive eigenvalues of the linear eigenvalue problem

$$L_1[w] = \mu\psi g_1(t), \qquad A_0[w] = A_1[w] = 0, \tag{4.2b}$$
$$L_2[\psi] = \mu w g_2(t), \qquad B_0[\psi] = B_1[\psi] = 0.$$

Naturally, we assume

$$\lambda_j < \lambda_{j+1}, \qquad \mu_j < \mu_{j+1}. \tag{4.2c}$$

Let $\lambda > 0$ be fixed. Let $(q_1(t), q_2(t)) \in A$. Let

$$\begin{aligned} \sigma_n &= \sigma_n(q_1, q_2), \\ U_n(t) &= U_n(t; q_1, q_2), \\ \Theta_n(t) &= \Theta_n(t; q_1, q_2) \end{aligned} \tag{4.3}$$

denote the $n$th positive eigenvalue and eigenfunction respectively of the linear eigenvalue problem

$$\begin{aligned} L_1[U_n] &= \lambda\sigma_n\Theta_n q_1(t), \qquad A_0[U_n] = A_1[U_n] = 0, \\ L_2[\Theta_n] &= \lambda\sigma_n U_n q_2(t), \qquad B_0[\Theta_n] = B_1[\Theta_n] = 0, \end{aligned} \tag{4.4a}$$

normalized so that

$$\max\{\|U_n\|_\infty, \|\Theta_n\|_\infty\} = 1. \tag{4.4b}$$

*Note.* Since the eigenvalues are all simple, this normalization determines $(U_n, \Theta_n)$ up to sign.

*Remark.* Each function $U_n(t), \Theta_n(t)$ has exactly $n$ nodal zeros in $(0, 1)$ and no other interior zeros.

Given $(q_1, q_2) \in A$, and hence $(U_n, \Theta_n)$, let $\alpha \in (0, \infty)$ and let

$$\begin{aligned} \rho_n &= \rho_n(q_1, q_2, \alpha), \\ V_n(t) &= V_n(t; q_1, q_2, \alpha), \\ \Psi_n(t) &= \Psi_n(t; q_1, q_2, \alpha) \end{aligned} \tag{4.5a}$$

denote the $n$th positive eigenvalue and eigenfunction respectively of the linear eigenvalue problem

$$\begin{aligned} L_1[V_n] &= \lambda\rho_n\Psi_n H_1(t, \alpha U_n(t), \alpha\Theta_n(t)), \\ L_2[\Psi_n] &= \lambda\rho_n V_n H_2(t, \alpha U_n(t), \alpha\Theta_n(t)), \\ A_0[V_n] &= A_1[V_n] = B_0[\Psi_n] = B_1[\Psi_n] = 0, \end{aligned} \tag{4.5b}$$

normalized so that

$$\max\{\|V_n\|_\infty \|\Psi_n\|_\infty\} = 1. \tag{4.5c}$$

*Note.* Because the functions $H_k(t, u, \theta)$ are even, we know the functions $H_k(t, \alpha U_n(t), \alpha\Theta_n(t))$ are well-defined.

LEMMA 4.1. *The quantities $U_n(t; q_1, q_2)$, $\Theta_n(t; q_1, q_2)$, $H_1(t, \alpha U_n, \alpha\Theta_n)$, $H_2(t, \alpha U_n, \alpha\Theta_n)$, $\rho_n(q_1, q_2, \alpha)$, $V_n(t, q_1, q_2, \alpha)$, $\Psi_n(t, q_1, q_2, \alpha)$ are continuous functions of $(q_1, q_2, \alpha)$ in the following sense. If*

$$\alpha_n^{(k)} \to \bar{\alpha} < \infty \quad \textit{as } k \to \infty,$$

$$q_j^{(k)}(t) \rightharpoonup \bar{q}_j(t) \in B \quad \textit{weakly in } L_2(0, 1) \textit{ as } k \to \infty,$$

*then*

$$U_n^{(k)}(t) = U_n(t; q_1^{(k)}, q_2^{k}) \to U_n(t; \bar{q}_1, \bar{q}_2) \quad \textit{uniformly on } [0,1], \tag{4.6a}$$
$$\Theta_n^{(k)}(t) = \Theta_n(t; q_1^{(k)}, q_2^{(k)}) \to \Theta_n(t; \bar{q}_1, \bar{q}_2) \quad \textit{uniformly on } [0,1],$$

$$H_j(t, \alpha^{(k)} U_n^{(k)}, \alpha^{(k)} \Theta_n^{(k)}) \to H_j(t, \bar{\alpha} U_n, \bar{\alpha}\Theta_n) \quad \textit{uniformly on } [0,1], \tag{4.6b}$$
$$\rho_n(q_1^{(k)}, q_2^{(k)}, \alpha^{(k)}) \to \rho(\bar{q}_1, \bar{q}_2, \bar{\alpha})$$

*and*

$$V_n(t; q_1^{(k)}, q_2^{(k)}, \alpha^{(k)}) \to V_n(t; \bar{q}_1, \bar{q}_2, \bar{\alpha}) \quad \textit{uniformly on } [0,1], \tag{4.6c}$$
$$\Psi_n(t; q_1^{(k)}, q_2^{(k)}, \alpha^{(k)}) \to \Psi_n(t; \bar{q}_1, \bar{q}_2, \bar{\alpha}) \quad \textit{uniformly on } [0,1].$$

*Proof.* Apply Theorem 3.1.

LEMMA 4.2. *Suppose*

$$\lambda_n < \lambda < \mu_n. \tag{4.7}$$

*Let $(q_1(t), q_2(t)) \in A$. Then there is at least one value of $\alpha \in (0, \infty)$ such that*

$$\rho_n(q_1, q_2, \alpha) = 1.$$

*Proof.* By Lemma 4.1, for fixed $(q_1, q_2) \in A$, $\rho_n(q_1, q_2, \alpha)$ is a continuous function of $\alpha$. The lemma follows from the observations that

$$\lim_{\alpha \to 0} \rho_n(q_1, q_2, \alpha) = \frac{\lambda_n}{\lambda} < 1 \tag{4.8a}$$

and

$$\lim_{\alpha \to \infty} \rho_n(q_1, q_2, \alpha) = \frac{\mu_n}{\lambda} > 1. \tag{4.8b}$$

LEMMA 4.3. *Let* (4.7) *hold. There is a positive constant $\alpha_1 > 0$ such that for all $(q_1(t), q_2(t)) \in A$ and all $\alpha \in (0, \alpha_1)$ we have*

$$\rho_n(q_1, q_2, \alpha) < 1. \tag{4.9}$$

*Proof.* Assume the lemma is false. Using the continuity of $\rho_n(q_1, q_2, \alpha)$ and condition (4.8a), we may assume that there is a sequence $(q_1^{(k)}(t); q_2^{(k)}(t)) \in A$ and a sequence $\alpha^{(k)} \in (0, \infty)$ such that

$$\alpha^{(k)} \to 0 \quad \text{as } k \to \infty, \tag{4.10a}$$

$$\rho_n(q_1^{(k)}, q_2^{(k)}, \alpha^{(k)}) = 1 \quad \text{for all } k = 1, 2, \cdots. \tag{4.10b}$$

However, we may extract a subsequence $(k')$ and a pair of functions $(\bar{q}_1(t), \bar{q}_2(t)) \in B$ such that

$$q_j^{(k')} \rightharpoonup \bar{q}_j(t) \quad \text{weakly in } L_2(0,1), \qquad u = 1, 2. \tag{4.11}$$

Applying Lemma 4.1 or Theorem 3.1 we see that

$$\lim_{k' \to \infty} \rho_n(q_1^{(k')}, q_2^{(k')}, \alpha^{(k')}) = \frac{\lambda_n}{\lambda} < 1,$$

which contradicts (4.10b).

LEMMA 4.4. *Let* (4.7) *hold. There is a finite positive constant* $\alpha_2$ *such that for all* $(q_1(t), q_2(t)) \in A$ *and all* $\alpha \in (\alpha_2, \infty)$ *we have*

$$\rho_n(q_1, q_2, \alpha) > 1. \tag{4.12}$$

*Proof.* Assume the lemma is false. Using Lemma 4.1 and condition (4.8b) we may assume that there is a sequence $(q_1^{(k)}(t), q_2^{(k)}(t)) \in A$ and a sequence $\alpha^{(k)} > 0$ such that

$$\alpha^{(k)} \to \infty \quad \text{as } k \to \infty, \tag{4.13a}$$

$$\rho_n(q_1^{(k)}, q_2^{(k)}, \alpha^{(k)}) = 1, \qquad k = 1, 2, \cdots. \tag{4.13b}$$

Using Lemma 4.1 and extracting enough subsequences we may also assume that there is a pair of functions $(\bar{q}_1(t), \bar{q}_2(t)) \in B$ such that

$$q_j^{(k)}(t) \rightharpoonup \bar{q}_j(t) \quad \text{weakly in } L_2(0, 1), \qquad j = 1, 2,$$

$$U_n^{(k)}(t) = U_n(t; q_1^{(k)}, q_2^{(k)}) \to U_n(t, \bar{q}_1, \bar{q}_2) \quad \text{uniformly},$$

$$\Theta_n^{(k)}(t) = U_n(t; q_1^{(k)}, q_2^{(k)}) \to \Theta_n(t, \bar{q}_1, \bar{q}_2) \quad \text{uniformly}.$$

Then

$$H_j(t, \alpha^{(k)} U_n^{(k)}(t), \alpha^{(k)} \Theta_n^k(t)) \to H_j(t, \infty, \infty) = g_j(t), \qquad j = 1, 2,$$

uniformly on all closed intervals not containing the zeros (at most $2n$) of

$$U_n(t; g_1, g_2) \Theta_n(t; g_1, g_2).$$

Thus, this convergence is $L_2(0, 1)$ convergence and

$$\rho_n(q_1^{(k)}, q_2^{(k)}, \alpha^{(k)}) \to \frac{\mu_n}{\lambda} > 1,$$

which contradicts (4.13b).

THEOREM 4.1. *Let inequality* (4.7) *hold. Let* $A_n, B_n$ *be the constants of* (3.9) *describing Assumption* 3. *Let*

$$\alpha_0 = \frac{\lambda}{B_n} \alpha_1, \tag{4.14a}$$

$$\alpha_3 = \frac{\lambda}{A_n} \alpha_2. \tag{4.14b}$$

*Then*

$$\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3. \tag{4.15}$$

*Also, let F be a mapping defined on*

$$S \equiv A \times [\alpha_0, \alpha_3]$$

*by*

$$F(q_1, q_2, \alpha) = \left\{ H_1(t, \alpha U_n, \alpha \Theta_n), H_2(t, \alpha U_n, \alpha \Theta_n), \frac{\alpha}{\rho_n(q_1, q_2, \alpha)} \right\}. \tag{4.16}$$

*Then F has a fixed point* $(\bar{q}_1, \bar{q}_2, \bar{\alpha})$. *Finally let*

$$\bar{\alpha} U_n(t, \bar{q}_1, \bar{q}_2) = u(t),$$

$$\bar{\alpha} \Theta_n(t, \bar{q}_1, \bar{q}_2) = \theta(t).$$

*Then* $(u(t), \theta(t))$ *is a solution of* (1.3), (1.3a), *and each function* $u(t)$ *or* $\theta(t)$ *has exactly* $n$ *interior nodal zeros in* $(0, 1)$.

*Proof.* The inequalities (4.15) follow immediately from the inequality (4.7). By Lemma 4.1, the mapping is continuous. Clearly, $S$ is convex. Moreover, standard estimates, together with the continuity of $H_j(t, u, \theta)$, show that $F$ is compact. Finally we shall show that $F$ maps $S$ into $S$. Clearly,

$$(H_1(t, \alpha U_n, \alpha \Theta_n), H_2(t, \alpha U_n, \alpha \Theta_n)) \in A.$$

Thus we need only show that

$$\alpha_0 \leqq \frac{\alpha}{\rho_n(q_1, q_2, \alpha)} \leqq \alpha_3. \tag{4.17}$$

If $\alpha \in [\alpha_0, \alpha_1]$, then from (4.9) we see that

$$\frac{\alpha}{\rho_n(q_1, q_2, \alpha)} \geqq \alpha_0. \tag{4.18a}$$

From (3.9) we have that $\alpha \in [\alpha_0, \alpha_2]$ implies

$$\frac{\alpha}{\rho_n(q_1, q_2, \alpha)} \leqq \frac{\alpha_2 \lambda}{A_n} = \alpha_3. \tag{4.18b}$$

If $\alpha \in [\alpha_1, \alpha_3]$, then (3.9) implies that

$$\alpha_0 = \frac{\alpha_1 \lambda}{B_n} \leqq \frac{\alpha}{\rho_n(q_1, q_2, \alpha)}. \tag{4.18c}$$

Finally, if $\alpha \in [\alpha_2, \alpha_1]$, then (4.12) implies that

$$\frac{\alpha}{\rho_n(q_1, q_2, \alpha)} \leqq \alpha_3. \tag{4.18d}$$

The inequalities (4.18a), (4.18b), (4.18c), (4.18d) show that $F$ maps $S$ into $S$.

The Schauder fixed-point theorem [3] asserts the existence of a fixed point $(\bar{q}_1(t), \bar{q}_2(t), \bar{\alpha})$. Then

$$\rho_n(\bar{q}_1, \bar{q}_2, \bar{\alpha}) = 1. \tag{4.19}$$

Moreover, using (4.4a) and (4.4b) together with the fact that

$$H_j(t, \bar{\alpha} U_n, \bar{\alpha} \Theta_n) = \bar{q}_j(t), \qquad j = 1, 2,$$

we see that

$$U_n(t) = V_n(t), \qquad \Theta_n(t) = \Psi_n(t)$$

and the functions $u(t), \theta(t)$ satisfy (1.3), (1.3a).

**5. Applications to other problems.** In many cases of interest Theorem 4.1 cannot be applied directly. For example, Assumption 2 does not hold when

$$\lim_{\substack{|u| \to \infty \\ |\theta| \to \infty}} H_1(t, u, \theta) H_2(t, u, \theta) = 0.$$

However, it often happens that one may modify the functions $H_k(t, u, \theta)$ for large $u$, $\theta$ without changing *the set of solutions* and the modified problem satisfies all the hypotheses of Theorem 4.1.

In a preliminary report [13] of this work we discussed a general case. Here we restrict ourselves to Problems A and B of Odeh and Tadjbakhsh [10].

Let

$$\lambda_0 < \lambda. \tag{5.1}$$

In [12] we proved the existence of a maximal solution $(u(t), \theta(t))$ of (1.4) and the boundary conditions (A) or (B) subject to the additional constraint (1.4a); that is,

$$u(t) \leqq 0 \leqq \theta(t) < \pi/2. \tag{5.2}$$

Moreover, if $(v(t), \psi(t))$ is any other solution of (1.4) which satisfies the appropriate boundary conditions and the constraint (1.4a), then

$$\begin{aligned} |v(t)| &\leqq -u(t) = |u(t)|, \\ |\psi(t)| &\leqq \theta(t). \end{aligned} \tag{5.3}$$

Let

$$\pi/2 > \theta_0 \geqq \max \theta(t), \qquad 0 < 1 - m < \tfrac{1}{4},$$

and set

$$\hat{H}_1(\theta) = \begin{cases} \dfrac{\sin \theta}{\theta}, & |\theta| \leqq \theta_0, \\[2ex] \dfrac{\sin \theta_0}{\theta_0}\{m + (1-m)e^{-\beta(|\theta| - \theta_0)}\}, & |\theta| \geqq \theta_0, \end{cases} \tag{5.4a}$$

$$\hat{H}_2(\theta) = \begin{cases} \cos \theta, & |\theta| \leqq \theta_0, \\ \cos \theta_0\{m + (1-m)e^{-\gamma(|\theta| - \theta_0)}\}, & |\theta| \geqq \theta_0, \end{cases} \tag{5.4b}$$

where $\beta$ and $\gamma$ have been chosen so that

$$\hat{H}_k(\theta) \in C^1, \quad -\infty < \theta < \infty, \qquad k = 1, 2; \tag{5.4c}$$

that is,

$$\beta = \frac{1}{1-m} \frac{\sin \theta_0 - \theta_0 \cos \theta_0}{\theta_0 \sin \theta_0} > 0, \tag{5.5a}$$

$$\gamma = \frac{1}{1-m} \frac{\sin \theta_0}{\cos \theta_0} > 0. \tag{5.5b}$$

LEMMA 5.1. *Let $\lambda$ be fixed and satisfy* (5.1). *Consider the system of differential equations*

$$\begin{aligned} u'' &= \lambda \theta \hat{H}_1(\theta), \\ \theta'' &= \lambda u \hat{H}_2(\theta) \end{aligned} \tag{5.6}$$

*subject to the appropriate boundary conditions* (A) *or* (B). *Then every solution of* (5.6) *is also a solution of* (1.4) *which also satisfies the constraint* (1.4a). *Conversely any solution of* (1.4) *and their corresponding boundary conditions which also satisfies the constraint* (1.4a) *is a solution of* (5.6).

*Proof*. Since $(u(t), \theta(t))$ is a maximal solution, we see that every solution of (4.1), (4.1a) is also a solution of (5.6). Similarly, using Theorems 4.3 and 5.3 of [12] we see that (5.6) has a unique positive solution $(\hat{u}(t), \hat{\theta}(t))$ which must also be a maximal solution. But then

$$\hat{u}(t) = u(t), \qquad \hat{\theta}(t) = \theta(t)$$

and all solutions of (5.6) satisfy (1.4), (1.4a).

Suppose

$$\lambda_n < \lambda.$$

Since $\cos\theta_0$ may be taken as small as we please, we may easily satisfy all the conditions and apply Theorem 4.1 to obtain a solution $(u_n(t), \theta_n(t))$ with exactly $n$ interior nodal zeros.

**Appendix.** This appendix is devoted to establishing Assumption 3 in two cases of special interest. The basic tools are Lemmas 3.1, 3.2 and 3.3.

We shall be concerned with three problems. In all three cases we take

$$L_1[\varphi] \equiv L_2[\varphi] \equiv L[\varphi].$$

The difficulties will arise from the boundary condition. Let $(q_1(t), q_2(t)) \in A$ and consider the differential equations

$$L[u] = \lambda\theta q_1, \qquad L[\theta] = \lambda u q_2. \tag{A.1}$$

*Problem* S. $u(0) = u(1) = \theta(0) = \theta(1) = 0$.

*Note*. This is a symmetrizable problem and we may apply Lemma 2.1.

*Problem* N. $u(0) = u(1) = 0, \quad \theta(0) = \theta'(1) = 0$.

*Problem* A. $u'(0) = u(1) = 0, \quad \theta(0) = \theta'(1) = 0$.

*Note*. The boundary conditions in Problem A are the boundary conditions A of Odeh and Tadjbakhsh [10]. The eigenvalues of these problems will be denoted by $\lambda_k(S)$, $\lambda_k(N)$, $\lambda_k(A)$, respectively.

We now turn our attention to a basic boundary value problem.

LEMMA A.1. *Let* $\lambda$ *be a fixed positive constant. Let* $(q_1(t), q_2(t)) \in A$. *There exists a unique pair* $(u(t), \theta(t))$ *which satisfies* (A.1) *and also satisfies the boundary conditions*

$$u(1) = u(0) = 0, \qquad \theta(0) = 0, \quad \theta'(0) = 1. \tag{A.2}$$

*Moreover, if* $t_0 \in (0, 1)$ *and* $u(t_0) = 0$, *then* $u'(t_0) \neq 0$. *Similarly, if* $t_0 \in (0, 1)$ *and* $\theta(t_0) = 0$, *then* $\theta'(t_0) \neq 0$.

*Proof.* Let $u_j(t), j = 0, 1, 2, 3$, be the basic solutions of (A.1) which satisfy the initial conditions

$$\text{(A.3)}\qquad \begin{aligned} &\left(\frac{d}{dt}\right)^k u_j(0) = \delta_{kj}, && k = 0, 1, \quad j = 0, 1, 2, 3, \\ &\left(\frac{d}{dt}\right)^k \left[\frac{1}{\lambda q_1} L[u_j](0)\right] = \delta_{k+2,j}, && k = 0, 1, \quad j = 0, 1, 2, 3. \end{aligned}$$

These functions exist. The existence of $u_0, u_1$ is clear when we view (A.1) as a fourth order equation for $u(t)$. The existence of $u_2, u_3$ is clear when we view (A.1) as a fourth order equation for $\theta(t)$. Moreover, they are linearly independent. A direct computation shows that

$$u(t) = -\left[\frac{u_3(1)}{u_1(1)}\right] u_1(t) + u_3(t)$$

is a solution. And, $\theta(t)$ is obtained from the differential equation. Suppose there were two solutions, say $u(t)$ and $v(t)$. Then $(u(t) - v(t)) = w(t)$ is a solution (of the fourth order equation in $u$) satisfying

$$w(0) = \frac{1}{\lambda q_1(0)} L[w](0) = \left(\frac{1}{\lambda q_1} L[w]\right)'(0) = 0.$$

Then, using Lemma 3.1 we would have

$$w'(0) \neq 0, \qquad w'(0)w(1) > 0.$$

But,

$$w(1) = 0.$$

The concluding remark of the lemma follows from Lemma 3.3.

Let

$$r(t) \equiv \frac{1}{\lambda q_1(t)},$$
$$P(t) \equiv \lambda q_2(t).$$

For the remainder of this section we let $r(t)$ and $P(t)$ be continuous functions of a parameter $\sigma$; that is, the coefficients of (A.1) are

$$\lambda q_1(t, \sigma) = [r(t, \sigma)]^{-1} \quad \text{and} \quad \lambda q_2(t, \sigma) = P(t, \sigma).$$

Let $u(t, \sigma)$, $\theta(t, \sigma)$ denote the solution of (A.1) which also satisfies the boundary conditions (A.2). With this notation we obtain a corollary to the preceding lemma.

COROLLARY A.1. *The functions $u(t, \sigma)$, $u'(t, \sigma)$, $\theta(t, \sigma)$ and $\theta'(t, \sigma)$ are continuous functions of $\sigma$.*

*Proof.* The functions $u_j(t, \sigma)$ satisfying (3.13) (for each $\sigma$) are continuous in $\sigma$. This follows from general theorems for $u_0(t), u_2(t)$. For $u_1(t)$ and $u_3(t)$ the continuity follows from the representations (3.2a), (3.2b). Also, those representations establish the continuity of $u'(t, \sigma)$, $\theta(t, \sigma)$, $\theta'(t, \sigma)$.

Following § 2, let $Z(u, \sigma)$ denote the number of interior zeros of $u(t, \sigma)$ while $Z(\theta, \sigma)$ denotes the number of interior zeros of $\theta(t, \sigma)$. Because $K_1(s, t)$ is an oscillation kernel,

$$Z(u, \sigma) \leqq Z(\theta, \sigma). \tag{A.4}$$

LEMMA A.2. *For every $\sigma_0$ there is an $\varepsilon = \varepsilon(\sigma_0) > 0$ such that*

$$|\sigma - \sigma_0| < \varepsilon \Rightarrow Z(\theta, \sigma) \geqq Z(\theta, \sigma_0). \tag{A.5}$$

*Proof*. Let $M = Z(\theta, \sigma_0)$. Let $\xi_0 = 0$, and for $j = 1, 2, \cdots, M$ let $\xi_j$ denote the ordered interior zeros of $\theta(t, \sigma_0)$, i.e.,

$$0 < \xi_j < \xi_{j+1} < 1, \qquad \theta(\xi_j, \sigma_0) = 0.$$

By Rolle's theorem there is a point $\eta_j$ with

$$\xi_j < \eta_j < \xi_{j+1}, \qquad j = 0, 1, \cdots, M - 1,$$

such that

$$\theta'(\eta_j, \sigma_0) = 0.$$

Let

$$\eta_M = \tfrac{1}{2}(\xi_M + 1).$$

Let

$$\rho = \min |\theta(\eta_j, \sigma_0)| > 0, \qquad j = 0, 1, 2, \cdots, M.$$

There is an $\varepsilon > 0$ such that $|\sigma - \sigma_0| < \varepsilon$ implies

$$|\theta(\eta_j, \sigma) - \theta(\eta_j, \sigma_0)| < \tfrac{1}{2}\rho.$$

Thus, there exist $M + 1$ points at which the continuous function $\theta(t, \sigma)$ alternates in sign. Hence, $\theta(t, \sigma)$ has at least $M$ zeros.

LEMMA A.3. *If $\theta(1, \sigma_0) \neq 0$, there exists an $\varepsilon = \varepsilon(\sigma_0) > 0$ such that $|\sigma - \sigma_0| < \varepsilon$ implies that*

$$Z(\theta, \sigma) = Z(\theta, \sigma_0). \tag{A.6}$$

*Proof*. Suppose not. Then there is a sequence $\sigma_n \to \sigma_0$ such that

$$Z(\theta, \sigma_n) > Z(\theta, \sigma_0).$$

Let $\xi_j(\sigma_n)$ denote the zeros of $\theta(t, \sigma_n)$ as in the above lemma. Consider the following vectors in $\mathfrak{R}^{M+1}$:

$$\xi^{(n)} = (\xi_1(\sigma_n), \xi_2(\sigma_n), \cdots, \xi_{M+1}(\sigma_n)), \qquad n = 1, 2, \cdots.$$

There is a subsequence $\xi^{(n')}$ which converges to a limit vector $\xi = (\bar{\xi}_1, \bar{\xi}_2, \cdots, \bar{\xi}_{M+1})$. We observe that

$$0 \leqq \bar{\xi}_j \leqq \bar{\xi}_{j+1} \leqq 1$$

and

$$\theta(\bar{\xi}_j, \sigma_0) = 0.$$

But $\theta(t, \sigma_0)$ has only $M$ interior zeros. Hence, one of the following cases must occur.

*Case* 1. $\bar{\xi}_1 = 0$. But then the point $\eta_1(\sigma_n)$ at which $\theta'(\eta_1(\sigma_n), \sigma_n) = 0$ must also converge to zero. Hence

$$\theta'(0, \sigma_0) = 0.$$

But, of course, $\theta'(0, \sigma_0) = 1$.

*Case* 2. $\bar{\xi}_{n+1} = 1$. But then

$$\theta(1, \sigma_0) = 0$$

contrary to our assumption.

*Case* 3. There is a $j$ such that

$$0 < \bar{\xi}_j = \bar{\xi}_{j+1} < 1.$$

But then $\eta_j(\sigma_n) \to \bar{\xi}_j$ and

$$\theta'(\bar{\xi}_j, \sigma_0) = \theta(\bar{\xi}_j, \sigma_0) = 0,$$

which is impossible.

DEFINITION. A value $\sigma$ will be called a *k-value* for the Problem A (the Problem N, the Problem S) if

$$1 = \lambda_k(A, \sigma), \quad (1 = \lambda_k(N, \sigma), \quad 1 = \lambda_k(S, \sigma)). \tag{A.7}$$

We shall let $A_k$, $N_k$, $S_k$ denote the set of all $k$-values; that is,

$$\begin{aligned} A_k &\equiv \{\sigma | 1 = \lambda_k(A, \sigma)\}, \\ N_k &\equiv \{\sigma | 1 = \lambda_k(N, \sigma)\}, \\ S_k &\equiv \{\sigma | 1 = \lambda_k(S, \sigma)\}. \end{aligned} \tag{A.8}$$

We observe that $S_k$ contains at most one element.

For the remainder of this section we assume that

$$\frac{\partial r(t, \sigma)}{\partial \sigma} \leqq 0, \qquad \frac{\partial P(t, \sigma)}{\partial \sigma} \geqq 0$$

and $\sigma_1 < \sigma_2$ implies that

$$P(t, \sigma_1) \not\equiv P(t, \sigma_2).$$

LEMMA A.4. *Suppose*

$$1 = \lambda_k(S, \sigma_0), \qquad 1 = \lambda_{k+1}(S, \sigma_2).$$

*Then*

$$\sigma_0 < \sigma_2. \tag{A.9}$$

*The set $N_{k+1}$ is not empty; and, if $\sigma \in N_{k+1}$, then*

$$\sigma_0 < \sigma < \sigma_2. \tag{A.10}$$

*Proof*. The inequality (A.9) follows from Lemma 2.1. Let $(u(t, \sigma_0), \theta(t, \sigma_0))$ be the solution of (A.1) which satisfies the boundary conditions (A.2). Then $(u(t, \sigma_0), \theta(t, \sigma_0))$ is (except for scalar multiples) the $k$th eigenfunction of Problem S. To see this, let $v(t)$ be a $k$th eigenfunction of Problem S. We need only verify that $\psi'(0) = r(0)L_1[v](0) \neq 0$. If $\psi'(0) = 0$, then Lemma 3.1 implies that $v(1) \neq 0$. However, $v(1) = 0$. We observe that because $K_1(s,t)$ and $K_2(s,t)$ are both oscillation kernels, we have

$$Z(u, \sigma_0) = Z(\theta, \sigma_0),$$

$$Z(u, \sigma_2) = Z(\theta, \sigma_2).$$

Let $\sigma$ increase from $\sigma_0$ to $\sigma_2$. Since all zeros of $\theta(t, \sigma)$ are nodal zeros, we see that

$$\operatorname{sgn} \theta'(1, \sigma_0) = (-1)^{k+1},$$

$$\operatorname{sgn} \theta'(1, \sigma_2) = (-1)^{k+2}.$$

Thus, there must be at least one value of $\sigma \in (\sigma_0, \sigma_2)$ such that

$$\theta'(1, \sigma) = 0. \tag{A.11}$$

Now there is an $\varepsilon_0 = \varepsilon(\sigma_0, \sigma_2) > 0$ such that

$$Z(\theta, \sigma_0) \leqq Z(\theta, \sigma), \qquad |\sigma - \sigma_0| < \varepsilon_0 \tag{A.12a}$$

and

$$Z(\theta, \sigma_2) \leqq Z(\theta, \sigma), \qquad |\sigma - \sigma_2| < \varepsilon_0. \tag{A.12b}$$

Moreover, for every point $\hat{\sigma}$ in the closed interval $[\sigma_0 + \varepsilon_0/2, \sigma_2 - \varepsilon_0/2]$ there is an $\varepsilon = \varepsilon(\hat{\sigma})$ such that

$$Z(\theta, \sigma) = Z(\theta, \hat{\sigma}), \qquad |\sigma - \hat{\sigma}| < \varepsilon(\hat{\sigma}).$$

Thus, we may apply the Heine–Borel theorem to conclude that

$$Z(\theta, \sigma) \equiv \text{const.}, \qquad \sigma_0 + \varepsilon_0/2 \leqq \sigma \leqq \sigma_2 - \varepsilon_0/2.$$

Thus, on letting $\varepsilon_0 \to 0$ we see that

$$Z(\theta, \sigma) \equiv \text{const.}, \qquad \sigma_0 < \sigma < \sigma_2.$$

This fact, combined with the inequalities (A.12a), (A.12b) and the fact that

$$Z(\theta, \sigma_2) = Z(\theta, \sigma_0) + 1,$$

implies that

$$Z(\theta, \sigma) = k + 1, \qquad \sigma_0 < \sigma \leqq \sigma_2.$$

Thus

$$N_{k+1} \neq \varnothing.$$

Suppose there are values $\sigma \in N_{k+1}$ which do not lie in the interval $(\sigma_0, \sigma_2)$.

*Case* 1. There is a value $\hat{\sigma} \in N_{k+1}$ and $\hat{\sigma} < \sigma_0$. Let $u(t, \hat{\sigma})$ be the solution of (A.1) which satisfies the boundary condition (A.2). Then as before $u(t, \hat{\sigma})$ must be the $(k + 1)$st eigenfunction of Problem N (except for scalar multiples).

Let $\sigma$ increase from $\hat{\sigma}$ to $\sigma_0$. The argument given above shows that

$$k = Z(\theta, \sigma_0) \geqq Z(\theta, \hat{\sigma}) = k + 1.$$

This is impossible.

*Case* 2. There is a value $\hat{\sigma} \in N_{k+1}$ and $\hat{\sigma} > \sigma_2$. But again, the argument given above shows that

$$k + 1 = Z(\theta, \hat{\sigma}) > Z(\theta, \sigma_2) = k + 1.$$

Thus, the lemma is proved.

COROLLARY A.2. *For any fixed value of* $\sigma$,

$$\lambda_k(S, \sigma) < \lambda_{k+1}(N, \sigma) < \lambda_{k+1}(S, \sigma). \tag{A.13}$$

*Proof.* Let $\sigma$ be fixed, and let $r(t, \sigma, \sigma') \equiv r(t, \sigma')$ while

$$P(t, \sigma, \sigma') = (\sigma')^2 P(t, \sigma).$$

Applying the above ideas to $r(t, \sigma, \sigma')$, $P(t, \sigma, \sigma')$ as functions of $\sigma'$ we obtain (in an obvious notation)

$$\lambda_k(S, \sigma, \sigma_0') = \lambda_{k+1}(S, \sigma, \sigma_2') = \lambda_{k+1}(N, \sigma, \sigma_1')$$

for some $\sigma_1' \in (\sigma_0', \sigma_2')$. But then

$$(\sigma_0')^2 = \lambda_k(S, \sigma) < (\sigma_1')^2 = \lambda_{k+1}(N, \sigma) < (\sigma_2')^2 = \lambda_{k+1}(S, \sigma).$$

We wish to obtain similar results for the $k$ values of Problem A and the eigenvalues related to Problem A. Hence we consider another special problem.

LEMMA A.5. *There is a unique function pair* $(v(t, \sigma), \psi(t, \sigma))$ *which satisfies* (A.1) (*under the identification* $v(t, \sigma) = u$, $\psi(t, \sigma) = \theta$) *and the boundary conditions*

$$v(1) = 0, \qquad v'(1) = 1,$$

$$\psi(0) = \psi'(1) = 0.$$

*Moreover, if* $t_0 \in (0, 1)$ *and* $v(t_0) = 0$, *then* $v'(t_0) \neq 0$. *Similarly, if* $t_0 \in (0, 1)$ *and* $\psi(t_0) = 0$, *then* $v'(t_0) \neq 0$.

*Proof.* Let $v_j(t)$, $j = 0, 1, 2, 3$, be the basic solutions of (A.1) which satisfy the initial (terminal) conditions

$$\begin{aligned} &\left(\frac{d}{dt}\right)^k v_j(1) = \delta_{k,j}, && k = 0, 1, \quad j = 0, 1, 2, 3, \\ &\left(\frac{d}{dt}\right)^k (rL_1[v_j])(1) = \delta_{k+2,j}, && k = 0, 1, \quad j = 0, 1, 2, 3. \end{aligned} \tag{A.14}$$

Then, a computation gives $v(t, \sigma)$ in the form

$$v(t, \sigma) = v_1(t, \sigma) + M v_2(t, \sigma).$$

The rest of the lemma follows exactly as the proof of Lemma A.1.

COROLLARY A.3. *The functions* $v(t, \sigma)$, $v'(t, \sigma)$, $\psi(t, \sigma)$, $\psi'(t, \sigma)$ *are all continuous functions of* $\sigma$.

Let $Z(v, \sigma)$, $Z(\psi, \sigma)$ denote the number of interior zeros of $v(t, \sigma)$ and $\psi(t, \sigma)$ respectively. Then, as before, because we are dealing with oscillation kernels

$$Z(\psi, \sigma) \leqq Z(v, \sigma).$$

LEMMA A.6. *For every* $\sigma_0$ *there is an* $\varepsilon = \varepsilon(\sigma_0) > 0$ *such that* $|\sigma - \sigma_0| < \varepsilon$ *implies that*

$$Z(v, \sigma) \geqq Z(v, \sigma_0).$$

*Proof.* The proof of this lemma is exactly the same as the proof of Lemma A.2.

LEMMA A.7. *For every $\sigma_0$ for which $v(0, \sigma_0) \neq 0$ there is an $\varepsilon = \varepsilon(\sigma_0) > 0$ such that $|\sigma - \sigma_0| < \varepsilon$ implies that*

$$Z(v, \sigma) = Z(v, \sigma_0).$$

*Proof.* The proof of this lemma is exactly the same as the proof of Lemma A.3.

LEMMA A.8. *Let*

$$\sigma_1 \equiv \sup\{\sigma : \sigma \in N_k\},$$

$$\sigma_3 \equiv \inf\{\sigma : \sigma \in N_{k+1}\}.$$

*We assume $-\infty < \sigma_3, \sigma_1 < \infty$. Then*

$$\sigma_1 < \sigma_3,$$

$$A_{k+1} \neq \varnothing,$$

*and, if $\sigma \in A_{k+1}$, then*

$$\sigma_1 < \sigma < \sigma_3.$$

The proof of this lemma is exactly the same as the proof of Lemma A.4.

COROLLARY A.4. *For every fixed value of $\sigma$,*

$$\lambda_k(N, \sigma) < \lambda_{k+1}(A, \sigma) < \lambda_{k+1}(N, \sigma) < \lambda_{k+1}(S, \sigma). \tag{A.15}$$

*Proof.* The proof follows the same argument as the proof of Corollary A.2.

THEOREM A.1. *For the special cases of Problem* A *and Problem* N, *Assumption* 3 *holds.*

*Proof.* Since Assumption 3 holds for Problem S, the upper bound on $\lambda_n$ follows from the inequalities (A.13), (A.15). The lower bound follows from an elementary argument based on the Krein–Rutman theory [8]. See [5] also.

## REFERENCES


[1] N. BAZLEY AND B. ZWAHLEN, *Remarks on the bifurcation of solutions of a nonlinear eigenvalue problem*, Arch. Rational Mech. Anal., 28 (1968), pp. 51–58.

[2] R. COURANT AND D. HILBERT, *Methods of Mathematical Physics*, vol. I, Interscience, New York, 1953.

[3] ———, *Methods of Mathematical Physics*, vol. II, Interscience, New York, 1953.

[4] F. R. GANTMACHER AND M. G. KREIN, *Oscillation matrices and kernels, and small vibrations of mechanical systems*, A.E.C. Transl. 4481, Office of Technical Services, Dept. of Commerce, Washington, D.C., 1961.

[5] S. KARLIN, *Positive operators*, J. Math. Mech., 8 (1959), pp. 907–932.

[6] ———. *Total positivity. II*, to appear.

[7] T. KATO, *Perturbation Theory for Linear Operators*, Springer Verlag, New York, 1966.

[8] M. G. KREIN AND M. A. RUTMAN, *Linear operators leaving invariant a cone in a Banach space*, Uspekhi Mat. Nauk, 3 (1948), no. 1 (23), pp. 3–95; English transl., Amer. Math. Soc. Transl., no. 26, 1950, 128 pp.

[9] W. LEIGHTON AND Z. NEHARI, *On the oscillation of solutions of self-adjoint linear differential equations of the fourth order*, Trans. Amer. Math. Soc., 89 (1958), pp. 325–377.

[10] F. ODEH AND I. TADJBAKHSH, *A nonlinear eigenvalue problem for rotating rods*, Arch. Rational Mech. Anal., 20 (1965), pp. 81–94.

[11] SEYMOUR V. PARTER, *A note on the eigenvalue problem for sublinear Hammerstein operators*, J. Math. Anal. Appl., to appear.

[12] ———, *Nonlinear eigenvalue problems for some fourth order equations. I: Maximal solutions*, this Journal, 1 (1970), pp. 437–457.

[13] ———, *Nonlinear eigenvalue problems for some fourth order equations. II*, Tech. Rep. 77, Computer Sciences Dept., Univ. of Wisconsin, Madison, 1969.

[14] GEORGE PIMBLEY, *A fixed-point method for eigenfunctions of sublinear Hammerstein operators*, Arch. Rational Mech. Anal., 31 (1968), pp. 357–363.

[15] ———, *Eigenfunction branches of nonlinear operators, and their bifurcations*, Lecture Notes in Mathematics No. 104, Springer-Verlag, Berlin, 1969.

[16] J. H. WOLKOWISKY, *A nonlinear Sturm–Liouville problem*, Bull. Amer. Math. Soc., 73 (1967), pp. 634–636.

SIAM J. MATH. ANAL.
Vol. 1, No. 4, November 1970

# ON BOUNDARY VALUE PROBLEMS FOR A SINGULARLY PERTURBED DIFFERENTIAL EQUATION WITH A TURNING POINT*

R. E. O'MALLEY, JR.†

**1. Introduction.** Let us consider boundary value problems for the linear equation

$$\varepsilon y'' + 2xA(x, \varepsilon)y' - A(x, \varepsilon)B(x, \varepsilon)y = 0 \tag{1.1}$$

on the interval $-1 \leqq x \leqq 1$ for $\varepsilon$ a small positive parameter. We suppose that $A(x, \varepsilon)$ is nonzero throughout $[-1, 1]$ for $\varepsilon$ sufficiently small, and that both $A$ and $B$ have asymptotic expansions in this interval as $\varepsilon$ tends to zero. Thus the reduced equation obtained by setting $\varepsilon = 0$ in (1.1) is singular at $x = 0$. Following Wasow [10] and others, we call $x = 0$ a *turning point* for (1.1). Our object is to determine under what conditions the solution $y(x, \varepsilon)$ of the given boundary value problem will converge as $\varepsilon \to 0$ to the solution of a reduced boundary value problem, i.e., to a solution of the reduced equation which satisfies some boundary condition. Where such convergence occurs and what boundary condition is appropriate for the reduced problem are of primary interest.

Wasow's fundamental work on asymptotic solutions of boundary value problems for linear ordinary differential equations without turning points is well known (see Wasow [9]). Specializing his results, we obtain the limiting behavior of the solution of the boundary value problem

$$\begin{aligned} &\varepsilon y'' + 2A(x, \varepsilon)y' + A(x, \varepsilon)B(x, \varepsilon)y = 0, \\ &y(-1) \text{ and } y(1) \text{ prescribed}, \qquad -1 \leqq x \leqq 1. \end{aligned} \tag{1.2}$$

Thus if $A(x, \varepsilon)$ is negative throughout $[-1, 1]$, $y$ converges to the solution of the reduced problem

$$2z' + B(x, \varepsilon)z = 0, \qquad z(-1) = y(-1)$$

as $\varepsilon \to 0$ everywhere away from the endpoint $x = 1$. Similarly, if $A$ is positive, $y$ converges to the solution of

$$2z' + B(x, \varepsilon)z = 0, \qquad z(1) = y(1)$$

as $\varepsilon \to 0$ except at $x = -1$. In each case, boundary layer behavior (nonuniform convergence) generally occurs at the excepted endpoint.

In his unpublished thesis [8], Wasow also considered a boundary value problem for an equation with a turning point. Specifically, he discussed the nonhomogeneous form of (1.1) for $A(x, \varepsilon) \equiv \alpha < 0$ and $B(x, \varepsilon) \equiv 0$. The equation was directly integrated and the integrals involved were expanded asymptotically

* Received by the editors June 19, 1969, and in revised form February 23, 1970.

† Courant Institute of Mathematical Sciences, New York University, New York, New York 10012. This research was supported by the National Science Foundation under Grant NSF GP 11458.

to give the limiting behavior of the solution as $\varepsilon \to 0$. Using a two-variable approach, Cochran [2] also solved a special problem, namely, (1.1) when $2A(x, \varepsilon) = A(x, \varepsilon)B(x, \varepsilon) = -\frac{1}{4}$. Otherwise, no asymptotic analysis of boundary value problems for (1.1) seems to have been previously reported. Asymptotic solutions of such equations have, however, been studied (see, e.g., Sibuya [7]). Finally, for certain special nonlinear problems of physical significance, some results are also known (cf. Dorr [3]).

We shall first consider boundary value problems for (1.1) where $A$ and $B$ are independent of $x$ and $\varepsilon$. The general solution of (1.1) can then be obtained in terms of appropriate Weber (or parabolic cylinder) functions (cf. Whittaker and Watson [11]) and boundary value problems can then be solved asymptotically by using the known asymptotic expansions of these special functions. We shall show that the general problem can be uniformly reduced to the case where $A$ and $B$ are constants. This follows from Lee's recent work [4] on systems which can be uniformly reduced to Weber's equation. Thus the problem can be analyzed in general. Principal results are contained in Theorems 1, 2 and 3.

**2. A special problem of fundamental significance.** Let us analyze the boundary value problem

$$\varepsilon y'' + 2\alpha x y' - \alpha\beta y = 0, \qquad y(\pm 1) \text{ prescribed}, \tag{2.1}$$

on the interval $-1 \leqq x \leqq 1$, where $\alpha$ and $\beta$ are constants and $\alpha > 0$. By analogy to Wasow's results for (1.2), we might expect the corresponding reduced boundary value problem to be

$$\begin{aligned} &2xz_1' - \beta z_1 = 0, \quad z_1(-1) = y(-1) \quad \text{on } [-1, 0), \\ &2xz_2' - \beta z_2 = 0, \quad z_2(1) = y(1) \qquad \text{on } (0, 1]. \end{aligned} \tag{2.2}$$

Behavior at the turning point $x = 0$ will most likely vary with the sign of $\beta$. If $\beta$ is positive, we expect $y(0, \varepsilon)$ to converge to $z_1(0) = z_2(0) = 0$. If $\beta = 0$, $z_1(x) \equiv y(-1)$ and $z_2(x) \equiv y(1)$, and we might expect $y(0, \varepsilon)$ to converge to their average value. If $\beta$ is negative, both $z_1(0)$ and $z_2(0)$ are unbounded and we expect $y(x, \varepsilon)$ to have complicated limiting behavior.

To solve (2.1), we set

$$y = z\, e^{-\alpha x^2/(2\varepsilon)},$$

where $z$ satisfies

$$\varepsilon^2 \frac{d^2 z}{dx^2} = (x^2\alpha^2 + \varepsilon\alpha(1 + \beta))z. \tag{2.3}$$

This transformation puts (2.1) into the form (2.3) usually studied and shows that $x = 0$ is a second order turning point (cf. Sibuya [7] and McKelvey [5]). Since the parabolic cylinder functions $D_n(t)$ and $D_{n-1}(it)$ are linearly independent solutions of

$$\frac{d^2 w}{dt^2} + \left(n + \frac{1}{2} - \frac{t^2}{4}\right) w = 0,$$

the general solution of the differential equation (2.1) is

$$y(x) = e^{-\alpha x^2/(2\varepsilon)}\left[c_1 D_{-1-\beta/2}\left(\sqrt{\frac{2\alpha}{\varepsilon}}x\right) + c_2 D_{\beta/2}\left(i\sqrt{\frac{2\alpha}{\varepsilon}}x\right)\right] \tag{2.4}$$

for $c_1$ and $c_2$ arbitrary (i.e., independent of $x$). Since $y(1)$ and $y(-1)$ are prescribed for problem (2.1),

$$\begin{aligned} y(1) &= e^{-\alpha/(2\varepsilon)}\left[c_1 D_{-1-\beta/2}\left(\sqrt{\frac{2\alpha}{\varepsilon}}\right) + c_2 D_{\beta/2}\left(i\sqrt{\frac{2\alpha}{\varepsilon}}\right)\right], \\ y(-1) &= e^{-\alpha/(2\varepsilon)}\left[c_1 D_{-1-\beta/2}\left(-\sqrt{\frac{2\alpha}{\varepsilon}}\right) + c_2 D_{\beta/2}\left(-i\sqrt{\frac{2\alpha}{\varepsilon}}\right)\right] \end{aligned} \tag{2.5}$$

provide two linear equations for the unknowns $c_1$ and $c_2$.

We note that the parabolic cylinder function $D_n(z)$ is an entire function having the asymptotic approximations

$$D_n(z) = \begin{cases} e^{-z^2/4}z^n(1 + o(1)) \quad \text{as } |z| \to \infty \quad \text{for } |\arg z| < 3\pi/4, \\ e^{-z^2/4}z^n(1 + o(1)) - \dfrac{\sqrt{2\pi}}{\Gamma(-n)}\dfrac{e^{n\pi i}e^{z^2/4}}{z^{n+1}}(1 + o(1)) \\ \qquad \text{as } |z| \to \infty \quad \text{for } \pi/4 < \arg z < 5\pi/4. \end{cases} \tag{2.6}$$

Thus for $x$ negative $D_n(x)$ is asymptotically exponentially large as $|x| \to \infty$ unless $n$ is a nonnegative integer. In the latter event $D_n(z) = e^{-z^2/4}\,\mathrm{He}_n(z)$, where $\mathrm{He}_n(z)$ is the $n$th Hermite polynomial and $D_n(x)$ is asymptotically exponentially small. This difference has substantial influence on the behavior of asymptotic solutions of (2.1).

*Case* (a). *Problem* (2.1) *with* $\beta \neq -2n$, $n = 1, 2, \cdots$. Setting

$$y(x) = m_1\frac{e^{-\alpha x^2/(2\varepsilon)}D_{-1-\beta/2}\left(\sqrt{\frac{2\alpha}{\varepsilon}}x\right)}{e^{-\alpha/(2\varepsilon)}D_{-1-\beta/2}\left(-\sqrt{\frac{2\alpha}{\varepsilon}}\right)} + m_2\frac{e^{-\alpha x^2/(2\varepsilon)}D_{\beta/2}\left(i\sqrt{\frac{2\alpha}{\varepsilon}}x\right)}{e^{-\alpha/(2\varepsilon)}D_{\beta/2}\left(-i\sqrt{\frac{2\alpha}{\varepsilon}}\right)}, \tag{2.7}$$

we use the boundary conditions (2.5) and the asymptotic expansions (2.6) and find that

$$m_1(\varepsilon) \to y(-1) - (-1)^{\beta/2}y(1), \qquad m_2(\varepsilon) \to (-1)^{\beta/2}y(1) \quad \text{as } \varepsilon \to 0.$$

Thus, using (2.6) again, we let $\varepsilon \to 0$ and obtain:

(i) for $x > 0$,

$$y(x) = m_1 O(e^{-\alpha x^2(1-\delta)/\varepsilon}) + m_2(-x)^{\beta/2}(1 + o(1))$$

for any small $\delta > 0$, so,

$$y(x) \to z_2(x) = y(1)x^{\beta/2}; \tag{2.8}$$

(ii) for $x < 0$,

$$y(x) = (-x)^{\beta/2}(m_1 + m_2)(1 + o(1)) \sim z_1(x) = y(-1)(-x)^{\beta/2}; \tag{2.9}$$

(iii) for $x = 0$,

$$\begin{aligned} y(0) = \left(-\sqrt{\frac{2\alpha}{\varepsilon}}\right)^{-\beta/2} \Bigg[ & \frac{m_1\Gamma(1 + \beta/2)}{2^{1+\beta/4}\Gamma(1 + \beta/4)}(1 + o(1)) \\ & + \frac{m_2\sqrt{\pi}(-2)^{\beta/4}}{\Gamma(1/2 - \beta/4)}(1 + o(1))\Bigg] = O(\varepsilon^{\beta/4}). \end{aligned} \tag{2.10}$$

In obtaining (2.10) use has been made of the relation $D_n(0) = \sqrt{\pi}2^{n/2}/\Gamma((1 - n)/2)$.

Thus if $\beta$ is not a negative even integer, $y$ converges to $z_1(x)$ on $[-1, 0)$ and to $z_2(x)$ on $(0, 1]$. If $\beta > 0$, $y$ actually converges uniformly on $[-1, 1]$ since $y(0) \to 0 = z_1(0) = z_2(0)$. If $\beta = 0$ and $y(1) \neq y(-1)$, convergence is nonuniform at $x = 0$ since

$$y(0) \to \frac{y(1) + y(-1)}{2} = \frac{z_1(0) + z_2(0)}{2}.$$

For $\beta < 0$, but not an even integer, both $z_1(0)$ and $z_2(0)$ are unbounded and $y(0)$ becomes unbounded in proportion to $\varepsilon^{\beta/4}$.

*Case* (b). *Problem* (2.1) *with* $\beta = -2n$, $n = 1, 2, \cdots$. Here it is convenient to write

$$y(x) = \frac{n_1 e^{-ax^2/\varepsilon}\,\mathrm{He}_{-1-\beta/2}(\sqrt{2\alpha/\varepsilon}\,x)}{e^{-\alpha/\varepsilon}\,\mathrm{He}_{-1-\beta/2}(\sqrt{2\alpha/\varepsilon})} + \frac{n_2 e^{-ax^2/(2\varepsilon)}D_{\beta/2}(i\sqrt{2\alpha/\varepsilon}\,x)}{e^{-\alpha/(2\varepsilon)}D_{\beta/2}(i\sqrt{2\alpha/\varepsilon})}, \tag{2.11}$$

where $\mathrm{He}_n(z) = z^n(1 + o(1))$ as $z \to \infty$. Thus, we find that

$$n_{1,2} \to \tfrac{1}{2}[y(1) \mp (-1)^{\beta/2}y(-1)].$$

Thus for $x \neq 0$,

$$y(x) = n_1\frac{e^{\alpha(1-x^2)/\varepsilon}}{x^{1+\beta/2}}(1 + o(1)) + n_2 x^{\beta/2}(1 + o(1)): \tag{2.12}$$

that is, away from $x = \pm 1$ the solution becomes exponentially large as $\varepsilon \to 0$ unless $n_1 = 0$. In this event $y(1)$ and $y(-1)$ cannot be prescribed independently. This peculiar behavior is illustrated by the special example

$$\varepsilon y'' + 2xy' + 2y = 0, \qquad y(\pm 1) \text{ prescribed},$$

which can be directly integrated.

Changing the sign of the coefficient of $y'$, we consider instead the problem

$$\varepsilon y'' - 2\gamma xy' + \gamma\beta y = 0, \quad -1 \leqq x \leqq 1, \quad y(\pm 1)\text{ prescribed}, \tag{2.13}$$

where $\gamma > 0$. By analogy with Wasow's results, we expect boundary layer behavior to occur near $x = \pm 1$ with convergence to a solution of the reduced equation taking place in the interior of the interval $(-1, 1)$. The general solution of the differential equation is of the form

$$y(x) = c_1 e^{\gamma x^2/(2\varepsilon)} D_{-1-\beta/2}(i\sqrt{2\gamma/\varepsilon}\, x) + c_2 e^{\gamma x^2/(2\varepsilon)} D_{\beta/2}(-\sqrt{2\gamma/\varepsilon}\, x), \tag{2.14}$$

and the constants $c_1$ and $c_2$ will again be determined by the prescribed boundary conditions. Two cases arise.

*Case* (c). *Problem* (2.13) *with* $\beta \neq 2m$, $m = 0, 1, 2, \cdots$. Proceeding as before, we see that $y$ has the form

$$y(x) = k_1 \frac{e^{\gamma x^2/(2\varepsilon)} D_{-1-\beta/2}(i\sqrt{2\gamma/\varepsilon}\, x)}{e^{\gamma/(2\varepsilon)} D_{-1-\beta/2}(i\sqrt{2\gamma/\varepsilon})} + k_2 \frac{e^{\gamma x^2/(2\varepsilon)} D_{\beta/2}(-\sqrt{2\gamma/\varepsilon}\, x)}{e^{\gamma/(2\varepsilon)} D_{\beta/2}(-\sqrt{2\gamma/\varepsilon})}, \tag{2.15}$$

where

$$k_1 \to -(-1)^{\beta/2} y(-1) \quad \text{and} \quad k_2 \to y(1) + (-1)^{\beta/2} y(-1).$$

Thus

$$y(x) = \frac{e^{-\gamma(1-x^2)/\varepsilon}}{x^{1+\beta/2}}(y(1) + o(1)) \quad \text{for } x > 0, \tag{2.16}$$

$$y(x) = \frac{y(-1)e^{-\gamma(1-x^2)/\varepsilon}(1 + o(1))}{(-x)^{1+\beta/2}} + O(e^{-\gamma(1-\delta)/\varepsilon}) \quad \text{for } x < 0, \tag{2.17}$$

and

$$y(0) = O(e^{-\gamma(1-\delta)/\varepsilon}) \quad \text{for any small } \delta > 0. \tag{2.18}$$

Hence, if $\beta$ is not zero or a positive even integer, $y(x) \to 0$ within $(-1, 1)$ as $\varepsilon \to 0$. Note that $z \equiv 0$ is the trivial solution of the reduced equation.

*Case* (d). *Problem* (2.13) *with* $\beta = 2m$, $m = 0, 1, 2, \cdots$. Here

$$y(x) = l_1 \frac{e^{\gamma x^2/(2\varepsilon)} D_{-1-\beta/2}(i\sqrt{2\gamma/\varepsilon}\, x)}{e^{\gamma/(2\varepsilon)} D_{-1-\beta/2}(i\sqrt{2\gamma/\varepsilon})} + l_2 \frac{\mathrm{He}_{\beta/2}(-\sqrt{2\gamma/\varepsilon}\, x)}{\mathrm{He}_{\beta/2}(-\sqrt{2\gamma/\varepsilon})}, \tag{2.19}$$

where

$$l_{1,2} \to \tfrac{1}{2}\{y(1) \mp (-1)^{\beta/2} y(-1)\}.$$

Thus for $x \neq 0$,

$$\begin{aligned} y(x) &= \frac{1}{2}(y(1) - (-1)^{\beta/2} y(-1)) \frac{e^{-\gamma(1-x^2)/\varepsilon}}{x^{1+\beta/2}}(1 + o(1)) \\ &\quad + \frac{1}{2}(y(1) + (-1)^{\beta/2} y(-1)) x^{\beta/2}(1 + o(1)) \end{aligned} \tag{2.20}$$

and

$$y(0) = O(\varepsilon^{\beta/4}); \tag{2.21}$$

so, away from the boundaries $x = \pm 1$, $y$ decays exponentially to the limiting solution

$$z(x) = l_2 x^{\beta/2}$$

which satisfies the reduced equation. A simple example of such convergence is furnished by the example

$$\varepsilon y'' - 2\alpha x y' = 0, \qquad y(\pm 1) \text{ prescribed}.$$

Case (d) is of independent interest, because the limiting solution cannot be easily obtained by using the more intuitive boundary layer methods. In fact, the result stated here contradicts the assertion in Pearson [6, Appendix I] that the solutions of such problems are very nearly zero except near each endpoint where there is a boundary layer.

It is interesting to generalize slightly and observe that the limiting solution of the boundary value problem

$$\begin{aligned} &\varepsilon y'' - 2\gamma x y' + \gamma\beta y = 0, \qquad -1 \leqq x \leqq a^2, \\ &y(-1) \text{ and } y(a^2) \text{ prescribed}, \\ &\beta = 2m, \qquad m = 0, 1, 2, \cdots, \end{aligned}$$

is

$$y(a^2)(x/a^2)^{\beta/2}$$

away from $x = -1$ if $0 < a^2 < 1$, while if $a^2 > 1$ the limiting solution is

$$y(-1)(-x)^{\beta/2}$$

away from $x = a^2$. Further discussion of related problems may be found in Ackerberg and O'Malley [1].

Before considering variable coefficient problems, we summarize the results obtained thus far.

The asymptotic solution of the boundary value problem

$$\begin{aligned} &\varepsilon y'' + 2\alpha x y' - \alpha\beta y = 0, \qquad -1 \leqq x \leqq 1, \\ &y(1), y(-1) \text{ prescribed}, \end{aligned}$$

is given in Table 1.

TABLE 1

| Case | Limiting solution as $\varepsilon \to 0$ |
|---|---|
| (a) $\alpha > 0$, $\beta \neq -2n$, $n = 1, 2, \cdots$. | $y(-1)(-x)^{\beta/2}$, $-1 \leqq x < 0$,<br>$O(\varepsilon^{\beta/4})$, $x = 0$,<br>$y(1)x^{\beta/2}$, $0 < x \leqq 1$. |
| (b) $\alpha > 0$, $\beta = -2n$, $n = 1, 2, \cdots$. | Solutions become exponentially large for $-1 < x < 1$. |
| (c) $\alpha < 0$, $\beta \neq 2m$, $m = 0, 1, 2, \cdots$. | 0, $-1 < x < 1$. |
| (d) $\alpha < 0$, $\beta = 2m$, $m = 0, 1, 2, \cdots$. | $\frac{1}{2}(y(1) + (-1)^{\beta/2}y(-1))x^{\beta/2}$, $-1 < x < 1$. |

**3. The uniform reduction theorem.** Instead of examining the equation

$$\varepsilon y'' + 2xA(x,\varepsilon)y' - A(x,\varepsilon)B(x,\varepsilon)y = 0 \tag{1.1}$$

for $x \in [-1, 1]$, it is convenient to introduce the new variable

$$\eta = \eta(x) = \left\{\frac{2}{\alpha}\int_0^x sA(s,0)ds\right\}^{1/2}, \tag{3.1}$$

where $\alpha = A(0,0)$. We note that $\eta'(x) \neq 0$ for $x \neq 0$ and that $\eta(x)$ is a monotonically increasing function with $\eta(0) = 0$ (since $A(x,0) \neq 0$). In terms of $\eta$, $y$ satisfies

$$\varepsilon y_{\eta\eta} + \left[2\alpha\eta\frac{A(x,\varepsilon)}{A(x,0)} - \varepsilon\frac{\eta_{xx}}{\eta_x^2}\right]y_\eta - \left[\frac{\alpha^2\eta^2A(x,\varepsilon)B(x,\varepsilon)}{x^2A^2(x,0)}\right]y = 0, \tag{3.2}$$

where $\eta_- \equiv \eta(-1) \leqq \eta \leqq \eta(1) \equiv \eta_+$. Then we have the following theorem.

THEOREM 1. *Let $y$ be a solution of the equation*

$$\varepsilon y'' + 2xA(x,\varepsilon)y' - A(x,\varepsilon)B(x,\varepsilon)y = 0, \tag{1.1}$$

*where $A(x, \varepsilon)$ and $B(x, \varepsilon)$ are holomorphic in $x$ and $\varepsilon$ and possess asymptotic expansions as $\varepsilon \to 0$ for $x$ in some complex neighborhood of the interval $[-1, 1]$ and for $\varepsilon$ in some sector $S$: $0 < |\varepsilon| \leqq \varepsilon_0$, $|\arg \varepsilon| \leqq \theta_0$, $\theta_0 > 0$. Suppose also that $A$ and $B$ are real when $x \in [-1, 1]$ and $\varepsilon$ is positive, and that $A(x, 0)$ is nonzero on the interval $[-1, 1]$. Further, let $w(\eta)$ satisfy the equation*

$$\varepsilon w_{\eta\eta} + 2\alpha\eta w_\eta - (\alpha\beta + \varepsilon\sigma(\varepsilon))w = 0, \tag{3.3}$$

*where $\alpha = A(0,0)$, $\beta = B(0,0)$ and $\eta_- \leqq \eta \leqq \eta_+$.*

*Then functions $M$, $N$ and $\sigma$ exist such that*

$$y = M(\eta,\varepsilon)w + \varepsilon N(\eta,\varepsilon)w_\eta, \tag{3.4}$$

*where $M$, $N$ and $\sigma$ are holomorphic functions having asymptotic power series expansions whose terms can be found recursively. Moreover, $M(0, 0) = 1$.*

*Note* 1. The coefficient of $w_\eta$ in (3.3) equals that of $y_\eta$ in (3.2) when $\varepsilon = 0$. Similarly, the coefficient of $w$ in (3.3) agrees with that of $y$ in (3.2) at the turning point $\eta = 0$ when $\varepsilon = 0$. For $\varepsilon \neq 0$, a nontrivial correction $\sigma(\varepsilon)$ is, in general, necessary (cf. analogous results in McKelvey [5]). If $A$ and $B$ are constants, then $M \equiv 1$, $N \equiv 0$ and $\sigma \equiv 0$ provides the trivial transformation.

*Note* 2. By the results of § 2, the general solution of (3.3) has the form

$$w(\eta) = e^{-\alpha\eta^2/(2\varepsilon)}[c_1D_{-1-\tilde\beta/2}(\sqrt{2\alpha/\varepsilon}\,\eta) + c_2D_{\tilde\beta/2}(i\sqrt{2\alpha/\varepsilon}\,\eta)], \tag{3.5a}$$

where $\tilde\beta = \beta + \varepsilon\sigma(\varepsilon)/\alpha$ and $c_1$ and $c_2$ are arbitrary functions of $\varepsilon$. Similarly,

$$w_\eta(\eta) = \sqrt{\frac{2\alpha}{\varepsilon}}e^{-\alpha\eta^2/(2\varepsilon)}\left[-c_1D_{-\tilde\beta/2}\left(\sqrt{\frac{2\alpha}{\varepsilon}}\eta\right) + \frac{ic_2\tilde\beta}{2}D_{-1-\tilde\beta/2}\left(i\sqrt{\frac{2\alpha}{\varepsilon}}\eta\right)\right] \tag{3.5b}$$

(cf. Whittaker and Watson [11]). When $w$ and $w_\eta$ are known, the general solution $y$ of (3.1) is determined by (3.4).

*Proof.* For convenience, we rewrite (3.2) as

$$\varepsilon y_{\eta\eta} + (2\alpha\eta + \varepsilon\zeta(\eta,\varepsilon))y_\eta - (\alpha\beta + 2\alpha\eta\theta(\eta) + \varepsilon\delta(\eta,\varepsilon))y = 0, \tag{3.6}$$

where

$$\theta(\eta) = \frac{1}{2\eta}\left(\frac{\alpha\eta^2 B(x,0)}{A(x,0)x^2} - \beta\right) \tag{3.7}$$

and $\delta(\eta, \varepsilon)$ and $\zeta(\eta, \varepsilon)$ have asymptotic power series expansions such that

$$\zeta(\eta, 0) = \frac{2\alpha\eta}{A(x,0)} A_\varepsilon(x,0) + \frac{\eta_{xx}}{\eta_x^2}. \tag{3.8}$$

Substituting (3.4) into (3.6), we obtain an equation of the form

$$Cw + \varepsilon D w_\eta = 0.$$

By setting $C$ and $D$ separately equal to zero, a system of two differential equations for $M$ and $N$ is obtained. Thus, we have

$$2\alpha\eta(M_\eta - \theta M) + \varepsilon[(\sigma - \delta)M + (\alpha\beta + \varepsilon\sigma)(2N_\eta + \delta N) + \zeta M_\eta + M_{\eta\eta}] = 0 \tag{3.9}$$

and

$$\begin{aligned} &-2\alpha\eta(N_\eta + \theta N) - 2\alpha N(1 + \eta\delta) + \zeta M + 2M_\eta \\ &\quad + \varepsilon[(\sigma - \delta)N + \zeta N_\eta + N_{\eta\eta}] = 0. \end{aligned} \tag{3.10}$$

In order to solve formally these equations asymptotically, we substitute

$$M(\eta, \varepsilon) \sim \sum_{k \geqq 0} M_k \varepsilon^k, \quad N(\eta, \varepsilon) \sim \sum_{k \geqq 0} N_k \varepsilon^k, \quad \sigma(\varepsilon) \sim \sum_{k \geqq 0} \sigma_k \varepsilon^k \tag{3.11}$$

and successively equate coefficients of like powers of $\varepsilon$ in the resulting equations.

When $\varepsilon = 0$, we obtain

$$\begin{aligned} &M_{0\eta} - \theta M_0 = 0, \\ &2\alpha(\eta N_{0\eta} + N_0 + \eta N_0(\theta + \zeta(\eta, 0))) = \zeta(\eta, 0)M_0 + 2M_{0\eta}. \end{aligned} \tag{3.12}$$

Since we want $M_0(0) = 1$ and $N_0(\eta)$ to be holomorphic, we obtain

$$\begin{aligned} M_0(\eta) &= \exp\left(\int_0^\eta \theta(t)\,dt\right), \\ 2\alpha\eta N_0(\eta) &= \exp\left(\int_0^\eta \theta(t)\,dt\right) - \exp\left(-\int_0^\eta (\theta(t) + \zeta(t, 0))\,dt\right). \end{aligned} \tag{3.13}$$

Note that $M_0(\eta) \equiv 1$ and $N_0(\eta) \equiv 0$ if $A(x, 0)$ and $B(x, 0)$ are constants.

In general, we find $M_l$, $N_l$ and $\sigma_{l-1}$ successively for $l = 1, 2, 3, \cdots$. By induction, suppose that $M_j$, $N_j$ and $\sigma_{j-1}$ are known for all $j \leqq l$. Equating coefficients of $\varepsilon^l$ in (3.9) and (3.10) to zero, we have

$$2\alpha\eta(M_{l\eta} - \theta(\eta)M_l) = -\sigma_{l-1}M_0 + K_{l-1}(\eta) \tag{3.14}$$

and

$$2\alpha[\eta N_{l\eta} + N_l + \eta N_l(\theta + \zeta(\eta, 0))] = \zeta(\eta, 0)M_l + 2M_{l\eta} + J_{l-1}(\eta), \tag{3.15}$$

where $K_{l-1}$ and $J_{l-1}$ are known. We then select

$$\sigma_{l-1} = K_{l-1}(0) \tag{3.16}$$

so that

$$-\sigma_{l-1}M_0(\eta) + K_{l-1}(\eta) \equiv 2\alpha\eta L_{l-1}(\eta). \tag{3.17}$$

Then we obtain the holomorphic solutions

$$M_l(\eta) = M_l(0) \exp\left(\int_0^\eta \theta(t)\,dt\right) + \int_0^\eta L_{l-1}(s) \exp\left(\int_s^\eta \theta(t)\,dt\right) ds \tag{3.18}$$

and

$$\begin{aligned} 2\alpha\eta N_l(\eta) = \int_0^\eta \exp\left(-\int_t^\eta (\theta(s) + \zeta(s,0))\,ds\right)&[(\zeta(t,0) + 2\theta(t))M_l(t) \\ &+ 2L_{l-1}(t) + J_{l-1}(t)]\,dt. \end{aligned} \tag{3.19}$$

With the stated hypotheses on $A$ and $B$, we can show that appropriate functions $M(\eta, \varepsilon)$, $N(\eta, \varepsilon)$ and $\sigma(\varepsilon)$ exist which have the formally determined expansions. We refer to Sibuya [5] where expansions analogous to those for $M$, $N$ and $\sigma$ are shown to be asymptotically correct in sectors with central angle less than $\pi$. The difficult problem of establishing the uniform validity (in $\eta$) of these expansions was achieved by Lee [4]. As he observes, such results are known only for equations reducible to Airy's equation or Weber's equation. For the special case considered here, the expansions obtained above were generated by a more direct procedure than that of Sibuya.

**4. A more general boundary value problem.**

THEOREM 2. *Let $y(x)$ satisfy the boundary value problem*

$$\varepsilon y'' + 2xA(x,\varepsilon)y' - A(x,\varepsilon)B(x,\varepsilon)y = 0, \qquad y(\pm 1) \text{ prescribed}, \tag{4.1}$$

*on the interval $-1 \leqq x \leqq 1$, where $A(x,0) > 0$. Suppose that the hypotheses of Theorem 1 hold and that $B(0,0) \neq -2n$, $n = 1, 2, \cdots$. Then*

$$y(x) = \begin{cases} z_1(x) + o(1) & \text{for } -1 \leqq x < 0, \\ O(\varepsilon^{\beta/4}) & \text{for } x = 0, \\ z_2(x) + o(1) & \text{for } 0 < x \leqq 1, \end{cases} \tag{4.2}$$

*where $z_1(x)$ and $z_2(x)$ satisfy the reduced boundary value problems*

$$\begin{aligned} 2xA(x,0)z_1' - A(x,0)B(x,0)z_1 &= 0, \qquad z_1(-1) = y(-1), \\ 2xA(x,0)z_2' - A(x,0)B(x,0)z_2 &= 0, \qquad z_2(1) = y(1). \end{aligned} \tag{4.3}$$

We shall not consider the exceptional case $B(0,0) = -2n$, $n = 1, 2, \cdots$. The reader may recall the discussion above that for the special case where $A$ and $B$ are constants the solution became exponentially large.

*Proof.* Introducing the new independent variable $\eta$ by (3.2), we see that (3.4) to (3.6) imply that $y$ has the form

$$\begin{aligned} y(\eta) = e^{-\alpha\eta^2/(2\varepsilon)}\Bigg[c_1\left[M(\eta,\varepsilon)D_{-1-\tilde\beta/2}\left(\sqrt{\frac{2\alpha}{\varepsilon}}\eta\right) - \varepsilon\sqrt{\frac{2\alpha}{\varepsilon}}N(\eta,\varepsilon)D_{-\tilde\beta/2}\left(\sqrt{\frac{2\alpha}{\varepsilon}}\eta\right)\right] \\ + c_2\Bigg[M(\eta,\varepsilon)D_{\tilde\beta/2}\left(i\sqrt{\frac{2\alpha}{\varepsilon}}\eta\right) \\ + \frac{i\varepsilon\tilde\beta}{2}\sqrt{\frac{2\alpha}{\varepsilon}}N(\eta,\varepsilon)D_{-1-\tilde\beta/2}\left(i\sqrt{\frac{2\alpha}{\varepsilon}}\eta\right)\Bigg]\Bigg]. \end{aligned} \tag{4.4}$$

Since $y(\eta_-)$ and $y(\eta_+)$ are prescribed, two linear equations are available for determining the constants $c_1$ and $c_2$. For convenience, we introduce

$$k_1 = c_1 e^{-\alpha\eta_-^2/(2\varepsilon)} D_{-1-\tilde{\beta}/2}(\sqrt{2\alpha/\varepsilon}\,\eta_-)$$

and

$$k_2 = c_2 e^{\alpha\eta_+^2/(2\varepsilon)} D_{\tilde{\beta}/2}(i\sqrt{2\alpha/\varepsilon}\,\eta_+).$$

Using the expansions (2.6), we have

$$y(\eta) \to \frac{k_1 e^{-\alpha\eta^2/\varepsilon}\Gamma(1+\beta/2)(M_0(\eta) - 2\alpha\eta N_0(\eta))}{\sqrt{2\pi}(\sqrt{2\alpha/\varepsilon}\,\eta)^{1+\beta/2}(-\sqrt{2\alpha/\varepsilon}\,\eta_-)^{\beta/2}} + k_2\left(\frac{\eta}{\eta_+}\right)^{\beta/2} M_0(\eta)$$

for $\eta > 0$ and

$$y(\eta) \to M_0(\eta)\eta^{\beta/2}\left[\frac{k_1}{(\eta_-)^{\beta/2}} + \frac{k_2}{(\eta_+)^{\beta/2}}\right]$$

for $\eta < 0$. Thus we obtain bounded coefficients $k_1$ and $k_2$ directly and

$$(4.5)\qquad \begin{aligned} y(\eta) &\to y(\eta_+)\frac{M_0(\eta)}{M_0(\eta_+)}\left(\frac{\eta}{\eta_+}\right)^{\beta/2} \equiv z_2(\eta) \quad \text{for } \eta > 0, \\ y(\eta) &\to y(\eta_-)\frac{M_0(\eta)}{M_0(\eta_-)}\left(\frac{\eta}{\eta_-}\right)^{\beta/2} \equiv z_1(\eta) \quad \text{for } \eta < 0. \end{aligned}$$

Since $M_0(\eta)$ is given by (3.13), we observe that $z_1(\eta)$ and $z_2(\eta)$ satisfy the reduced boundary value problems

$$(4.6)\qquad \begin{aligned} &2\alpha\eta z_{i\eta} - (\alpha\beta + 2\alpha\eta\theta(\eta))z_i = 0, \qquad i = 1, 2, \\ &z_1(\eta_-) = y(\eta_-), \qquad z_2(\eta_+) = y(\eta_+). \end{aligned}$$

Finally, we have

$$(4.7)\qquad \begin{aligned} y(0) \sim \left(\sqrt{\frac{2\alpha}{\varepsilon}}\right)^{-\beta/2} &\left[\left(\frac{y(\eta_-)}{M_0(\eta_-)} - \frac{y(\eta_+)}{M_0(\eta_+)}\left(\frac{\eta_-}{\eta_+}\right)^{\beta/2}\right)\right. \\ &\left.\cdot\left(\frac{\Gamma(1+\beta/2)D_{-1-\beta/2}(0)}{\sqrt{2\pi}(-\eta_-)^{\beta/2}}\right) + \frac{y(\eta_+)}{M_0(\eta_+)}\frac{D_{\beta/2}(0)}{(i\eta_+)^{\beta/2}}\right]. \end{aligned}$$

Reintroducing the independent variable $x$ we obtain (4.2).

Similarly, we have the following theorem.

THEOREM 3. *Let $y(x)$ satisfy the boundary value problem*

$$(4.8)\qquad \varepsilon y'' - 2xC(x,\varepsilon)y' + C(x,\varepsilon)B(x,\varepsilon)y = 0, \qquad y(\pm 1) \textit{ prescribed},$$

*for $x \in [-1, 1]$ where $C(x, 0) > 0$. Suppose that $C(x, \varepsilon)$ and $B(x, \varepsilon)$ satisfy the hypotheses of Theorem 1 and that $B(0, 0) \neq 2m$, $m = 0, 1, 2, \cdots$. Then*

$$(4.9)\qquad y(x) \to 0 \quad \textit{for } -1 < x < 1 \quad \textit{as } \varepsilon \to 0.$$

*Note* 1. The limiting solution trivially satisfies the reduced differential equation within $(-1, 1)$. Since it fails, in general, to satisfy the boundary conditions, nonuniform convergence of the solution can be expected as $\varepsilon \to 0$ at $x = 1$ and $x = -1$.

*Note* 2. In cases where $B(0,0) = 2m, m = 0, 1, 2, \cdots$, the limiting solution within $(-1, 1)$ may be nontrivial; see Ackerberg and O'Malley [1]. More careful analysis is necessary in the variable coefficient case.

*Proof*. Introducing the new variable

$$\eta = \eta(x) = \left\{ \frac{2}{\gamma} \int_0^x sC(s,0)\,ds \right\}^{1/2}, \tag{4.10}$$

where $\gamma = C(0,0)$, we proceed as before and find that as a function of $\eta$ the general solution of the differential equation is given by

$$y(\eta) = c_1 R(\eta) + c_2 S(\eta), \tag{4.11}$$

where

$$\begin{aligned} R(\eta) &= e^{\gamma\eta^2/(2\varepsilon)}\left[ M(\eta,\varepsilon) D_{-1-\tilde{\beta}/2}\left( i\sqrt{\frac{2\gamma}{\varepsilon}}\eta \right) - i\varepsilon\sqrt{\frac{2\gamma}{\varepsilon}} N(\eta,\varepsilon) D_{\tilde{\beta}/2}\left( i\sqrt{\frac{2\gamma}{\varepsilon}}\eta \right) \right], \\ S(\eta) &= e^{\gamma\eta^2/(2\varepsilon)}\left[ M(\eta,\varepsilon) D_{\tilde{\beta}/2}\left( -\sqrt{\frac{2\gamma}{\varepsilon}}\eta \right) \right. \\ &\qquad \left. - \frac{\varepsilon\tilde{\beta}}{2}\sqrt{\frac{2\gamma}{\varepsilon}} N(\eta,\varepsilon) D_{-1-\tilde{\beta}/2}\left( -\sqrt{\frac{2\gamma}{\varepsilon}}\eta \right) \right]. \end{aligned} \tag{4.12}$$

Here, $\tilde{\beta} = \beta - \varepsilon\sigma(\varepsilon)/\gamma$ and $M(\eta,\varepsilon)$, $N(\eta,\varepsilon)$ and $\sigma(\varepsilon)$ are the functions whose existence is proved by Theorem 1. We note that $d\eta/dx > 0$ for $x \in [-1, 1]$, $x \neq 0$, and define $\eta_- = \eta(-1)$ and $\eta_+ = \eta(1)$. The prescribed values $y(\eta_-)$ and $y(\eta_+)$ are available for determining the constants $c_1$ and $c_2$ in (4.11). We find that the boundary value problem has the unique solution

$$y(\eta) = \frac{y(\eta_-)}{D}(S(\eta_+)R(\eta) - S(\eta)R(\eta_+)) - \frac{y(\eta_+)}{D}(S(\eta_-)R(\eta) - S(\eta)R(\eta_-)), \tag{4.13}$$

where

$$D = S(\eta_+)R(\eta_-) - S(\eta_-)R(\eta_+).$$

By (2.6),

$$R(\eta) \to \frac{e^{\gamma\eta^2/\varepsilon}(M_0(\eta) + 2\gamma\eta N_0(\eta))}{(i\sqrt{2\gamma/\varepsilon}\,\eta)^{1+\beta/2}} \qquad \text{for } \eta \neq 0,$$

$$S(\eta) \to \frac{e^{\gamma\eta^2/\varepsilon}\sqrt{2\pi}(M_0(\eta) + 2\gamma\eta N_0(\eta))}{\Gamma(-\beta/2)(\sqrt{2\gamma/\varepsilon}\,\eta)^{1+\beta/2}} \qquad \text{for } \eta > 0$$

and

$$S(\eta) \to \left( -\sqrt{\frac{2\gamma}{\varepsilon}}\eta \right)^{\beta/2} M_0(\eta) \quad \text{for } \eta < 0.$$

Noting that $M_0(\eta) + 2\gamma\eta N_0(\eta)$ is nonzero throughout $[\eta_-, \eta_+]$ by (3.13), we have

$$y(\eta) = y(-1)e^{\gamma(\eta^2-\eta_-^2)/\varepsilon}O(1) + y(1)e^{-\gamma\eta_+^2/\varepsilon}O(1) \quad \text{for } \eta < 0 \tag{4.14}$$

and

$$\begin{aligned} y(\eta) &= y(-1)[S(\eta_+)R(\eta) - S(\eta)R(\eta_+)]e^{-\gamma(\eta_+^2+\eta_-^2)/\varepsilon}O(1) \\ &\qquad + y(1)e^{\gamma(\eta^2-\eta_+^2)/\varepsilon}O(1) \quad \text{for } \eta > 0. \end{aligned} \tag{4.15}$$

Since $y(0) \to 0$ and

$$[S(\eta_+)R(\eta) - S(\eta)R(\eta_+)]e^{-\gamma(\eta_+^2+\eta_-^2)/\varepsilon} \to 0 \quad \text{for } \eta > 0,$$

$$y(\eta) \to 0 \quad \text{for } \eta_- < \eta < \min(\eta_+, -\eta_-).$$

If $\eta_+ \leqq -\eta_-$, this proves (4.9) and, in addition, indicates the boundary layer behavior which occurs at the endpoints. Otherwise, we note that the differential equation for $y(\eta)$ (cf. (3.2)) has no turning point on the $\eta$-interval $0 < -\eta_-/2 \leqq \eta \leqq \eta_+$. Applying Wasow's theory for (1.2), we see that the solution $y(\eta)$ converges asymptotically to the solution of the reduced boundary value problem

$$2\gamma\eta z_\eta - \gamma^2 \frac{\eta^2 B(x,0)}{x^2 C(x,0)} z = 0, \qquad z(-\eta_-/2) = y(-\eta_-/2)$$

in the interior of this interval as $\varepsilon \to 0$. Since $y(-\eta_-/2) \to 0$, however, $y$ is asymptotically zero throughout the open interval $\eta_- < \eta < \eta_+$.

Instead of referring to (1.2), we could show directly that the coefficient of $y(-1)$ in (4.15) is asymptotically zero for $0 < \eta < \eta_+$. In the case where $C(x, \varepsilon) \equiv \gamma$ and $B(x, \varepsilon) \equiv \beta$, for example, this is equivalent to showing that

$$\exp\left(-\frac{\gamma\eta_+^2}{2\varepsilon} + \frac{\gamma\eta^2}{2\varepsilon}\right)\left[D_{\tilde\beta/2}\left(-\sqrt{\frac{2\gamma}{\varepsilon}}\eta_+\right)D_{\tilde\beta/2}\left(\sqrt{\frac{2\gamma}{\varepsilon}}\eta\right) - D_{\tilde\beta/2}\left(-\sqrt{\frac{2\gamma}{\varepsilon}}\eta\right)D_{\tilde\beta/2}\left(\sqrt{\frac{2\gamma}{\varepsilon}}\eta_+\right)\right] = o(e^{\gamma\eta^2/\varepsilon}).$$

**Acknowledgment.** The author wishes to thank Professors W. A. Harris, Jr. and J. B. Keller for suggestions which led to improvements in this paper.

REFERENCES

[1] R. C. Ackerberg and R. E. O'Malley, Jr., *Boundary layer problems exhibiting resonance*, Studies in Appl. Math., 49 (1970), pp. 277–295.

[2] J. A. Cochran, *Problems in singular perturbation theory*, Doctoral thesis, Stanford University, Stanford, Calif., 1962.

[3] F. W. Dorr, Jr., *The asymptotic behavior and numerical solution of singular perturbation problems with turning points*, Doctoral thesis, University of Wisconsin, Madison, 1969.

[4] R. Y. Lee, *On uniform simplification of linear differential equations in a full neighborhood of a turning point*, J. Math. Anal. Appl., 27 (1969), pp. 501–510.

[5] R. W. McKelvey, *The solution of second order linear ordinary differential equations about a turning point of order two*, Trans. Amer. Math. Soc., 79 (1955), pp. 103–123.

[6] C. E. Pearson, *On a differential equation of boundary layer type*, J. Math. and Phys., 67 (1968), pp. 134–154.

[7] Y. Sibuya, *Asymptotic solutions of a system of linear ordinary differential equations containing a parameter*, Funkcial. Ekvac., 4 (1962), pp. 83–113.

[8] W. Wasow, *On boundary layer problems in the theory of ordinary differential equations*, Doctoral thesis, New York University, New York, 1941.

[9] ———, *On the asymptotic solution of boundary value problems for ordinary differential equations containing a parameter*, J. Math. and Phys., 23 (1944), pp. 173–183.

[10] ———, *Asymptotic Expansions for Ordinary Differential Equations*, Interscience, New York, 1965.

[11] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed. (reprinted), Cambridge University Press, London, 1965.

SIAM J. MATH. ANAL.
Vol. 1, No. 4, November 1970

# A REPRESENTATION OF LINEAR CONTINUOUS OPERATORS ON TESTING FUNCTIONS AND DISTRIBUTIONS*

VACLAV DOLEZAL†

It is well known that, due to the famous Schwarz's kernel theorem, every linear continuous operator from $\mathscr{D}$ into $\mathscr{D}'$ (space of testing functions and its dual, respectively) can be represented by a distributional kernel. In this paper we consider linear continuous operators from $\mathscr{D}$ into $\mathscr{D}$, and operators from $\mathscr{D}'$ into $\mathscr{D}'$. Unlike in the case of Schwarz's kernel theorem, whose proof is based on topological vector space theory, we use entirely elementary methods, i.e., only the concept of convergence in the respective spaces. First, theorems characterizing linear continuous operators from $\mathscr{D}$ into $\mathscr{D}$ and $\mathscr{D}'$ into $\mathscr{D}'$ are proved. Using these results, necessary and sufficient conditions for an operator to be shift-invariant are found. Then, convergence of sequences of operators is studied and it is shown that the spaces of operators are sequentially complete. Finally, certain modifications and extensions of the presented results are discussed.

**1.** Let $\mathscr{D}$ denote the set of all complex-valued testing functions defined on $R^m$; as usual, we write $\varphi_n \to 0$ in $\mathscr{D}$, if $\varphi_n \in \mathscr{D}$ for $n = 1, 2, \cdots$; the support of every $\varphi_n$ is contained in a fixed bounded subset of $R^m$ and, for every multi-index $k$, $D^k\varphi_n \to 0$ uniformly.

Let $\mathscr{D}'$ signify the set of all distributions, i.e., the set of all linear and continuous functionals (in the sense of the above convergence) on $\mathscr{D}$; furthermore, let $\mathscr{D}'_*$ stand for the set of all distributions of finite order.

Suppose we wished to introduce the weak convergence into $\mathscr{D}$; i.e., we would write $\varphi_n \to 0$ weakly, if $\langle f, \varphi_n\rangle \to 0$ for every $f \in \mathscr{D}'$. However, surprisingly enough, we would not obtain anything new; actually, we have the following theorem.

THEOREM 1.1. *Let $\varphi_n \in \mathscr{D}$, $n = 1, 2, \cdots$, be a sequence such that $\langle f, \varphi_n\rangle \to 0$ for every $f \in \mathscr{D}'_*$; then $\varphi_n \to 0$ in $\mathscr{D}$.*

*Proof.* First observe that if the sequence $\varphi_n$ has the property that $\langle f, \varphi_n\rangle \to 0$ for every $f \in \mathscr{D}'_*$, then any subsequence $\varphi_{n_i}$ also has the same property. Furthermore, if $p$ is a fixed multi-index, then the sequence $D^p\varphi_n$ also has this property, because if $f \in \mathscr{D}'_*$, then $(-1)^{\hat{p}}D^pf \in \mathscr{D}'_*$, and consequently, $\langle(-1)^{\hat{p}}D^pf, \varphi_n\rangle = \langle f, D^p\varphi_n\rangle \to 0$. In particular, if $\xi \in R^m$ and $p$ is a fixed multi-index, we obtain $D^p\varphi_n(\xi) \to 0$ by setting $f = \delta_\xi \in \mathscr{D}'_*$.

(i) First, we are going to show that the supports of all $\varphi_n$ are contained in a bounded subset of $R^m$. Suppose that this is not true; i.e., for every $a > 0$ we can find and index $n_a \geqq 1$ and a point $t_a \in R^m$ with $|t_a| \geqq a$ such that $\varphi_{n_a}(t_a) \neq 0$. Because each $\varphi_n$ has a compact support it follows that $n_a \to \infty$ as $a \to \infty$.

Let us now construct a sequence $t_i \in R^m$ and a sequence of indices $n_1 < n_2 < \cdots$ as follows:

---

* Received by the editors June 10, 1969, and in revised form April 20, 1970.

† Department of Applied Analysis, State University of New York at Stony Brook, Stony Brook, New York 11790.

Find $t_1 \in R^m$ with $|t_1| > 1$ and $n_1 \geqq 1$ such that $\lambda_1 = \varphi_{n_1}(t_1) \neq 0$. Next, find $t_2 \in R^m$ with $|t_2| > 2$, $t_2 \notin \operatorname{supp} \varphi_{n_1}$ and $n_2 > n_1$ such that $\lambda_2 = \varphi_{n_2}(t_2) \neq 0$ and

$$|\varphi_k(t_1)| < \tfrac{1}{2}|\lambda_1|2^{-1} \tag{1.1}$$

for all $k \geqq n_2$. Such a pair $t_2, n_2$ actually exists, because due to the above fact there exist arbitrarily large indices $n_a$ and points $t_a$ such that $\varphi_{n_a}(t_a) \neq 0$; moreover, since $\varphi_n(\xi) \to 0$ for any point $\xi \in R^m$, inequality (1.1) can be satisfied.

Generally, if indices $n_1, n_2, \cdots, n_{q-1}$ and points $t_1, t_2, \cdots, t_{q-1}$ are already found, we construct $n_q > n_{q-1}$ and $t_q$ so as to have

$$|t_q| > q, \qquad t_q \notin \operatorname{supp} \varphi_{n_i} \quad \text{for } i = 1, 2, \cdots, q-1, \tag{1.2}$$
$$\lambda_q = \varphi_{n_q}(t_q) \neq 0,$$

and

$$|\varphi_k(t_{q-p})| < \tfrac{1}{2}|\lambda_{q-p}|2^{-p}, \qquad p = 1, 2, \cdots, q-1, \tag{1.3}$$

whenever $k \geqq n_q$.

Since $|t_i| \to \infty$, the functional $f = \sum_{i=1}^{\infty} |\lambda_i|^{-1}\delta_{t_i}$ belongs to $\mathscr{D}'_*$; thus, by assumption,

$$z_j = \langle f, \varphi_j \rangle = \sum_{i=1}^{\infty} |\lambda_i|^{-1}\varphi_j(t_i) \to 0 \tag{1.4}$$

as $j \to \infty$.

However, let $j = n_q$, when $n_q$ is a term of the sequence $n_1 < n_2 < \cdots$ we constructed; then we have, due to (1.2) and (1.3),

$$\begin{aligned} |z_{n_q}| &= \left| \sum_{i=1}^{\infty} |\lambda_i|^{-1}\varphi_{n_q}(t_i) \right| = \left| \sum_{i=1}^{q} |\lambda_i|^{-1}\varphi_{n_q}(t_i) \right| \\ &\geqq |\lambda_q|^{-1}|\varphi_{n_q}(t_q)| - \sum_{i=1}^{q-1} |\lambda_i|^{-1}|\varphi_{n_q}(t_i)| \\ &> 1 - \sum_{i=1}^{q-1} \tfrac{1}{2} \cdot 2^{i-q} > \tfrac{1}{2}, \end{aligned}$$

which contradicts (1.4). Hence, all supports of $\varphi_n$ are contained in a set $I = \{t : |t| \leqq C\}, C > 0$.

(ii) Now, let us show that, for any multi-index $k$, we have $D^k\varphi_n \to 0$ uniformly on $I$. Suppose that this is not true; i.e., there exists a multi-index $p$, number $\varepsilon > 0$, a subsequence $\varphi_{n_i}$ and points $t_i \in I$ such that

$$|D^p\varphi_{n_i}(t_i)| \geqq \varepsilon. \tag{1.5}$$

Because $I$ is compact, we may assume that we have already selected the indices $n_i$ and the corresponding points $t_i$ so that $t_i \to \xi \in I$. Moreover, we can assume that $n_i$ and $t_i$ are chosen so that we have

$$|t_i - \xi| < i^{-3} \quad \text{for } i = 1, 2, \cdots. \tag{1.6}$$

Observe also that the sequence $t_i$ contains infinitely many distinct points, since otherwise (1.5) would contradict the pointwise convergence of $D^p\varphi_{n_i}$. Finally, since $D^p\varphi_{n_i}(\xi) \to 0$, we may assume that

$$(1.7) \qquad |D^p\varphi_{n_i}(\xi)| < \varepsilon/2$$

for all $n_i$, $i = 1, 2, \cdots$.

Next, denote $t_i = (t_i^1, t_i^2, \cdots, t_i^m)$, $\xi = (\xi^1, \xi^2, \cdots, \xi^m)$, and let $D_q$ stand for $\partial/\partial t^q$, $1 \leqq q \leqq m$. Then we have, by the mean value theorem,

$$(1.8) \qquad D^p\varphi_{n_i}(t_i) - D^p\varphi_{n_i}(\xi) = \sum_{q=1}^{m} (D_q D^p \varphi_{n_i}(\bar{t}_i)\cdot(t_i^q - \xi^q),$$

where $\bar{t}_i = \lambda t_i + (1 - \lambda)\xi$ with $\lambda \in (0, 1)$. Thus, by (1.6),

$$(1.9) \qquad |\bar{t}_i - \xi| < i^{-3}.$$

However, (1.8) yields, by (1.5), (1.6) and (1.7),

$$(1.10) \qquad \varepsilon/2 < |D^p\varphi_{n_i}(t_i) - D^p\varphi_{n_i}(\xi)| < \sum_{q=1}^{m} |D_q D^p\varphi_{n_i}(\bar{t}_i)| \cdot i^{-3}.$$

Consequently, (1.10) shows that, for every $i$, at least one partial derivative $D_q$, say $D_{q_i}$, satisfies the inequality

$$(1.11) \qquad |D_{q_i}D^p\varphi_{n_i}(\bar{t}_i)| > \frac{\varepsilon}{2m} i^3.$$

On the other hand, since $q_i$ assumes only values $1, 2, \cdots, m$, we may assume that our subsequence $\varphi_{n_i}$ has already been chosen so that $q_i = q'$ for all $i$, and denote $D_{q'}D^p = D^{p'}$. Due to (1.9) we may finally assume that $\varphi_{n_i}$ and $\bar{t}_i$ are such that $|\bar{t}_i - \xi| > |\bar{t}_{i+1} - \xi|$ for every $i$.

Thus, summarizing our results, we see that our original sequence $\varphi_k$ contains a subsequence $\varphi_{k_n}$ such that, with some fixed multi-index $p'$, the sequence $v_n = D^{p'}\varphi_{k_n}$ has the following properties:

There exists a sequence $\bar{t}_n \in R^m$ with $\bar{t}_n \to \xi$ and

$$(1.12) \qquad |\bar{t}_n - \xi| > |\bar{t}_{n+1} - \xi|$$

for all $n$ such that

$$(1.13) \qquad |v_n(\bar{t}_n)| > \frac{\varepsilon}{2m} n^3.$$

Next, recalling our remark at the beginning of the proof, we are going to construct a subsequence $v_{n_i}$ of $v_n$ as follows:

Put $A = \varepsilon/(4m)$; because $v_n(\xi) \to 0$, we can find $n_1 \geqq 1$ so that $|v_r(\xi)| < A/2$ for all $r \geqq n_1$. Since $v_{n_1}$ is continuous at $\xi$, there exists a $\lambda_1 > 0$ such that $|v_{n_1}(t)| < A$ for every $t \in I_1 = \{t : |t - \xi| < \lambda_1\}$ and $\bar{t}_{n_1} \notin I_1$.

Next, find $n_2 > n_1$ such that $\bar{t}_{n_2} \in I_1$ and $|v_{n_2}(\bar{t}_{n_1})| < A/2$; such an $n_2$ exists due to (1.12) and the fact that $v_p(\bar{t}_{n_1}) \to 0$ as $p \to \infty$. By continuity of $v_{n_2}$ there exists a $\lambda_2 > 0$, $\lambda_2 < \lambda_1$ such that $|v_{n_2}(t)| < A$ for every $t \in I_2 = \{t : |t - \xi| < \lambda_2\}$ and $\bar{t}_{n_2} \notin I_2$.

Generally, if $n_1 < n_2 < \cdots < n_{k-1}$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_{k-1}$ are established, we find $n_k > n_{k-1}$ so that

$$(1.14) \qquad \bar{t}_{n_k} \in I_{k-1} = \{t : |t - \xi| < \lambda_{k-1}\}$$

and

$$|v_{n_k}(\bar{t}_{n_{k-p}})| < \frac{A}{2} \cdot 2^{1-p}, \qquad p = 1, 2, \cdots, k-1;$$

such an $n_k$ exists due to (1.12) and pointwise convergence of $v_p$ at points $\bar{t}_{n_1}, \bar{t}_{n_2}, \cdots, \bar{t}_{n_{k-1}}$. By continuity of $v_{n_k}$ we then establish $\lambda_k > 0$, $\lambda_k < \lambda_{k-1}$ so that $|v_{n_k}(t)| < A$ for $t \in I_k$ and $\bar{t}_{n_k} \notin I_k$, etc.

Now define a functional $f$ on $\mathscr{D}$ by

$$f = \sum_{j=1}^{\infty} n_j^{-2} \delta_{\bar{t}_{n_j}}; \tag{1.15}$$

i.e., for any $\varphi \in \mathscr{D}$,

$$\langle f, \varphi \rangle = \sum_{j=1}^{\infty} n_j^{-2} \varphi(\bar{t}_{n_j}). \tag{1.16}$$

Since $\bar{t}_{n_j} \to \xi$ as $j \to \infty$ and each $\varphi$ is bounded, it follows from (1.16) that the series converges; furthermore, each partial sum of (1.15) is in $\mathscr{D}'$, and consequently (cf. [1, p. 37]), $f \in \mathscr{D}'$. Finally, it is clear that $f \in \mathscr{D}'_*$.

By assumption, $\langle f, v_{n_k} \rangle \to 0$ as $k \to \infty$; i.e.,

$$\mu_k = \sum_{j=1}^{\infty} n_j^{-2} v_{n_k}(\bar{t}_{n_j}) \to 0. \tag{1.17}$$

However,

$$\mu_k = a + n_k^{-2} v_{n_k}(\bar{t}_{n_k}) + b, \tag{1.18}$$

where

$$a = \sum_{j=1}^{k-1} n_j^{-2} v_{n_k}(\bar{t}_{n_j}), \qquad b = \sum_{j=k+1}^{\infty} n_j^{-2} v_{n_k}(\bar{t}_{n_j}).$$

Using the inequalities (1.14) we see that

$$|a| \leqq \sum_{j=1}^{k-1} |v_{n_k}(\bar{t}_{n_j})| < \sum_{j=1}^{k-1} \frac{A}{2} \cdot 2^{1+j-k} < A.$$

On the other hand, because $\bar{t}_{n_j} \in I_k$ for $j \geqq k+1$, we have, by our construction, $|v_{n_k}(\bar{t}_{n_j})| < A$ for $j \geqq k+1$, and consequently,

$$|b| < \sum_{j=k+1}^{\infty} n_j^{-2} A < A.$$

Thus, (1.18) yields finally, by (1.13),

$$|\mu_k| \geqq |n_k^{-2} v_{n_k}(\bar{t}_{n_k})| - |a| - |b| > n_k^{-2} \cdot \frac{\varepsilon}{2m} n_k^3 - 2A = \frac{\varepsilon}{2m}(n_k - 1).$$

This, however, contradicts (1.17) which completes the proof.

Let us now introduce some further concepts.

For every $a \in R^m$ let $f_a \in \mathscr{D}'$; if, for every $a \in R^m$ and $\varphi \in \mathscr{D}$ there exists a finite limit

$$\lim_{\varkappa \to 0} \left\langle \frac{1}{\varkappa}(f_{a+h} - f_a), \varphi \right\rangle, \qquad \varkappa \neq 0, \tag{1.19}$$

where $h = (0, 0, \cdots, 0, \varkappa, 0, \cdots, 0) \in R^m$ ($\varkappa$ stands at the $i$th place), then, by the theorem on completeness of $\mathscr{D}'$ (cf. [1, p. 37]), (1.19) defines a distribution $g_a$ in $\mathscr{D}'$ for every $a \in R^m$, and we write $g_a = \partial f_a / \partial a_i$.

From this definition it follows immediately that the following propositions are true (see [3, p. 148] or [2]).

LEMMA 1.1. *Let* $f_a \in \mathscr{D}'$ *for every* $a \in R^m$, *and let* $\psi_\varphi(a) = \langle f_a, \varphi \rangle$, $\varphi \in \mathscr{D}$; *then* $\partial f_a/\partial a_i$ *exists if and only if* $(\partial/\partial a_i)\psi_\varphi(a)$ *exists at every point* $a \in R^m$ *and* $\varphi \in \mathscr{D}$. *Moreover, then*

$$\left\langle \frac{\partial f_a}{\partial a_i}, \varphi \right\rangle = \frac{\partial}{\partial a_i}\langle f_a, \varphi \rangle.$$

LEMMA 1.2. *Let* $f_a \in \mathscr{D}'$ *for every* $a \in R^m$; *then* $\partial f_a/\partial a_i$ *exists if and only if*

$$(a_n^i - a^i)^{-1}(f_{a_n} - f_a) \to L$$

*in* $\mathscr{D}'$ *for all convergent sequences* $a_n^i \to a^i$, $a_n^i \neq a^i$, *where* $a_n = (a^1, a^2, \cdots, a^{i-1}, a_n^i, a^{i+1}, \cdots, a^m)$, $a = (a^1, a^2, \cdots, a^m)$. *Moreover,* $\partial f_a/\partial a_i = L$.

If $\partial f_a/\partial a_i$ exists and $(\partial/\partial a_j)(\partial f_a/\partial a_i)$ also exists, then the latter derivative will be denoted by $\partial^2 f_a/\partial a_j \partial a_i$, and similarly for derivatives of higher order.

We can easily verify that the following proposition holds.

LEMMA 1.3. *Let* $f_a \in \mathscr{D}'$ *for every* $a \in R^m$, *let* $\psi_\varphi(a) = \langle f_a, \varphi \rangle$ *for every* $\varphi \in \mathscr{D}$ *and let* $k$ *be a multi-index. Then* $D_a^k f_a$ *exists and is independent of the order of differentiation if and only if* $D_a^k \psi_\varphi(a)$ *exists for every* $a \in R^m$ *and* $\varphi \in \mathscr{D}$ *and is independent of the order of differentiation. Then*

$$\langle D_a^k f_a, \varphi \rangle = D_a^k \langle f_a, \varphi \rangle. \tag{1.20}$$

Next, let $\{f_a\}$ be a family such that $f_a \in \mathscr{D}'$ for every $a \in R^m$; the family $\{f_a\}$ is called continuous if, for every $\varphi \in \mathscr{D}$, the function $\psi_\varphi(a) = \langle f_a, \varphi \rangle$ is continuous on $R^m$.

Clearly, $\{f_a\}$ is continuous if and only if $f_{a_n} \to f_a$ in $\mathscr{D}'$ for any sequence $a_n \to a$ in $R^m$.

DEFINITION. Let $F$ be the set of all families $\{f_a\}$, where $f_a \in \mathscr{D}'$ for every $a \in R^m$, such that, for $\{f_a\} \in F$, the function $\psi_\varphi(a) = \langle f_a, \varphi \rangle$ is in $\mathscr{D}$ for every $\varphi \in \mathscr{D}$.

Observe that $F$ is not empty, because $\{\delta_a\} \in F$; moreover, every $\{f_a\} \in F$ is clearly a continuous family.

Then we have the following lemma.

LEMMA 1.4. $\{f_a\} \in F$ *if and only if*:

(a) $D_a^k f_a$ *exists for every multi-index* $k$ *and* $\{D_a^k f_a\}$ *is a continuous family*;

(b) $\alpha_n f_{a_n} \to 0$ *in* $\mathscr{D}'$ *for every sequence of numbers* $\alpha_n$ *and points* $a_n$ *such that* $|a_n| \to \infty$ *as* $n \to \infty$.

*Proof.* By Lemma 1.3, condition (a) is satisfied if and only if $\psi_\varphi(a)$ is infinitely smooth for every $\varphi \in \mathscr{D}$. As for (b), let $\psi_\varphi(a)$ vanish identically outside of a bounded set in $R^m$, for any $\varphi \in \mathscr{D}$. Let $\alpha_n$, $a_n$ be sequences satisfying the hypothesis; then, for any $\varphi \in \mathscr{D}$, $\langle \alpha_n f_{a_n}, \varphi \rangle = \alpha_n \langle f_{a_n}, \varphi \rangle = \alpha_n \psi_\varphi(a_n) \to 0$ as $n \to \infty$.

Conversely, let $\alpha_n f_{a_n} \to 0$ in $\mathscr{D}'$ for any sequences $\alpha_n$, $a_n$ with the above properties. Suppose that there exists $\varphi \in \mathscr{D}$ such that $\psi_\varphi(a)$ does not vanish outside of a bounded set; i.e., there exists a sequence $a_n \in R^m$ with $|a_n| \to \infty$ such that $\psi_\varphi(a_n) \neq 0$. Setting $\alpha_n = (\psi_\varphi(a_n))^{-1}$, we obtain $\langle \alpha_n f_{a_n}, \varphi \rangle = \alpha_n \langle f_{a_n}, \varphi \rangle = \alpha_n \psi_\varphi(a_n) = 1$, i.e., a contradiction to the assumption made.

From Lemma 1.3 we also have the following obvious assertion.

LEMMA 1.5. *If* $\{f_a\} \in F$ *and* $k$ *is a multi-index, then* $\{D_a^k f_a\} \in F$.

Now we are ready to state the following result.

THEOREM 1.2. *Let* $\{f_a\} \in F$ *and let* $\psi_\varphi(a) = \langle f_a, \varphi \rangle$ *for every* $a \in R^m$ *and* $\varphi \in \mathscr{D}$; *then the mapping* $\varphi \to \psi_\varphi$ *is linear and continuous* (*in the sense of convergence in* $\mathscr{D}$).

*Proof.* Linearity being obvious, let us turn to continuity. Thus, let $\varphi_n \to 0$ in $\mathscr{D}$ and denote for brevity $\psi_n(a) = \psi_{\varphi_n}(a) = \langle f_a, \varphi_n \rangle$; clearly $\psi_n(\xi) \to 0$ at any point $\xi \in R^m$.

(i) First show that the supports of all the $\psi_n$ are contained in a bounded subset of $R^m$.

To prove it suppose that this is not true; i.e., for every $a > 0$ there exists an index $n_a \geqq 1$ and a point $t_a \in R^m$ with $|t_a| \geqq a$ such that $\psi_{n_a}(t_a) \neq 0$. The compactness of supports implies that $n_a \to \infty$ as $a \to \infty$.

Construct a sequence of indices $n_1 < n_2 < \cdots$ and points $t_n \in R^m$ as follows: Find $t_1$ with $|t_1| > 1$ and $n_1 \geqq 1$ such that $\lambda_1 = \psi_{n_1}(t_1) \neq 0$. Next, find $t_2$ with $|t_2| > 2$, $t_2 \notin \operatorname{supp} \psi_{n_1}$ and $n_2 > n_1$ such that $\lambda_2 = \psi_{n_2}(t_2) \neq 0$ and

$$|\psi_k(t_1)| < \tfrac{1}{2}|\lambda_1| \tag{1.21}$$

for all $k \geqq n_2$.

Generally, if $n_1 < n_2 < \cdots < n_{q-1}$ and $t_1, t_2, \cdots, t_{q-1}$ are already found, we establish $n_q > n_{q-1}$ and $t_q$ so as to have

$$|t_q| > q, \qquad t_q \notin \operatorname{supp} \psi_{n_i} \quad \text{for } i = 1, 2, \cdots, q-1, \tag{1.22}$$

$$\lambda_q = \psi_{n_q}(t_q) \neq 0 \tag{1.23}$$

and

$$|\psi_k(t_{q-1})| < \tfrac{1}{2}|\lambda_{q-1}| \quad \text{for all } k \geqq n_q. \tag{1.24}$$

Such $n_q$ and $t_q$ clearly exist due to the pointwise convergence of $\psi_k$ and the fact that $n_a \to \infty$ as $a \to \infty$.

Now, define functions $v_k$, $k = 1, 2, \cdots$, and $\varphi$ by

$$v_k = \sum_{i=1}^{k} 2^{-i}\varphi_{n_i}, \qquad \varphi = \sum_{i=1}^{\infty} 2^{-i}\varphi_{n_i}. \tag{1.25}$$

Since $\varphi_{n_i} \to 0$ in $\mathscr{D}$, the infinite series in (1.25) converges uniformly, and consequently, $\varphi$ is continuous; moreover, since all $\varphi_n$ vanish outside a bounded set in $R^m$, so does $\varphi$. On the other hand, for every multi-index $r$ we clearly have $D^r v_k \to \sum_{i=1}^{\infty} 2^{-i} D^r \varphi_{n_i}$ uniformly. Hence, $\varphi \in \mathscr{D}$ and $v_k \to \varphi$ in $\mathscr{D}$.

By assumption, $\psi_\varphi = \langle f_a, \varphi \rangle \in \mathscr{D}$. If $a \in R^m$, we have $\langle f_a, v_k \rangle \to \langle f_a, \varphi \rangle = \psi_\varphi(a)$ as $k \to \infty$; i.e., by (1.25),

$$\psi_\varphi(a) = \sum_{i=1}^{\infty} 2^{-i}\psi_{n_i}(a). \tag{1.26}$$

Putting $a = t_q$, we obtain, due to (1.22),

$$\psi_\varphi(t_q) = \sum_{i=1}^{\infty} 2^{-i}\psi_{n_i}(t_q) = 2^{-q}\psi_{n_q}(t_q) + \sum_{i=q+1}^{\infty} 2^{-i}\psi_{n_i}(t_q);$$

consequently, by (1.24), (1.23),

$$
\begin{aligned}
|\psi_\varphi(t_q)| &\geqq 2^{-q}|\lambda_q| - \sum_{i=q+1}^{\infty} 2^{-i}|\psi_{n_i}(t_q)| \\
&> 2^{-q}|\lambda_q| - \sum_{i=q+1}^{\infty} 2^{-i}\tfrac{1}{2}|\lambda_q| = 2^{-q-1}|\lambda_q| > 0.
\end{aligned} \tag{1.27}
$$

However, since $|t_q| \to \infty$ as $q \to \infty$, (1.27) shows that $\psi_\varphi$ does not have a compact support, which contradicts the fact that $\psi_\varphi \in \mathscr{D}$. Hence, there exists $C > 0$ such that, for every $n$, $\operatorname{supp} \psi_n \subset I = \{t : |t| \leqq C\}$.

(ii) Now we are going to show that for any multi-index $k$, the sequence $D_a^k \psi_n \to 0$ uniformly. Thus, supposing that this is not true we conclude as in the second part of the proof of Theorem 1.1 that there exist an $\varepsilon > 0$, a fixed multi-index $s$, a subsequence $\psi_{r_{\bar n}}$ of our original sequence $\psi_r$ and a sequence of points $\bar t_{\bar n} \in R^m$ with

$$
|\bar t_{\bar n} - \xi| > |\bar t_{\bar n+1} - \xi|, \qquad \bar t_{\bar n} \to \xi \in I, \tag{1.28}
$$

such that

$$
|D_a^s \psi_{r_{\bar n}}(\bar t_{\bar n})| > \varepsilon \bar n^3/2m \tag{1.29}
$$

for all $\bar n$.

However, by Lemma 1.3,

$$
D_a^s \psi_{r_{\bar n}} = D_a^s \langle f_a, \varphi_{r_{\bar n}} \rangle = \langle D_a^s f_a, \varphi_{r_{\bar n}} \rangle,
$$

and, by Lemma 1.5, $\{D_a^s f_a\} \in F$. Since also $\varphi_{r_{\bar n}} \to 0$ in $\mathscr{D}$, write for simplicity $\varphi_n$ instead of $\varphi_{r_{\bar n}}$ and put $D_a^s \psi_{r_{\bar n}} = v_{\bar n}$, $g_a = D_a^s f_a$. Thus, summarizing,

$$
\begin{gathered}
\varphi_n \to 0 \quad \text{in } \mathscr{D}, \qquad v_n(a) = \langle g_a, \varphi_n \rangle \in \mathscr{D} \quad \text{for } n = 1, 2, \cdots, \\
|v_n(\bar t_n)| > \varepsilon \bar n^3/2m,
\end{gathered} \tag{1.30}
$$

where the $\bar t_n$ satisfy (1.28).

Next, let us construct a subsequence $v_{n_i}$ of $v_n$ as follows: Since $v_n(a) \to 0$ at any point $a \in R^m$, choose $n_1 \geqq 1$ so large that $|v_k(\xi)| < A/(2\mu)$ for every $k \geqq n_1$, where $A = \varepsilon/(4m)$ and $\mu = \sum_{i=1}^{\infty} i^{-2}$.

By continuity of $v_{n_1}$, there exists $\lambda_1 > 0$ such that $|v_{n_1}(t)| < A/\mu$ for every $t \in I_1 = \{t : |t - \xi| < \lambda_1\}$ and $\bar t_{n_1} \notin I_1$.

Further, find $n_2 > n_1$ so that

$$
\bar t_{n_2} \in I_1 \quad \text{and} \quad |v_{n_2}(\bar t_{n_1})| < A/2; \tag{1.31}
$$

by continuity of $v_{n_2}$, there exists $\lambda_2 > 0$, $\lambda_2 < \lambda_1$ such that $|v_{n_2}(t)| < A/\mu$ for all $t \in I_2 = \{t : |t - \xi| < \lambda_2\}$ and $\bar t_{n_2} \notin I_2$.

Generally, if indices $n_1 < n_2 < \cdots < n_{k-1}$ and numbers $\lambda_1 > \lambda_2 > \cdots > \lambda_{k-1}$ are already established, we find $n_k > n_{k-1}$ so as to have

$$
\bar t_{n_k} \in I_{k-1} = \{t : |t - \xi| < \lambda_{k-1}\} \tag{1.32}
$$

and

$$
|v_{n_k}(\bar t_{n_{k-p}})| < \frac{A}{2} \cdot 2^{1-p}, \qquad p = 1, 2, \cdots, k-1;
$$

such an $n_k$ clearly exists due to the pointwise convergence of $v_n$ at any point. By continuity of $v_{n_k}$ we then find $\lambda_k > 0$, $\lambda_k < \lambda_{k-1}$ such that $|v_{n_k}(t)| < A/\mu$ for $t \in I_k$ and $\bar{t}_{n_k} \notin I_k$, etc.

Now, define functions

$$(1.33) \qquad \rho_r = \sum_{i=1}^{r} \varphi_{n_i} \cdot n_i^{-2}, \qquad \varphi = \sum_{i=1}^{\infty} \varphi_{n_i} \cdot n_i^{-2}.$$

As before, we conclude easily that $\varphi \in \mathscr{D}$ and $\rho_r \to \varphi$ in $\mathscr{D}$. Thus, by the fact that $\{g_a\} \in F$, $v = \langle g_a, \varphi \rangle \in \mathscr{D}$.

On the other hand, for any $a \in R^m$,

$$(1.34) \qquad \langle g_a, \rho_r \rangle \to \langle g_a, \varphi \rangle = v(a).$$

By (1.33) and (1.30),

$$\langle g_a, \rho_r \rangle = \sum_{i=1}^{r} n_i^{-2} \langle g_a, \varphi_{n_i} \rangle = \sum_{i=1}^{r} n_i^{-2} v_{n_i}(a);$$

consequently, by (1.34),

$$v(a) = \sum_{i=1}^{\infty} v_{n_i}(a) n_i^{-2}.$$

Choose $a = \bar{t}_{n_k}$; then

$$(1.35) \qquad v(\bar{t}_{n_k}) = \sum_{i=1}^{\infty} v_{n_i}(\bar{t}_{n_k}) n_i^{-2} = \alpha + v_{n_k}(\bar{t}_{n_k}) n_k^{-2} + \beta,$$

where

$$\alpha = \sum_{i=1}^{k-1} v_{n_i}(\bar{t}_{n_k}) n_i^{-2}, \qquad \beta = \sum_{i=k+1}^{\infty} v_{n_i}(\bar{t}_{n_k}) n_i^{-2}.$$

However, because by our construction $\bar{t}_{n_k} \in I_{k-1} \subset I_{k-2} \subset \cdots \subset I_1$, we have $|v_{n_i}(\bar{t}_{n_k})| < A/\mu$ for $i = 1, 2, \cdots, k-1$; hence,

$$|\alpha| \leqq \frac{A}{\mu} \sum_{i=1}^{k-1} n_i^{-2} < A.$$

Similarly, by (1.32),

$$|\beta| \leqq \sum_{i=k+1}^{\infty} |v_{n_i}(\bar{t}_{n_k})| < A \sum_{i=1}^{\infty} 2^{-i} = A.$$

Thus, by (1.35) and (1.30),

$$(1.36) \qquad |v(\bar{t}_{n_k})| > \frac{\varepsilon}{2m} n_k^3 \cdot n_k^{-2} - 2A = \frac{\varepsilon}{2m}(n_k - 1).$$

However, $\bar{t}_{n_k} \to \xi$ and $n_k \to \infty$ as $k \to \infty$, so that (1.36) contradicts the continuity of $v \in \mathscr{D}$. Hence the proof.

The theorem just proved has the following converse.

THEOREM 1.3. *Let $T: \mathscr{D} \to \mathscr{D}$ be a linear and continuous mapping (in the sense of convergence in $\mathscr{D}$); then there exists a family $\{f_a\} \in F$ such that $(T\varphi)(a) = \langle f_a, \varphi \rangle$ for any $\varphi \in \mathscr{D}$.*

*Proof.* For any $a \in R^m$ define a functional $f_a$ on $\mathscr{D}$ by

$$(1.37) \qquad \langle f_a, \varphi \rangle = (T\varphi)(a).$$

Obviously, $f_a$ is linear; moreover, if $\varphi_n \to 0$ in $\mathscr{D}$, then $T\varphi_n \to 0$ in $\mathscr{D}$, and consequently, $\langle f_a, \varphi_n \rangle \to 0$ by (1.37). Hence, $f_a \in \mathscr{D}'$.

On the other hand, since $T\varphi \in \mathscr{D}$ for every $\varphi \in \mathscr{D}$, (1.37) shows that $\{f_a\} \in F$.

Thus, Theorems 1.2 and 1.3 prove that every linear continuous operator from $\mathscr{D}$ into itself has the form (1.37).

**2.** Let us now discuss some corollaries of Theorems 1.1, 1.2 and 1.3, which concern linear continuous operators mapping $\mathscr{D}'$ into $\mathscr{D}'$.

By continuity of an operator $A:\mathscr{P} \to \mathscr{D}'$, where $\mathscr{P} \subset \mathscr{D}'$, we understand that $Ax_n \to Ax$ whenever $x_n, x \in \mathscr{P}$ and $x_n \to x$ in $\mathscr{D}'$.

THEOREM 2.1. *Let* $\{f_a\} \in F$; *for every* $x \in \mathscr{D}'$ *define the functional* $Ax$ *on* $\mathscr{D}$ *by*

$$\langle Ax, \varphi\rangle = \langle x, \langle f_a, \varphi\rangle\rangle. \tag{2.1}$$

*Then* $Ax \in \mathscr{D}'$ *for every* $x \in \mathscr{D}'$ *and* $A$ *is a linear continuous operator from* $\mathscr{D}'$ *into itself.*

*Note.* Writing the "arguments" explicitly, (2.1) should read

$$\langle (Ax)(t), \varphi(t)\rangle = \langle x(a), \langle f_a(t), \varphi(t)\rangle\rangle;$$

however, bearing this license in mind, the short-hand notation used in (2.1) can hardly lead to a misunderstanding.

*Proof.* (i) The linearity of the functional $Ax$ is obvious. If $\varphi_n \to 0$ in $\mathscr{D}$, then, by Theorem 1.2, $\langle f_a, \varphi_n\rangle \to 0$ in $\mathscr{D}$, and consequently, by (2.1) and continuity of $x$, $\langle Ax, \varphi_n\rangle \to 0$; hence $Ax \in \mathscr{D}'$.

(ii) Linearity of $A$ is obvious. If $x_n \to x$ in $\mathscr{D}'$, then (2.1) shows that $Ax_n \to Ax$ which completes the proof.

For proving the next theorem we will need the following lemma (cf. [2]).

LEMMA. 2.1. *Let* $A:\mathscr{P} \to \mathscr{D}'$ *be a linear continuous operator, where* $\mathscr{P}$ *is a linear subset of* $\mathscr{D}'$; *furthermore, let* $\{f_a\} \in F$ *and let* $D_a^k f_a \in \mathscr{P}$ *for any multi-index* $k$. *Then* $\{Af_a\} \in F$ *and*

$$D_a^k(Af_a) = A(D_a^k f_a). \tag{2.2}$$

*Proof.* Choose a point $a \in R^m$, a sequence of numbers $a_n^i \to a^i$, $a_n^i \neq a^i$, and put $a_n = (a^1, a^2, \cdots, a^{i-1}, a_n^i, a^{i+1}, \cdots, a^m)$, where $a = (a^1, a^2, \cdots, a^m)$. By Lemma 1.2 we have

$$(a_n^i - a^i)^{-1}(f_{a_n} - f_a) \to \partial f_a/\partial a_i$$

in $\mathscr{D}'$. Since $\partial f_a/\partial a_i \in \mathscr{P}$, we have, by linearity and continuity of $A$,

$$(a_n^i - a^i)^{-1}(Af_{a_n} - Af_a) \to A\partial f_a/\partial a_i.$$

Thus, again by Lemma 1.2, $\partial(Af_a)/\partial a_i$ exists and equals $A(\partial f_a/\partial a_i)$. Repeating this argument proves formula (2.2).

Next, by Lemma 1.4, $\{g_a\}$ with $g_a = D_a^k f_a$ is a continuous family; i.e., $g_{a_n} \to g_a$ in $\mathscr{D}'$ for any sequence $a_n \to a$ in $R^m$; consequently, by continuity of $A$, $Ag_{a_n} \to Ag_a$ in $\mathscr{D}'$. Thus, by (2.2), $\{D_a^k(Af_a)\}$ is a continuous family.

Finally, let $\alpha_n$ be a sequence of numbers and $a_n \in R^m$ a sequence of points in $R^m$ such that $|a_n| \to \infty$ as $n \to \infty$. Since $\{f_a\} \in F$, we have, by Lemma 1.4, $\alpha_n f_{a_n} \to 0$ in $\mathscr{D}'$. Hence, by linearity and continuity of $A$, $\alpha_n(Af_{a_n}) \to 0$ in $\mathscr{D}'$. By invoking Lemma 1.4 again we conclude the proof.

THEOREM 2.2. *Let $\mathscr{P}$ be a linear subset of $\mathscr{D}'$ such that $\mathscr{D}'_* \subset \mathscr{P} \subset \mathscr{D}'$, and let $A:\mathscr{P} \to \mathscr{D}'$ be linear and continuous; denote $f_a = A\delta_a$ for every $a \in R^m$. Then $\{f_a\} \in F$, and (2.1) holds for every $x \in \mathscr{P}$ and $\varphi \in \mathscr{D}$.*

*Proof.* We have $\{\delta_a\} \in F$; it can be easily verified that, for every multi-index $k$, $D_a^k\delta_a = (-1)^k D_t^k \delta_a \in \mathscr{D}'_* \subset \mathscr{P}$. Hence, the assumptions of Lemma 2.1 are met and we have $\{f_a\} \in F$.

Next, define a linear continuous operator $\tilde{A}:\mathscr{P} \to \mathscr{D}'$ by

$$\langle \tilde{A}x, \varphi\rangle = \langle x, \langle f_a, \varphi\rangle\rangle. \tag{2.3}$$

Then we have, for any $b \in R^m$, $\langle \tilde{A}\delta_b, \varphi\rangle = \langle \delta_b, \langle f_a, \varphi\rangle\rangle = \langle f_b, \varphi\rangle$; i.e., $\tilde{A}\delta_b = f_b$.

Let $\mu \in \mathscr{P}$ have a bounded support; then there exists a sequence $\mu_n \to \mu$ in $\mathscr{D}'$ such that, for each $n$,

$$\mu_n = \sum_{j=1}^{k_n} \alpha_j^n \delta_{b_j^n} \tag{2.4}$$

(cf. [1, p. 145]). This shows that $(A - \tilde{A})\mu_n = 0$ for every $n$, and consequently, by continuity, $(A - \tilde{A})\mu = 0$.

Next, if $x \in \mathscr{P}$, then there exists a sequence $x_n \to x$ in $\mathscr{D}'$ such that each $x_n$ has a bounded support, and consequently, $x_n \in \mathscr{P}$. Hence, $(A - \tilde{A})x = 0$ by continuity, which completes the proof.

Observe that Theorems 2.1 and 2.2 show that every linear and continuous operator $\mathscr{P} \to \mathscr{D}'$ has the form (2.1).

Let us now discuss a generalization of a time-invariant operator. Let $k$ be a fixed integer, $1 \leqq k \leqq m$; for any $\xi \in R^m$, $\xi = (\xi_1, \xi_2, \cdots, \xi_m)$, let $\xi^* = (\xi_1, \xi_2, \cdots, \xi_{k-1}, \xi_k, 0, 0, \cdots, 0)$ and $\xi^+ = \xi - \xi^*$. Clearly, $(\xi^*)^* = \xi^*$, $(\xi^+)^+ = \xi^+$, $(\xi^*)^+ = (\xi^+)^* = 0$ for any $\xi \in R^m$.

Let $\mathscr{P}$ be a linear subset of $\mathscr{D}'$; $\mathscr{P}$ will be called shift*invariant, if $P_{\xi^*}x \in \mathscr{P}$ whenever $x \in \mathscr{P}$ and $\xi \in R^m$. (Here, $P_b$ signifies the operator of shifting; i.e., $\langle P_b x, \varphi(t)\rangle = \langle x, \varphi(t + b)\rangle$ for all $x \in \mathscr{D}'$ and $\varphi \in \mathscr{D}$.)

Similarly, if $\mathscr{P}$ is shift*invariant and $A:\mathscr{P} \to \mathscr{D}'$, then $A$ will be called shift* invariant, if $P_{\xi^*}A = AP_{\xi^*}$ for any $\xi \in R^m$.

THEOREM 2.3. *Let $\mathscr{D}'_* \subset \mathscr{P} \subset \mathscr{D}'$ and let $\mathscr{P}$ be shift*invariant; let $A:\mathscr{P} \to \mathscr{D}'$ be linear and continuous. Then $A$ is shift*invariant if and only if $f_a = A\delta_a = P_{a^*}A\delta_{a^+}$ for all $a \in R^m$ and $\{f_a\} \in F$.*

COROLLARY. *The operator $A$ satisfies the condition $P_\xi A = AP_\xi$ for every $\xi \in R^m$ if and only if there exists $f \in \mathscr{D}'$ such that $A\delta_a = P_a f$ for all $a \in R^m$ and $\{P_a f\} \in F$.*

*Proof.* (i) First, let $f_a = A\delta_a = P_{a^*}A\delta_{a^+}$ for all $a \in R^m$ and $\{f_a\} \in F$. By Theorem 2.2,

$$\langle Ax, \varphi\rangle = \langle x, \langle f_a, \varphi\rangle\rangle \tag{2.5}$$

for any $x \in \mathscr{P}$ and $\varphi \in \mathscr{D}$. Choosing $\xi \in R^m$, we have

$$\langle AP_{\xi^*}x, \varphi\rangle = \langle P_{\xi^*}x, \langle f_a, \varphi\rangle\rangle. \tag{2.6}$$

Denote

$$\psi_{\varphi(t)}(a) = \langle f_a, \varphi\rangle = \langle P_{a^*}A\delta_{a^+}, \varphi\rangle. \tag{2.7}$$

Then

$$\psi_{\varphi(t)}(a^+) = \langle P_0 A\delta_{a^+}, \varphi\rangle = \langle A\delta_{a^+}, \varphi\rangle, \tag{2.8}$$

and consequently, by (2.7),

$$\psi_{\varphi(t)}(a) = \langle A\delta_{a^+}, \varphi(t + a^*)\rangle = \psi_{\varphi(t+a^*)}(a^+). \tag{2.9}$$

Thus, by (2.6) and (2.9),

$$\begin{aligned}\langle AP_{\xi^*}x, \varphi\rangle &= \langle P_{\xi^*}x, \psi_{\varphi(t)}(a)\rangle \\ &= \langle x, \psi_{\varphi(t)}(a + \xi^*)\rangle = \langle x, \psi_{\varphi(t+a^*+\xi^*)}(a^+)\rangle,\end{aligned} \tag{2.10}$$

because $(a + \xi^*)^+ = a^+$ and $(a + \xi^*)^* = a^* + \xi^*$.

On the other hand,

$$\begin{aligned}\langle P_{\xi^*}Ax, \varphi\rangle &= \langle Ax, \varphi(t + \xi^*)\rangle = \langle x, \langle f_a, \varphi(t + \xi^*)\rangle\rangle \\ &= \langle x, \psi_{\varphi(t+\xi^*)}(a)\rangle = \langle x, \psi_{\varphi(t+a^*+\xi^*)}(a^+)\rangle\end{aligned}$$

by (2.9). Hence, $P_{\xi^*}A = AP_{\xi^*}$.

(ii) Conversely, let $A$ be shift*invariant. Since $\delta_a \in \mathscr{P}$, we have, for every $a \in R^m$, $f_a = A\delta_a = AP_{a^*}\delta_{a^+} = P_{a^*}A\delta_{a^+}$, which is what we wished to show.

The corollary follows from Theorem 2.3 by setting $k = m$ and realizing that that $a^* = a$, $a^+ = 0$; i.e., we have $A\delta_a = P_a f$ with $f = A\delta_0$.

Finally, let us present a characterization of the convergence of linear continuous operators.

Let $A_n$, $A$, $n = 1, 2, \cdots$, be linear continuous operators mapping $\mathscr{P} \subset \mathscr{D}'$ into $\mathscr{D}'$; we say that $A_n \to A$, if $A_n x \to Ax$ in $\mathscr{D}'$ for every $x \in \mathscr{P}$.

As we have seen, the family $\{A\delta_a\}$ plays a crucial role in establishing the properties of an operator $A$. The above definition shows that if $A_n \to A$, then $A_n\delta_a \to A\delta_a$ for any $a \in R^m$. Can this implication be inverted, i.e., does $A_n\delta_a \to A\delta_a$ imply that $A_n \to A$? The answer to this question is furnished by the following proposition.

THEOREM 2.4. *Let $A_n$, $A$, $n = 1, 2, \cdots$, be continuous linear operators from $\mathscr{P} \to \mathscr{D}'$, where $\mathscr{P}$ is a linear set satisfying the inclusion $\mathscr{D}'_* \subset \mathscr{P} \subset \mathscr{D}'$; then $A_n \to A$ if and only if, for every $\varphi \in \mathscr{D}$,*

$$\psi_n(a) = \langle (A_n - A)\delta_a, \varphi\rangle \to 0 \quad in\ \mathscr{D}. \tag{2.11}$$

*Proof.* Denote $f^n_a = A_n\delta_a$, $f_a = A\delta_a$ for every $a \in R^m$. Then, by Theorem 2.2,

$$\langle A_n x, \varphi\rangle = \langle x, \langle f^n_a, \varphi\rangle\rangle, \qquad \langle Ax, \varphi\rangle = \langle x, \langle f_a, \varphi\rangle\rangle \tag{2.12}$$

for every $x \in \mathscr{P}$ and $\varphi \in \mathscr{D}$.

(i) Let (2.11) be true; i.e., for any $\varphi \in \mathscr{D}$, $\langle f^n_a, \varphi\rangle \to \langle f_a, \varphi\rangle$ in $\mathscr{D}$. Then, for any $x \in \mathscr{P}$,

$$\langle x, \langle f^n_a, \varphi\rangle\rangle \to \langle x, \langle f_a, \varphi\rangle\rangle, \tag{2.13}$$

and consequently, by (2.12), $A_n \to A$.

(ii) Conversely, let $A_n \to A$; then (2.13) holds for any choice of $\varphi \in \mathscr{D}$ and $x \in \mathscr{P}$. Hence, for any $x \in \mathscr{D}'_* \subset \mathscr{P}$,

$$\langle x, \langle f^n_a - f_a, \varphi\rangle\rangle \to 0.$$

Consequently, by Theorem 1.1, $\langle f_a^n - f_a, \varphi \rangle \to 0$ in $\mathscr{D}$; i.e., (2.11) holds. Hence the proof.

Now we are ready to summarize our previous results; for this purpose, we introduce the following notations:

Let $\mathscr{P}$ be a fixed linear subset satisfying the inclusion $\mathscr{D}'_* \subset \mathscr{P} \subset \mathscr{D}'$, and let $\mathfrak{U}'$ stand for the set of all linear and continuous operators mapping $\mathscr{P}$ into $\mathscr{D}'$. Furthermore, let $\mathfrak{U}$ signify the set of all linear and continuous operators from $\mathscr{D}$ into itself. Finally, let $B, B_n \in \mathfrak{U}$, $n = 1, 2, \cdots$; we will write $B_n \to B$ if $B_n\varphi \to B\varphi$ in $\mathscr{D}$ for every $\varphi \in \mathscr{D}$.

The following proposition gives a picture of the relation between $\mathfrak{U}'$ and $\mathfrak{U}$.

THEOREM 2.5. *Let the mapping $\mathscr{T} : \mathfrak{U}' \to \mathfrak{U}$ be defined by*

$$\langle x, (\mathscr{T}A)(\varphi) \rangle = \langle Ax, \varphi \rangle \tag{2.14}$$

*for all $A \in \mathfrak{U}'$, $x \in \mathscr{P}$ and $\varphi \in \mathscr{D}$. Then $\mathscr{T}$ is one-to-one from $\mathfrak{U}'$ onto $\mathfrak{U}$, and both mappings $\mathscr{T}$ and $\mathscr{T}^{-1}$ are linear and continuous.*

*Moreover, $\mathscr{T}(AB) = (\mathscr{T}B)(\mathscr{T}A)$ for every $A, B \in \mathfrak{U}'$ and $\mathscr{T}I = I \in \mathfrak{U}$.*

*Proof.* Let $A \in \mathfrak{U}'$; then by Theorem 2.2 there exists a family $\{f_a\} \in F$ such that (2.1) holds. However, due to Theorem 1.2, $\{f_a\}$ generates an operator $A^* \in \mathfrak{U}$ such that (2.1) becomes (2.14) with $A^* = \mathscr{T}A$. Moreover, the value $\mathscr{T}A$ is determined uniquely, because if there existed $\bar{A}^* \in \mathfrak{U}$ satisfying (2.14), we would have $\langle x, (A^* - \bar{A}^*)\varphi \rangle = 0$ for all $x \in \mathscr{P}$ and $\varphi \in \mathscr{D}$, which implies $A^* - \bar{A}^* = 0$.

Next, if $A_1, A_2 \in \mathfrak{U}'$ are such that $\mathscr{T}A_1 = \mathscr{T}A_2$, then (2.14) implies that $\langle (A_1 - A_2)x, \varphi \rangle = 0$ for all $x \in \mathscr{P}$ and $\varphi \in \mathscr{D}$, and consequently, $A_1 - A_2 = 0$; hence, $\mathscr{T}$ is one-to-one.

If $A^* \in \mathfrak{U}$, then by Theorem 1.3 there exists a family $\{f_a\} \in F$ such that $(A^*\varphi)(a) = \langle f_a, \varphi \rangle$; then due to Theorem 2.1 the operator $A$ defined by (2.1) is in $\mathfrak{U}'$ and (2.1) is (2.14) with $\mathscr{T}A = A^*$. Consequently, $\mathscr{T}$ is onto.

The linearity of $\mathscr{T}$ and $\mathscr{T}^{-1}$ is obvious. As for continuity, observe that, by Theorem 2.4, $A_n \to A$ for $A_n, A \in \mathfrak{U}'$ exactly if

$$\langle f_a^n, \varphi \rangle \to \langle f_a, \varphi \rangle \tag{2.15}$$

in $\mathscr{D}$ for any $\varphi \in \mathscr{D}$, where $f_a^n = A_n\delta_a$, $f_a = A\delta_a$. However, $\langle f_a^n, \varphi \rangle = \langle \mathscr{T}A_n)\varphi$ and $\langle f_a, \varphi \rangle = (\mathscr{T}A)\varphi$, and consequently, (2.15) is equivalent to $\mathscr{T}A_n \to \mathscr{T}A$.

To verify the last assertion is a matter of simple routine; hence, the proof.

**3.** In the preceding section we have defined the concept of convergence of operators either from $\mathfrak{U}'$ or $\mathfrak{U}$ by assuming that the limit is also a linear continuous operator. This, however, is no loss of generality; as a matter of fact, the sets $\mathfrak{U}'$ and $\mathfrak{U}$ are sequentially complete, and this section is devoted to proving this fact.

As usual, if $\varphi_n \in \mathscr{D}$, $n = 1, 2, \cdots$, we say that the sequence $\varphi_n$ converges in $\mathscr{D}$, provided the supports of all $\varphi_n$ are contained in a bounded set and the sequence $D^k\varphi_n$ converges uniformly for any multi-index $k$. Then, as known (cf. [1, p. 5]), there exists a unique $\varphi \in \mathscr{D}$ such that $(\varphi_n - \varphi) \to 0$ in $\mathscr{D}$; i.e., $\mathscr{D}$ is sequentially complete.

Similarly, if $f_n \in \mathscr{D}'$, $n = 1, 2, \cdots$, we say that the sequence $f_n$ converges in $\mathscr{D}'$ if, for any $\varphi \in \mathscr{D}$, the sequence of numbers $\langle f_n, \varphi \rangle$ converges; then again (cf. [1, p. 37]) there exists a unique $f \in \mathscr{D}'$ such that $f_n \to f$ in $\mathscr{D}'$; i.e., $\mathscr{D}'$ is sequentially complete.

Using these ideas, let us state the following definitions:

Let $B_n \in \mathfrak{U}$, $n = 1, 2, \cdots$; we say that the sequence $B_n$ converges if for every $\varphi \in \mathscr{D}$ the sequence $B_n\varphi$ converges in $\mathscr{D}$.

Let $A_n \in \mathfrak{U}'$, $n = 1, 2, \cdots$; we say that the sequence $A_n$ converges if for every $x \in \mathscr{D}'$ the sequence $A_n x$ converges in $\mathscr{D}'$. Then we have the following proposition.

THEOREM 3.1. *If $B_n \in \mathfrak{U}$, $n = 1, 2, \cdots$, and the sequence $B_n$ converges, then there exists a unique $B \in \mathfrak{U}$ such that $B_n \to B$.*

*Proof.* By our definition, for every $\varphi \in \mathscr{D}$, the sequence $B_n\varphi$ converges in $\mathscr{D}$; i.e., by completeness of $\mathscr{D}$, there exists a testing function $B\varphi \in \mathscr{D}$ such that

$$B_n\varphi \to B\varphi \quad \text{in } \mathscr{D}. \tag{3.1}$$

However, by Theorem 1.3, there exists a family $\{f_a^n\} \in F$ for every $n$ such that $(B_n\varphi)(a) = \langle f_a^n, \varphi\rangle$ for every $\varphi \in \mathscr{D}$ and $a \in R^m$. Consequently, we have, by (3.1) for any fixed $a$,

$$\langle f_a^n, \varphi\rangle \to (B\varphi)(a). \tag{3.2}$$

Hence, by completeness of $\mathscr{D}'$, there exists an $f_a \in \mathscr{D}'$ for every $a \in R^m$ such that

$$(B\varphi)(a) = \langle f_a, \varphi\rangle; \tag{3.3}$$

since $B\varphi \in \mathscr{D}$ for any $\varphi \in \mathscr{D}$, we have $\{f_a\} \in F$. Consequently, by Theorem 1.2, $B \in \mathfrak{U}$ which completes the proof.

For proving an analogous result concerning the set $\mathfrak{U}'$, we will need the following slight extension of Theorem 1.1.

THEOREM 3.2. *Let $\varphi_n \in \mathscr{D}$, $n = 1, 2, \cdots$, be a sequence such that, for any $f \in \mathscr{D}'_*$, the sequence $\langle f, \varphi_n\rangle$ converges; then the sequence $\varphi_n$ converges in $\mathscr{D}$.*

The proof of this theorem is almost the same as that of Theorem 1.1. Supposing that the supports of $\varphi_n$ are not contained in a bounded set we construct the same subsequence $\varphi_{n_i}$ as before. Now, we define $\tilde{f} \in \mathscr{D}'_*$ by

$$\tilde{f} = \sum_{i=1}^{\infty} (-1)^i \lambda_i^{-1} \delta_{t_i}; \tag{3.4}$$

then, by assumption, the sequence

$$z_{n_q} = \langle \tilde{f}, \varphi_{n_q}\rangle = \sum_{i=1}^{\infty} (-1)^i \lambda_i^{-1} \varphi_{n_q}(t_i)$$

should converge as $q \to \infty$. However, we see as before that $|z_{n_q}| > \frac{1}{2}$ and $\operatorname{sgn} z_{n_q} = (-1)^q$, which is a contradiction.

The proof of uniform convergence of $D^k\varphi_n$ is identical with part (ii) of the proof of Theorem 1.1.

Now, we can state the following theorem.

THEOREM 3.3. *If $A_n \in \mathfrak{U}'$, $n = 1, 2, \cdots$, and the sequence $A_n$ converges, then there exists a unique $A \in \mathfrak{U}'$ such that $A_n \to A$.*

*Proof.* By our definition and completeness of $\mathscr{D}'$, for every $x \in \mathscr{P}$ there exists a distribution $Ax \in \mathscr{D}'$ such that, for any $\varphi \in \mathscr{D}$,

$$\langle A_n x, \varphi\rangle \to \langle Ax, \varphi\rangle. \tag{3.5}$$

However, by Theorem 2.2, there exists a family $\{f_a^n\} \in F$ for every $n$ such that $\langle A_n x, \varphi\rangle = \langle x, \langle f_a^n, \varphi\rangle\rangle$. Hence, by (3.5),

$$\langle x, \langle f_a^n, \varphi\rangle\rangle \to \langle Ax, \varphi\rangle, \tag{3.6}$$

and consequently, by Theorem 3.2, $\langle f_a^n, \varphi\rangle$ converges in $\mathscr{D}$ for any $\varphi \in \mathscr{D}$.

On the other hand, by Theorems 1.2, 3.1 and 1.3 there exists $\{f_a\} \in F$ such that, for any $\varphi \in \mathscr{D}$,

$$\langle f_a^n, \varphi\rangle \to \langle f_a, \varphi\rangle \tag{3.7}$$

in $\mathscr{D}$. Hence, by (3.6), $\langle Ax, \varphi\rangle = \langle x, \langle f_a, \varphi\rangle\rangle$ for all $x \in \mathscr{P}$ and $\varphi \in \mathscr{D}$; then Theorem 2.1 shows that $A \in \mathfrak{U}'$ and the proof is complete.

Thus, both sets $\mathfrak{U}$ and $\mathfrak{U}'$ are sequentially complete.

**4.** We conclude the paper by making a few remarks on possible extensions of presented results.

**4.1.** For $i = 1, 2, \cdots, m$, let $I_i$ be either $(-\infty, \infty)$ or $[0, \infty)$, and put $S = I_1 \times I_2 \times \cdots \times I_m$. Let $\mathscr{D}_S$ stand for the set of all smooth functions defined on $S$ which have a compact support; if we introduce the convergence in $\mathscr{D}_S$ in the same way as in $\mathscr{D}$, denote $\mathscr{D}'_S$ the dual space of $\mathscr{D}_S$. Finally, let $\mathscr{D}_S^{*\prime} \subset \mathscr{D}'_S$ consist of all distributions of finite order.

Examining the proof of Theorem 1.1, we see that it works for spaces $\mathscr{D}_S, \mathscr{D}_S^{*\prime}$, too; thus, we have the following theorem.

THEOREM 4.1. *Let $\varphi_n \in \mathscr{D}_S$, $n = 1, 2, \cdots$, be a sequence such that $\langle f, \varphi_n\rangle \to 0$ for every $f \in \mathscr{D}_S^{*\prime}$; then $\varphi_n \to 0$ in $\mathscr{D}_S$.*

Similarly, if $f_a \in \mathscr{D}'_S$ for every $a \in S$, we can define the partial derivatives $D_a^k f_a$ by (1.19) and show that Lemmas 1.1–1.5 remain true. By $\{f_a\} \in F_S$ we now understand that $\psi_\varphi(a) = \langle f_a, \varphi\rangle \in \mathscr{D}_S$ whenever $\varphi \in \mathscr{D}_S$. Then again we may use the same argument as in the proof of Theorems 1.2, 1.3 and obtain the next theorem.

THEOREM 4.2. *The mapping $T: \mathscr{D}_S \to \mathscr{D}_S$ is linear and continuous exactly if there exists a family $\{f_a\} \in F_S$ such that $(T\varphi)(a) = \langle f_a, \varphi\rangle$ for all $\varphi \in \mathscr{D}_S$ and $a \in S$.*

Using this and the proofs of Theorems 2.1 and 2.2 we can easily verify that the following proposition is true.

THEOREM 4.3. *Let $\mathscr{P}_S$ be a linear subset such that $\mathscr{D}_S^{*\prime} \subset \mathscr{P}_S \subset \mathscr{D}'_S$; then the operator $A: \mathscr{P}_S \to \mathscr{D}'_S$ is linear and continuous if and only if*

$$\langle Ax, \varphi\rangle = \langle x, \langle f_a, \varphi\rangle\rangle \tag{4.1}$$

*for every $x \in \mathscr{D}'_S$ and $\varphi \in \mathscr{D}_S$, where $f_a = A\delta_a$ for every $a \in S$ and $\{f_a\} \in F$.*

Furthermore, it is not hard to verify that Theorem 2.3 remains true, provided shift-invariance is understood in such a sense that $P_{\xi*}A = AP_{\xi*}$ for every $\xi \in S$.

Finally, defining the convergence of operators in the same way as in § 2, we conclude that the following assertion is true.

THEOREM 4.4. *Let $A_n, A, n = 1, 2, \cdots$, be continuous linear operators from $\mathscr{P}_S \to \mathscr{D}'_S$, where $\mathscr{P}_S$ is a linear set such that $\mathscr{D}_S^{*\prime} \subset \mathscr{P}_S \subset \mathscr{D}'_S$; then $A_n \to A$ if and only if, for every $\varphi \in \mathscr{D}_S$,*

$$\langle (A_n - A)\delta_a, \varphi\rangle \to 0 \quad \text{in } \mathscr{D}_S.$$

As a result, Theorem 2.5 remains true without any change.

In the same manner as in § 3 we can conclude that here also the sets $\mathfrak{U}'$ and $\mathfrak{U}$ are sequentially complete.

*Example.* Let $m = 1$ and $S = [0, \infty)$; furthermore, let $W(t, \tau)$ be defined on the region $R = \{(t, \tau): 0 \leqq \tau \leqq t < \infty\}$ and let $W(t, \tau)$ have a continuous partial derivative of any order on $R$. (At the boundaries $\tau = 0$ and $\tau = t$ we understand derivatives from the inside of $R$, and the corresponding limits at (0, 0).)

For $a \geqq 0$, put $f_a = W(t, a)H_a$, where $H_a(t) = 1$ for $t \geqq a$ and $H_a(t) = 0$ elsewhere. Then $\{f_a\} \in F_S$. Actually, for any $\varphi \in \mathscr{D}_S$ we have

$$\psi_\varphi(a) = \langle f_a, \varphi \rangle = \int_a^\infty W(t, a)\varphi(t)\,dt; \tag{4.2}$$

however, (4.2) shows that $\psi_\varphi$ is infinitely smooth and vanishes for $a \geqq \sup \operatorname{supp} \varphi$. Hence, $\psi_\varphi \in \mathscr{D}_S$.

Thus, by Theorem 4.3, the equation

$$\langle Ax, \varphi \rangle = \left\langle x, \int_a^\infty W(t, a)\varphi(t)\,dt \right\rangle, \qquad \varphi \in \mathscr{D}_S, \tag{4.3}$$

defines a linear continuous operator from $\mathscr{D}'_S \to \mathscr{D}'_S$.

It is easy to verify that if $x$ is a regular distribution corresponding to a locally integrable function $x(t)$, i.e., there exists a locally integrable $x(t)$ such that $\langle x, \varphi \rangle = \int_0^\infty x(t)\varphi(t)\,dt$ for all $\varphi \in \mathscr{D}_S$, then $Ax$ is also regular and corresponds to the function $\int_0^t W(t, \tau)x(\tau)\,d\tau$.

Next, let $W_n(t, \tau)$, $n = 1, 2, \cdots$, be a sequence of smooth functions defined for $0 \leqq \tau \leqq t < \infty$ such that, for every pair of indices $i, j \geqq 0$ and $T > 0$, we have

$$\frac{\partial^{i+j} W_n(t, \tau)}{\partial t^i \partial \tau^j} \to 0$$

uniformly on $0 \leqq \tau \leqq t \leqq T$; for every $n$ define an operator $A_n : \mathscr{D}'_S \to \mathscr{D}'_S$ by (4.3). Then (4.2) shows that, for any $\varphi \in \mathscr{D}'_S$, $\langle f_a^n, \varphi \rangle \to 0$ in $\mathscr{D}_S$; consequently, by Theorem 4.4, $A_n \to 0$.

**4.2.** In the theory developed in § 1–§ 3 we have dealt with a single space $\mathscr{D}$ of testing functions on $R^m$ and the corresponding dual $\mathscr{D}'$. However, it can be readily seen that Theorem 1.2 remains true if $\varphi \to \psi_\varphi$ is a mapping from $\mathscr{D}_m$ into $\mathscr{D}_r$, $r \neq m$, where $\mathscr{D}_k$ stands for the set of testing functions defined on $R^k$. Similarly, Theorem 1.3 can be extended in this way. As a result, Theorems 2.1–2.4 and all theorems in § 3 may be extended to the case that $A$ is a continuous linear operator from $\mathscr{D}'_m$ into $\mathscr{D}'_r$.

**4.3.** Considering the situation that the testing functions $\varphi \in \mathscr{D}$ have values in a Banach space $B$ and $\langle f, \varphi \rangle$ also have values in $B$, we can verify that the above proofs still go through; thus, analogous theorems are true for this version.

**Acknowledgment.** The author wishes to acknowledge the valuable suggestions made by Professor I. Babuška from the University of Maryland.

## REFERENCES

[1] A. H. Zemanian, *Distribution Theory and Transform Analysis*, McGraw-Hill, New York, 1965.

[2] B. Pondelicek, *A contribution to the foundation of network theory using the distribution theory*, Czechoslovak Math. J. (94), 19 (1969), pp. 697–710.

[3] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Academic Press, New York, 1969.

SIAM J. MATH. ANAL.
Vol. 1, No. 4, November 1970

# ON THE ASYMPTOTIC BEHAVIOR OF AUTONOMOUS DIFFERENTIAL EQUATIONS*

R. H. MARTIN, JR.†

**1. Introduction.** Let $E$ be a Banach space over the field of real or complex numbers and let $|\cdot|$ denote the norm on $E$. In this paper it is assumed that $E$ has the strong topology. If $A$ is a continuous function from $E$ into $E$, we will be concerned with studying the behavior as $t$ tends to $+\infty$ of the solutions to the autonomous differential equation

$$\text{(ADE)} \qquad u'(t) = Au(t), \qquad u(0) = z,$$

where $z$ is in $E$ and $t$ is in $[0, \infty)$. The principal tool used in this paper is the right derivative of the norm on $E$. If $x$ and $y$ are in $E$, then we define

$$D_+[x, y; A] = \lim_{h \to +0} (|x - y + h[Ax - Ay]| - |x - y|)/h. \tag{1.1}$$

This notion will be used to study the set of attraction of the solutions to (ADE) and to give sufficient conditions for (ADE) to have a unique critical point which is globally asymptotically stable.

Let $\rho$ be a nonincreasing function from $[0, \infty)$ into $(0, \infty)$ such that

$$D_+[x, y; A] \leqq -\rho(r)|x - y| \tag{1.2}$$

whenever $x$ and $y$ are in $E$ with $|x|, |y| \leqq r$. In [2] N. N. Krasovskii shows that if $\rho(r) \geqq \rho_0 > 0$, $E$ is a finite-dimensional Hilbert space, and $A$ is continuously differentiable, then (ADE) has a unique critical point which is globally asymptotically stable. L. Markus and H. Yamabe [3, Theorem 1] improve this result by requiring that

$$\int_0^\infty \exp\left(-\varepsilon \int_0^s \rho(r)\, dr\right) ds < \infty$$

for each $\varepsilon > 0$. In [5, Theorem 1] the author establishes the same result assuming that $E$ is a Banach space, $A$ is continuous, and

$$\int_0^\infty \rho(r)\, dr = \infty.$$

The techniques used here are similar to those in [5]. However, in this paper, we are interested in obtaining results similar to those in [2], [3] and [5] in certain "singular" situations.

**2. Basic definitions and lemmas.** If $\Omega$ is a subset of $E$ and $x$ is in $E$, let $d(x, \Omega)$ denote the distance from $x$ to $\Omega$ (i.e., $d(x, \Omega) = \inf\{|x - y| : y \in \Omega\}$). If $R$ is a positive number, let $S(\Omega, R) = \{x \in E : d(x, \Omega) < R\}$ and if $z$ is in $E$ and $\Omega = \{z\}$, let $S(z, R)$ denote $S(\Omega, R)$.

* Received by the editors January 15, 1970, and in revised form April 4, 1970.
† Department of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332.

DEFINITION 2.1. If $\Omega$ is a subset of $E$ and $v$ is a function from $E$ into $[0, \infty)$, then $v$ is called *positive definite* with respect to $\Omega$ if $v(x) = 0$ for all $x$ in $\Omega$ and for each $\varepsilon > 0$ and each bounded subset $Q$ of $E$, there is a positive number $\delta = \delta(Q, \varepsilon)$ such that $v(x) \geqq \delta$ for all $x$ in $Q - S(\Omega, \varepsilon)$.

*Remark* 2.1. It will be convenient to allow the set $\Omega$ in Definition 2.1 to be the empty set. In this case, for each bounded subset $Q$ of $E$ there is a positive number $\delta = \delta(Q)$ such that $v(x) \geqq \delta$ for all $x$ in $Q$.

DEFINITION 2.2. A solution $u$ to (ADE) is said to *approach a subset* $\Omega$ of $E$ if for each positive number $\varepsilon$ there is a positive number $T$ such that $u(t)$ is in $S(\Omega, \varepsilon)$ for all $t$ in $[T, \infty)$.

LEMMA 2.1. *If $T > 0$, $u$ and $v$ are solutions to* (ADE) *which are defined on* $[0, T)$, *and $p(t) = |u(t) - v(t)|$ for each $t$ in* $[0, T)$, *then* $p'_+(t) = D_+[u(t), v(t); A]$.

*Proof*. It is shown in [1, p. 3] that

$$p'_+(t) = \lim_{h \to +0} (|u(t) - v(t) + h[u'(t) - v'(t)]| - |u(t) - v(t)|)/h,$$

and the lemma is immediate.

DEFINITION 2.3. Suppose that $A$ is a function from $E$ into $E$ and that there is a number $M$ such that $D_+[x, y; A] \leqq M|x - y|$ for all $x$ and $y$ in $E$. Then for each $z$ in $E$ define

$$\lambda_+[z; A] = \lim_{R \to +0} [\sup \{|x - y|^{-1} D_+[x, y; A] : x, y \in S(z, R), x \neq y\}].$$

*Remark* 2.2. Note that the term in the limit in Definition 2.3 is bounded above by $M$ and is nonincreasing as $R \to +0$ so that $\lambda_+[z; A]$ is either a number or $-\infty$.

*Example* 2.1. Suppose $U$ is a continuous linear function from $E$ into $E$, $\|U\| = \sup \{|Ux| : |x| = 1\}$, and $\mu[U] = \lim_{h \to +0} (\|I + hU\| - 1)/h$ (where $I$ is the identity function on $E$). Then $\mu[U] = \sup \{D_+[x, 0; U] : |x| = 1\}$ (see [5, Example 1]). Furthermore, if $V$ is a continuous linear function from $E$ into $E$, then $\mu[U + V] \leqq \mu[U] + \mu[V]$, $|\mu[U]| \leqq \|U\|$, and $|\mu[U] - \mu[V]| \leqq \|U - V\|$ (see [1, p. 41] and references cited there). Also, if $E$ is finite-dimensional and $U$ is associated with a square matrix, then Coppel [1, p. 41] gives formulas for computing $\mu[U]$ for three different norms on $E$.

*Example* 2.2. Suppose that $A$ is a function from $E$ into $E$ for which there is a number $M$ such that $D_+[x, y; A] \leqq M|x - y|$ for all $x$ and $y$ in $E$. Suppose further that $A$ has a Fréchet derivative $dA(x)$ for each $x$ in $E$ and that $dA$ is continuous on $E$. If $z$ is in $E$ and $R > 0$ is sufficiently small so that $dA$ is bounded on $S(z, R)$, then, by [5, Example 3] (see also [4, Theorem 3.1]),

$$\sup \{|x - y|^{-1} D_+[x, y; A] : x, y \in S(z, R), x \neq y\} \leqq \sup \{\mu[dA(x)] : x \in S(z, R)\}.$$

By Example 2.1, $|\mu[dA(x)] - \mu[dA(y)]| \leqq \|dA(x) - dA(y)\|$, and so, since $dA$ is continuous, $\lambda_+[z; A] \leqq \mu[dA(z)]$. Using the techniques in [4, Theorem 3.1], one can show that $\lambda_+[z; A] = \mu[dA(z)]$.

LEMMA 2.2 *Suppose that $A$ is a function from $E$ into $E$ for which there is a number $M$ such that $D_+[x, y; A] \leqq M|x - y|$ for all $x$ and $y$ in $E$. If $x$ and $y$ are in $E$, $z(\xi) = (1 - \xi)x + \xi y$ for each $\xi$ in* $[0, 1]$, *and $\rho$ is a* (Riemann) *integrable function* *from* [0, 1] *into the real numbers such that* $\lambda_+[z(\xi); A] \leqq \rho(\xi)$ *for each* $\xi$ *in* [0, 1], *then*

(i) $$D_+[x, y; A] \leqq |x - y| \int_0^1 \rho(s)\, ds$$

*and*

(ii) $$|Ax - Ay| \geqq -|x - y| \int_0^1 \rho(s)\, ds.$$

*Proof.* Since $|D_+[x, y; A]| \leqq |Ax - Ay|$, (ii) is an immediate consequence of (i). Furthermore, for each positive number $\varepsilon$ there is a continuous function $\sigma$ on [0, 1] such that $\sigma(\xi) > \rho(\xi)$ for each $\xi$ in [0, 1] and $\int_0^1 (\sigma(s) - \rho(s))\, ds < \varepsilon$. Thus (i) will follow if it is shown that if $\sigma$ is continuous on [0, 1] and $\lambda_+[z(\xi); A] < \sigma(\xi)$ for each $\xi$ in [0, 1], then $D_+[x, y; A] \leqq |x - y| \int_0^1 \sigma(s)\, ds$. Let $t$ be in [0, 1]. Since $\sigma$ is continuous and $\lambda_+[z(t); A] < \sigma(t)$, there is a neighborhood $U(t)$ of $t$ such that

$$D_+[z(\xi_1), z(\xi_2); A] \leqq |x - y| \int_{\xi_1}^{\xi_2} \sigma(s)\, ds \tag{2.1}$$

for each $\xi_1$ and $\xi_2$ in $U(t)$ with $\xi_1 \leqq \xi_2$. Here we have used the definition of $\lambda_+$, the fact that $|z(\xi_2) - z(\xi_1)| = |x - y|(\xi_2 - \xi_1)$, and the fact that $\int_{\xi_1}^{\xi_2} \sigma(s)\, ds = \sigma(\xi)(\xi_2 - \xi_1)$ for some $\xi$ in $[\xi_1, \xi_2]$. Consider the function

$$\psi(\xi) = D_+[x, z(\xi); A] - |x - y| \int_0^{\xi} \sigma(s)\, ds \tag{2.2}$$

for each $\xi$ in [0, 1]. Note that if $w_1$ and $w_2$ are in $E$ and $w_3$ is on the line segment from $w_1$ to $w_2$, then $D_+[w_1, w_2; A] \leqq D_+[w_1, w_3; A] + D_+[w_3, w_2; A]$ (this follows easily from (1.1) since $|w_2 - w_1| = |w_2 - w_3| + |w_3 - w_1|$). Consequently, if $0 \leqq \xi_1 \leqq \xi_2 \leqq 1$, then

$$\psi(\xi_2) - \psi(\xi_1) \leqq D_+[z(\xi_1), z(\xi_2); A] - |x - y| \int_{\xi_1}^{\xi_2} \sigma(s)\, ds. \tag{2.3}$$

By (2.1), for each $\xi$ in [0, 1] there is a neighborhood $U(\xi)$ of $\xi$ such that $\psi$ is nonincreasing in $U(\xi)$. Thus $\psi$ is nonincreasing in [0, 1] and so $\psi(1) - \psi(0) = \psi(1) \leqq 0$. This completes the proof of the lemma.

**3. The main results.** In this section we will establish two theorems concerning the asymptotic behavior of the solutions to (ADE). The basis conditions imposed on the function $A$ are the following:

(C1) $A$ is continuous on $E$.

(C2) For each number $M$ there is a number $N$ such that if $x$ is in $E$ with $|x| \geqq N$ then $|Ax| \geqq M$.

(C3) $D_+[x, y; A] \leqq 0$ for each $x$ and $y$ in $E$.

(C4) For each $z$ in $E$ there is a solution $u_z$ to (ADE) defined on some interval $[0, T_z)$ such that $u_z(0) = z$.

LEMMA 3.1. *Suppose $A$ satisfies the conditions* (C1)–(C4). *Then for each $z$ in $E$ there is a unique solution $u_z$ to* (ADE) *on $[0, \infty)$ such that $u_z(0) = z$. Furthermore, $u_z$ is bounded on $[0, \infty)$, the function $t \to |u_z'(t)|$ is nonincreasing on $[0, \infty)$, and $|u_z(t) - u_z(s)| \leqq |Az|\,|t - s|$ for each $s$ and $t$ in $[0, \infty)$. Also, if $w$ is in $E$, then $|u_z(t) - u_w(t)| \leqq |u_z(s) - u_w(s)| \leqq |z - w|$ whenever $0 \leqq s \leqq t$.*

*Proof.* Suppose that $u_z$ is defined on $[0, T)$ and $T < \infty$. Let $0 < h < T$ and for each $t$ in $[0, T - h)$ define $p(t) = |u_z(t + h) - u_z(t)|$. By Lemma 2.1 and condition (C3), $p'_+(t) = D_+[u_z(t + h), u_z(t); A] \leqq 0$, so that if $0 \leqq s \leqq t$, then

$$|u_z(t + h) - u_z(t)| \leqq |u_z(s + h) - u_z(s)|. \tag{3.1}$$

Dividing each side of (3.1) by $h$ and letting $h \to +0$, it follows that

$$|u_z'(t)| \leqq |u_z'(s)| \tag{3.2}$$

whenever $0 \leqq s \leqq t < T$. Letting $s = 0$ we have by (3.2) that $|u_z(t_1) - u_z(t_2)| \leqq |u_z'(0)|\,|t_1 - t_2|$ and it follows that $u_z(t)$ tends to a limit as $t$ tends to $T$ from below. By condition (C4), $u_z$ can be continued past $T$ and it follows that $u_z$ can be extended to $[0, \infty)$. It is immediate from (3.2) that the function $t \to |u'(t)|$ is nonincreasing and, if $t \geqq 0$, $|Au(t)| = |u'(t)| \leqq |u'(0)| = |Az|$, so $u$ is bounded on $[0, \infty)$ by condition (C2). If $w$ is in $E$ and $p(t) = |u_z(t) - u_w(t)|$, then by Lemma 2.1 and condition (C3), $p'_+(t) = D_+[u_z(t), u_w(t); A] \leqq 0$ and the last assertion of the lemma and the uniqueness of $u_z$ is immediate.

THEOREM 3.1. *In addition to the conditions* (C1)–(C4) *suppose there is a closed subset $\Omega$ of $E$ and a function $v$ from $E$ into $[0, \infty)$ which is positive definite with respect to $\Omega$ such that $\lambda_+[x; A] \leqq -v(x)$ for each $x$ in $E$. Then exactly one of the following occurs:*

(i) *Each solution $u_z$ to* (ADE) *tends to $\Omega$.*

(ii) *There is a unique point $x_c$ in $E - \Omega$ such that $Ax_c = 0$ and $\lim_{t\to\infty} u_z(t) = x_c$ for each $z$ in $E$.*

*Remark* 3.1. In this theorem we allow $\Omega$ to be the empty set and use the convention noted in Remark 2.1. In this case, conclusion (ii) must hold.

This theorem will be proved with a sequence of lemmas each of which is under the suppositions of Theorem 3.1. For the proof we suppose that (i) does not hold and show that (ii) must hold. Note that (i) and (ii) cannot hold simultaneously since $\Omega$ is closed.

LEMMA 3.2. *If* (i) *does not hold, there are a positive number $\alpha$ less than* 1, *a solution $u$ to* (ADE) *on $[0, \infty)$, and a sequence $(t_k)_1^\infty$ in $[0, \infty)$ such that $t_{k+1} \geqq t_k + 1$ and $d(u(t_k), \Omega) \geqq \alpha$. Furthermore, there is a positive number $\beta$ less than one such that if $k \geqq 1$ and $s$ is in $[t_k, t_k + \beta]$, then $u(s)$ is in $S(u(t_k), \alpha/2)$.*

*Proof.* Since (i) does not hold the existence of $u$ and the sequence $(t_k)_1^\infty$ such that $d(u(t_k), \Omega) \geqq \alpha > 0$ is obvious. By Lemma 3.1, if $\beta = \alpha(1 + 2|Au(0)|)^{-1}$, $k \geqq 1$, and $s$ is in $[t_k, t_k + \beta]$, then $|u(s) - u(t_k)| \leqq |Au(0)|\,|s - t_k| \leqq |Au(0)|\beta < \alpha/2$ and the last assertion of the lemma is true.

LEMMA 3.3. *If $u$ is as in Lemma* 3.2, *then* $\lim_{t\to\infty} u'(t) = 0$.

*Proof.* Let $\alpha$, $\beta$ and $(t_k)_1^\infty$ be as in Lemma 3.2. By Lemma 3.1, $u$ is bounded on $[0, \infty)$; so, let $R > 1$ be such that $|u(t)| < R - 1$ for all $t \geqq 0$ and let $Q = S(0, R)$. Then there is a positive number $\delta$ such that if $x$ is in $Q - S(\Omega, \alpha/2)$, $v(x) \geqq \delta$. Now let $0 < h < \beta/2$ and for each $t$ in $[0, \infty)$ let $p(t) = |u(t + h) - u(t)|$. By

Lemma 2.1, $p'_+(t) = D_+[u(t + h), u(t); A]$. Define the function $f$ from $[0, \infty)$ into the real numbers by $f(t) = -\delta$ if $t$ is in $[t_k, t_k + \beta/2]$ for some positive integer $k$ and $f(t) = 0$ otherwise. Since $S(u(t_k), \alpha/2)$ is contained in $Q - S(\Omega, \alpha/2)$ for each $k \geqq 1, \lambda_+[x; A] \leqq -\nu(x) \leqq -\delta$ for all $x$ in $S(u(t_k), \alpha/2)$. By part (i) of Lemma 2.2, $D_+[x, y; A] \leqq -\delta|x - y|$ for each $x$ and $y$ in $S(u(t_k), \alpha/2)$. By the choice of $\beta$ (see Lemma 3.2), if $t$ is in $[t_k, t_k + \beta/2]$ and $0 < h < \beta/2$, then $u(t)$ and $u(t + h)$ are in $S(u(t_k), \alpha/2)$. Thus $D_+[u(t + h), u(t); A] \leqq -\delta|u(t + h) - u(t)|$ whenever $t$ is in $[t_k, t_k + \beta/2]$ and $D_+[u(t + h), u(t); A] \leqq 0$ otherwise. In particular, $p'_+(t) \leqq f(t)p(t)$ for all $t$ in $[0, \infty)$ so that

$$|u(t + h) - u(t)| \leqq |u(h) - u(0)| \exp\left(\int_0^t f(s)\,ds\right) \tag{3.3}$$

for each $t$ in $[0, \infty)$ and $0 < h < \beta/2$. Dividing each side of (3.3) by $h$ and letting $h \to +0$, we have $|u'(t)| \leqq |u'(0)| \exp\left(\int_0^t f(s)\,ds\right)$. It follows easily from the definition of $f$ that $\lim_{t\to\infty} \int_0^t f(s)\,ds = -\infty$, and the assertion of the lemma follows.

LEMMA 3.4. *There is an $x_c$ in $E$ such that $d(x_c, \Omega) \geqq \alpha$ and $\lim_{t\to\infty} u(t) = x_c$ (where $u$ and $\alpha$ are as in Lemma* 3.2).

*Proof.* We will use the notations of Lemma 3.3. Since $|u(s)| < R$ for all $s$ in $[0, \infty)$, if $k \geqq 1$ and $z(\xi) = (1 - \xi)u(t_k) + \xi u(t)$ for each $\xi$ in $[0, 1]$, then $|z(\xi) - u(t_k)| = \xi|u(t) - u(t_k)| < 2\xi R$. *Thus if* $0 \leqq \xi \leqq \alpha/(4R)$, *then* $z(\xi)$ *is in* $S(u(t_k), \alpha/2)$ so that $\lambda_+[z(\xi); A] \leqq -\delta$. By taking $\rho(s) = 0$, if $s$ is in $[\alpha/(4R), 1]$ and $\rho(s) = -\delta$ if $s$ is in $[0, \alpha/(4R))$, then $\lambda_+[z(\xi); A] \leqq \rho(\xi)$ for each $\xi$ in $[0, 1]$. Since $\int_0^1 \rho(s)\,ds = -\delta\alpha/(4R)$, we have by part (ii) of Lemma 2.2 that

$$|Au(t) - Au(t_k)| \geqq \delta\alpha|u(t) - u(t_k)|/(4R) \tag{3.4}$$

for all $t$ in $[0, \infty)$ and $k \geqq 1$. Now let $\varepsilon$ be a positive number. Since $t_k \to \infty$ as $k \to \infty$ we have by Lemma 3.3 that there is a positive integer $n$ such that if $t \geqq t_n$ then $|u'(t)| \leqq \delta\alpha\varepsilon/(16R)$. Thus if $t, s \geqq t_n$, then by (3.4) and the choice of $t_n$,

$$\begin{aligned} |u(t) - u(s)| &\leqq |u(t) - u(t_n)| + |u(t_n) - u(s)| \\ &\leqq 4R(|Au(t) - Au(t_n)| + |Au(s) - Au(t_n)|)/(\delta\alpha) \\ &\leqq 4R(|u'(t)| + |u'(s)| + 2|u'(t_n)|)/(\delta\alpha) \\ &\leqq \varepsilon. \end{aligned}$$

Consequently the net $(u(t))_{t\geqq 0}$ is Cauchy and so it tends to a limit $x_c$ in $E$ as $t$ tends to $\infty$. Since $\lim_{k\to\infty} u(t_k) = x_c$ and $d(u(t_k), \Omega) \geqq \alpha$ for each $k \geqq 1$ it is immediate that $d(x_c, \Omega) \geqq \alpha$, and the lemma is proved.

LEMMA 3.5. *With the notations of Lemma* 3.4, $Ax_c = 0$ *and if $z$ is in $E$ and $u_z$ is the solution to* (ADE) *such that $u_z(0) = z$, then $\lim_{t\to\infty} u_z(t) = x_c$.*

*Proof.* By condition (C1) and Lemmas 3.3 and 3.4,

$$Ax_c = \lim_{t\to\infty} Au(t) = \lim_{t\to\infty} u'(t) = 0.$$

Let $z$ be in $E$ and let $u_z$ be the solution to (ADE) defined on $[0, \infty)$ such that $u_z(0) = z$.

Since $u_z$ is bounded on $[0, \infty)$ let $R > |x_c|$ be such that $|u_z(t)| < R$ for all $t$ in $[0, \infty)$. Since $d(x_c, \Omega) \geqq \alpha$, there is a $\delta > 0$ such that if $x$ is in $S(x_c, \alpha/2)$, then $v(x) \geqq \delta$, and hence $\lambda_+[x; A] \leqq -\delta$. As in the proof of Lemma 3.4, if $z(\xi) = (1 - \xi)x_c + \xi u_z(t)$ and $\rho(\xi) = -\delta$ if $\xi$ is in $[0, \alpha/(4R))$, $\rho(\xi) = 0$ if $\xi$ is in $[\alpha/(4R), 1]$ and $p(t) = |u_z(t) - x_c|$, then $p'_+(t) = D_+[u_z(t), x_c; A] \leqq p(t) \int_0^1 \rho(s)\, ds \leqq -\delta\alpha p(t)/(4R)$ for all $t$ in $[0, \infty)$. Consequently,

$$|u_z(t) - x_c| \leqq |z - x_c| \exp(-\delta\alpha t/(4R)),$$

and it follows that $\lim_{t\to\infty} u_z(t) = x_c$, and the lemma is true.

Thus, if conclusion (i) does not hold, conclusion (ii) must hold, and the proof of Theorem 3.1 is complete.

*Example* 3.1. Let $R$ denote the space of real numbers and let $E = R^2$ with the norm $|\cdot|$ on $R^2$ defined by $|(x, y)| = \max\{|x|, |y|\}$ for each $(x, y)$ in $R^2$. Define $A(x, y) = (-x^3 + 6\exp(y - x), \sin(x) - 2y)$ for all $(x, y)$ is $R^2$. Then $A$ is Fréchet differentiable on $R^2$ and $dA(x, y)$ is associated with the matrix

$$\begin{pmatrix} -3x^2 - 6\exp(y - x) & 6\exp(y - x) \\ \cos(x) & -2 \end{pmatrix}.$$

Using the formula in [1, p. 41] we have

$$\begin{aligned} \mu[dA(x, y)] &= \max\{-3x^2 - 6\exp(y - x) + 6\exp(y - x), -2 + |\cos(x)|\} \\ &\leqq \max\{-3x^2, -1\}. \end{aligned}$$

If $v(x, y) = 1$ if $x^2 \geqq \frac{1}{3}$ and $v(x, y) = 3x^2$ if $x^2 < \frac{1}{3}$, then $v$ is positive definite with respect to $\Omega = \{(0, y) : y \in R\}$. By Example 2.2, $\lambda_+[(x, y); A] \leqq \mu[dA(x, y)] \leqq -v(x, y)$ for each $(x, y)$ in $R^2$, and since $|A(x, y)| \to \infty$ as $|(x, y)| \to \infty$, each of the conditions of Theorem 3.1 is fulfilled. One can easily check that there is an $(x_c, y_c)$ in $R^2$ such that $1 < x_c < \exp(1)$ and $y_c = 3\ln(x_c) + x_c - \ln(6)$ and $A(x_c, y_c) = 0$, so that conclusion (ii) of Theorem 3.1 prevails.

*Remark* 3.2. If, in Example 3.1, we use the norm $|\cdot|$ on $R^2$ defined by $|(x, y)| = (x^2 + y^2)^{1/2}$, then by the formula in [1, p. 41],

$$\mu[dA(0, 0)] = (-8 + \sqrt{65})/2 > 0,$$

so that the conditions of Theorem 3.1 are not satisfied with this norm.

LEMMA 3.6. *Assume that each of the conditions of Theorem* 3.1 *holds and suppose that there are two subsets $\Omega_1$ and $\Omega_2$ of $E$ such that $\Omega = \Omega_1 \cup \Omega_2$ and $d(\Omega_1, \Omega_2) = \inf\{d(x, \Omega_2) : x \in \Omega_1\} = \gamma > 0$. Then if conclusion* (i) *to Theorem* 3.1 *holds, either every solution to* (ADE) *tends to $\Omega_1$ or every solution to* (ADE) *tends to $\Omega_2$.*

*Proof.* It is easy to see that if $u$ is a solution to (ADE) then either $u$ tends to $\Omega_1$ or $u$ tends to $\Omega_2$. Suppose that $u$ tends to $\Omega_1$ and, for contradiction, assume that $v$ is a solution to (ADE) such that $v$ tends to $\Omega_2$. Let $T > 0$ be such that if $t$ is in $[T, \infty)$, then $u(t)$ is in $S(\Omega_1, \gamma/3)$ and $v(t)$ is in $S(\Omega_2, \gamma/3)$. Then if $t \geqq T$,

$$|u(t) - v(t)| \geqq \gamma/3. \tag{3.5}$$

Suppose that $|u(t)|, |v(t)| < R$. Let $\delta > 0$ be such that if $x$ is in $S(0, R) - S(\Omega, \gamma/3)$, then $v(x) \geqq \delta$. Let $\sigma = \gamma/(6R)$; and define $f(t) = 0$ if $t$ is in $[0, T)$, and let $f(t)$

$= -\delta\sigma$ if $t$ is in $[T, \infty)$. For each $t$ in $[0, \infty)$ let $p(t) = |u(t) - v(t)|$. By Lemma 2.1 and condition (C3), if $t$ is in $[0, T)$, then $p'_+(t) = D_+[u(t), v(t); A] \leqq 0 = f(t)p(t)$. Now let $t$ be in $[T, \infty)$, and for each $\xi$ in $[0, 1]$, let $z(\xi) = (1 - \xi)u(t) + \xi v(t)$, and let $g(\xi) = d(z(\xi), S(\Omega_1, \gamma/3))$. Then $g$ is continuous, $g(0) = 0$, and $g(1) \geqq \gamma/3$. Let $\xi_0$ be the member of $[0, 1)$ such that $g(\xi_1) = 0$ and $g(\xi) > 0$ for each $\xi$ in $(\xi_1, 1]$. Since $|z(\xi) - z(\xi_1)| \leqq |\xi - \xi_1|(|u(t)| + |v(t)|) \leqq |\xi - \xi_1|2R$, if $\xi$ is in $[\xi_1, \xi_1 + \sigma)$, then $z(\xi)$ is in $S(0, R) - S(\Omega, \gamma/3)$ and hence $\lambda_+[z(\xi); A] \leqq -v(z(\xi)) \leqq -\delta$. Consequently, if $\rho(\xi) = -\delta$ for $\xi$ in $(\xi_1, \xi_1 + \sigma)$ and $\rho(\xi) = 0$ if $\xi$ is in $[0, 1] - (\xi_1, \xi_1 + \sigma)$, then $\int_0^1 \rho(s)\, ds = -\delta\sigma$; so, by part (i) of Lemma 2.2, $D_+[u(t), v(t); A] \leqq -\delta\sigma|u(t) - v(t)|$. By Lemma 2.1, $p'_+(t) \leqq -\delta\sigma p(t) = f(t)p(t)$, and it follows that $|u(t) - v(t)| \leqq |u(0) - v(0)| \exp\left(\int_0^t f(s)\, ds\right)$ for all $t$ in $[0, \infty)$. In particular, if $t$ is in $[T, \infty)$, then $|u(t) - v(t)| \leqq |u(0) - v(0)| \exp(-\delta\sigma(t - T))$. But this implies that $\lim_{t\to\infty} |u(t) - v(t)| = 0$, which is a contradiction to (3.5). This contradiction proves the lemma.

THEOREM 3.2. *In addition to the suppositions of Theorem* 3.1 *suppose that the intersection of* $\Omega$ *with each bounded subset of* $E$ *is finite. Then there is a unique point* $x_c$ *in* $E$ *such that* $Ax_c = 0$ *and* $\lim_{t\to\infty} u_z(t) = x_c$ *for each solution* $u_z$ *to* (ADE).

*Proof.* If conclusion (ii) of Theorem 3.1 prevails, the theorem is obvious. If conclusion (i) prevails, then it is an immediate consequence of Lemma 3.6 that there is a unique member $x_c$ of $\Omega$ such that

$$\lim_{t\to\infty} u_z(t) = x_c \tag{3.6}$$

for all $z$ in $E$. Since (3.6) implies that $Az \neq 0$ if $z \neq x_c$, it remains to show that $Ax_c = 0$. If $z$ is in $E$, then by (3.6) and condition (C1), $\lim_{t\to\infty} u'_z(t) = \lim_{t\to\infty} Au(t) = Ax_c$. By Lemma 3.1, the function $t \to |u'_z(t)|$ is nonincreasing so that $|Az| = |u'_z(0)| \geqq |u'_z(t)| = |Au_z(t)|$ for all $t$ in $[0, \infty)$. Thus by (3.6) and condition (C1), $|Az| \geqq |Ax_c|$ for all $z$ in $E$. Taking $z = x_c$, we have that $|Ax_c| \leqq |Au_{x_c}(t)| = |u'_{x_c}(t)|$ for all $t$ in $[0, \infty)$, and by Lemma 3.1, $|Ax_c| = |u'_{x_c}(0)| \geqq |u'_{x_c}(t)|$, so that

$$|u'_{x_c}(t)| = |Ax_c| \tag{3.7}$$

for all $t$ in $[0, \infty)$. As in the proof of Lemma 3.3, we can show that the function $t \to |u'_{x_c}(t)|$ is strictly decreasing when $u_{x_c}(t)$ is not in $\Omega$. Thus by (3.7), $u_{x_c}(t)$ must remain in $\Omega$ and it is immediate that $u_{x_c}(t) = x_c$ for all $t$ in $[0, \infty)$. Consequently, $Ax_c = u'_{x_c}(0) = 0$ and the proof of Theorem 3.2 is complete.

*Example* 3.2. Suppose that $E$ is the space of real numbers and $Ax = -x^3$ for all $x$ in $E$. Then $\lambda_+[z; A] = -3x^2$, and if $\Omega = \{0\}$ and $v(x) = 3x^2$, each of the suppositions of Theorem 3.2 is fulfilled.

*Example* 3.3. Let $R$ be the space of real numbers and let $E = R^2$ with the norm $|\cdot|$ on $R^2$ defined by $|(x, y)| = (x^2 + y^2)^{1/2}$. Define $A(x, y) = (-x + \sin(y), -y + \cos(x))$ for each $(x, y)$ in $R^2$. Then $A$ is Fréchet differentiable on $R^2$ and $dA(x, y)$ is associated with the matrix

$$\begin{pmatrix} -1 & \cos(y) \\ -\sin(x) & -1 \end{pmatrix}.$$

Using the formula in [1, p. 41], we have

$$\mu[dA(x, y)] = \max\{-1 + (\cos(y) - \sin(x))/2, -1 - (\cos(y) - \sin(x))/2\}$$
$$= -1 + |\cos(y) - \sin(x)|/2.$$

Thus if $\Omega = \{(\pi/2 + n\pi, m\pi)$: $m$ even and $n$ odd, or $m$ odd and $n$ even$\}$ and $v(x, y) = 1 - |\cos(y) - \sin(x)|/2$, then each of the suppositions of Theorem 3.2 are fulfilled.

*Remark* 3.3. If, in Example 3.3, we use the norm $|\cdot|$ on $R^2$ defined by $|(x, y)| = \max\{|x|, |y|\}$, then $\mu[dA(x, y)] = \max\{-1 + |\cos(y)|, -1 + |\sin(x)|\}$ so that $\Omega = \{(x, y): x \in R,\ y = n\pi$, or $y \in R,\ x = \pi/2 + n\pi$, $n$ an integer$\}$. Thus if $v(x, y) = -\mu[dA(x, y)]$, the suppositions of Theorem 3.1 are fulfilled but those of Theorem 3.2 are not.

*Remark* 3.4. The conditions (C1)–(C4) given at the beginning of this section are not all independent. If $E$ is finite-dimensional, then (C1) implies (C4). If $E$ is not finite-dimensional, then (C1) does not imply (C4); however, in [6, Theorem 1], it is shown that (C1) and (C3) imply (C4). It should also be noted that if $A$ satisfies the suppositions of Theorem 3.1 (respectively, Theorem 3.2), $y$ is in $E$, and $A_y x = Ax - y$ for each $x$ in $E$, then $A_y$ satisfies the suppositions of Theorem 3.1 (respectively, Theorem 3.2). In particular, if $A$ satisfies the suppositions of Theorem 3.2, then $A$ is a bijection since, for each $y$ in $E$, there is a unique $x_y$ in $E$ such that $Ax_y - y = 0$.

**Acknowledgment.** The author would like to express his appreciation to the referee for several helpful suggestions, especially concerning § 2. The proof of Lemma 2.2 given here was suggested by the referee.

REFERENCES

[1] W. A. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, Heath, Boston, 1965.
[2] N. N. Krasovskii, *On the stability in the large of solutions of a nonlinear system of nonlinear differential equations*, Prikl. Mat. Mekh., 18 (1954), pp. 735–737.
[3] L. Markus and H. Yamabe, *Global stability criteria for differential systems*, Osaka J. Math., 12 (1960), pp. 305–317.
[4] R. H. Martin, Jr., *Bounds for solutions of a class of nonlinear differential equations*, J. Differential Equations, 8 (1970), in press.
[5] ———, *A theorem on critical points and global asymptotic stability*, J. Math. Anal. Appl., to appear.
[6] ———, *A global existence theorem for autonomous differential equations in a Banach space*, Proc. Amer. Math. Soc., to appear.

SIAM J. MATH. ANAL.
Vol. 1, No. 4, November 1970

# NONCONTINUOUS LYAPUNOV FUNCTIONS AND EXTENDABILITY OF SOLUTIONS*

STEPHEN R. BERNFELD†

**1. Introduction.** A characterization of boundedness, equiboundedness, and uniform boundedness of solutions of

$$\text{(E)} \qquad \dot{x} = f(t, x)$$

in terms of Lyapunov functions has been given by Yoshizawa [6]. In particular, for $f(t, x)$ continuous, he has constructed Lyapunov functions in terms of the solution funnels. In order to obtain the behavior of the Lyapunov function along solutions it is necessary to know what the solutions are. In contrast, when $f(t, x)$ is Lipschitz, a characterization of uniform boundedness in terms of Lipschitz Lyapunov functions has also been obtained by Yoshizawa [6]: moreover, the behavior of the Lyapunov function along solutions can be found without knowing the solutions. However, even for $f(t, x)$ Lipschitz, there has been no characterization of the boundedness and equiboundedness of the solutions of (E) in terms of Lyapunov functions whose behavior along solutions can be found without knowledge of the solutions. With the use of a result of J. Yorke [5], we construct a noncontinuous Lyapunov function which will serve as a test for the boundedness, equiboundedness and uniform boundedness of the solutions of (E) when they are unique to the right; and the behavior of the Lyapunov function along solutions can be found without knowledge of the solution.

In addition, a characterization in terms of Lyapunov functions of the global existence of solutions is provided which extends the results of Kato and Strauss [3] to systems which only have uniqueness to the right. The construction of these Lyapunov functions are more "natural" than those constructed before and seem to have more applications.

**2. Preliminaries.** Let $R^n$ denote Euclidean $n$-space. $|\cdot|$ will denote the Euclidean norm. For $x, y \in R^n$ define $d(x, y) = |x - y|$. Denote a solution of (E) through $(t_0, x_0) \in R \times R^n$ by $x(\cdot, t_0, x_0)$. A solution through $(t_0, x_0)$ exists in the future if $x(t, t_0, x_0)$ exists for all $t > t_0$ and exists in the past if $x(t, t_0, x_0)$ exists for all $t < t_0$. A solution exists forever if it exists in the past and in the future.

For $(t_0, x_0) \in R \times R^n$ define the positive and negative solution funnels as

$$F^+_{t_0,x_0} = \{(t, x(t)) : t \geqq t_0, x(t_0) = x_0\} \subset R^{n+1}$$

and

$$F^-_{t_0,x_0} = \{(t, x(t)) : t \leqq t_0, x(t_0) = x_0\} \subset R^{n+1},$$

* Received by the editors February 3, 1970.

† Department of Mathematics, University of Missouri, Columbia, Missouri 65201. This research was supported in part by the National Science Foundation under Grant GP-6167.

respectively, where $\dot{x}(t) = f(t, x(t))$. The solution funnel through $(t_0, x_0)$, denoted by $F_{t_0,x_0}$, is defined as

$$F_{t_0,x_0} = F^+_{t_0,x_0} \cup F^-_{t_0,x_0}.$$

We define the $\tau$ cross section $F_{t_0,x_0}(\tau) = F_{t_0,x_0} \cap (\tau \times R^n) \subset R^n$, and define the set $F_{t_0,x_0}[a, b] = \bigcup_{\tau\in[a,b]} F_{t_0,x_0}(\tau) \subset R^n$.

We now define the following distances between points and sets and between sets by:

$$\begin{aligned} d(x, T) &= \inf\{d(x, y); y \in T\}, \\ \rho^*(S, T) &= \sup\{d(x, T); x \in S\}, \qquad (1) \\ \rho(S, T) &= \max\{\rho^*(S, T), \rho^*(T, S)\}. \end{aligned}$$

If the sets are compact, then $\rho$ is the Hausdorff metric.

The following properties of solution funnels will prove useful and are slight generalizations of known results (see [4]).

LEMMA 1. *Suppose all solutions of* (E) *exist forever. If we consider any point* $(t, x) \in R \times R^n$ *and any closed interval* $[a, b]$, *then for each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\rho^*(F_{\tau,\mu}[a, b], F_{t,x}[a, b]) < \varepsilon$$

*whenever*

$$d(\mu, x) + |t - \tau| < \delta.$$

LEMMA 2. *Consider the system* (E). *For each point* $(t_0, x_0) \in R \times R^n$ *and each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$\rho(F_{t_1,x_1}[t_1 - s, t_1 + s], (x_1)) < \varepsilon$$

*for all s such that* $|s| < \delta$, *and all* $(t_1, x_1)$ *such that* $d(x_1, x_0) + |t_1 - t_0| < \delta$.

We now define various types of boundedness (see [6]).

DEFINITION 1. The solutions of (E) are *bounded* if for each $(t_0, x_0) \in R \times R^n$ there exists a $\beta > 0$ such that $|x(t, t_0, x_0)| < \beta$ for all $t \geqq t_0$, where $\beta$ may depend on the solution.

DEFINITION 2. The solutions of (E) are *equibounded* if for any $\alpha > 0$ and $t_0 \in R$ there exists $\beta(t_0, \alpha)$ such that whenever $|x_0| < \alpha$ then $|x(t, t_0, x_0)| < \beta$ for all $t \geqq t_0$.

DEFINITION 3. The solutions of (E) are *uniformly bounded* if the $\beta$ in Definition 2 is independent of $t_0$.

DEFINITION 4. Let $V: R \times R^n \to R$. We say $V(t, x)$ is *radially unbounded* if

$$V(t, x) \to \infty \text{ as } |x| \to \infty \qquad (2)$$

uniformly in $t$ for $t$ in $R$. $V(t, x)$ is *mildly unbounded* if (2) holds uniformly in $t$ for $t$ in compact sets of $R$. We say $V(t, x)$ is *bounded on bounded sets* if

$$V(t, x) \leqq K(t, \alpha) \quad \text{for } |x| \leqq \alpha \qquad (3)$$

for some $K: R \times R \to R$.

$V$ is *uniformly bounded above* if $K$ is independent of $t$.

In the usual Lyapunov theory, $V(t, x)$ is locally Lipschitz and the time derivative of $V$ along solutions of (E) can be found without knowledge of the solution. J. Yorke [5] developed a more general derivative of the function $V$, which is only assumed to be lower semicontinuous, along solutions of (E) by defining

$$\overset{*}{V}(t, x) = \liminf_{\substack{y \to f(t,x) \\ h \to 0^+}} h^{-1}(V(t + h, x + hy) - V(t, x)). \tag{4}$$

Using the following result of Yorke [5], we are able to discuss the behavior of solutions along lower semicontinuous functions.

THEOREM 1. *Let $V: R \times R^n \to R$ be lower semicontinuous. Assume solutions of* (E) *are unique to the right. Then the following are equivalent:*

$$\overset{*}{V}(t, x) \leqq 0 \tag{5}$$

*and*

(6) $V(t, x(t))$ *is a nonincreasing function of* $t$, *where* $x(t)$ *is a solution of* (E).

**3. Results.** We are now able to characterize boundedness, equiboundedness and uniform boundedness of the solutions of (E) with the use of a single Lyapunov function whose behavior along solutions can be found without knowing the solutions.

THEOREM 2. *Assume all solutions of* (E) *are unique to the right. Then all solutions are bounded if and only if there exists a lower semicontinuous function $V(t, x)$ satisfying*

(a) *$V(t, x)$ is radially unbounded, and*
(b) $\overset{*}{V}(t, x) \leqq 0$.

*Furthermore, solutions are equibounded if and only if $V(t, x)$ is bounded on bounded sets. Finally solutions are uniformly bounded if and only if $V(t, x)$ is uniformly bounded above.*

*Proof.* Assume there exists a lower semicontinuous function $V(t, x)$ satisfying (a) and (b). Using Theorem 1 we have that $V(t, x(t))$ is a nonincreasing function of $t$. Assume there exists a point $(t_0, x_0)$ and a solution $x(\cdot, t_0, x_0)$ such that $x(t, t_0, x_0)$ is not bounded for $t > t_0$. Then there exists a sequence of points $\{t_n\}$ such that $t_n > t_0$ and

$$|x(t_n, t_0, x_0)| \to \infty \quad \text{as } n \to \infty.$$

Using (a) we have $V(t_n, x(t_n, t_0, x_0)) \to \infty$; but since

$$V(t_n, x(t_n, t_0, x_0)) \leqq V(t_0, x_0)$$

we arrive at a contradiction.

Conversely, define

$$V(t, x) = \sup_{s \geqq 0} |x(t + s, t, x)|.$$

Since $V(t, x) \geqq |x|$ we have that $V$ is radially unbounded. We now show that $V$ is lower semicontinuous. Consider any point $(t, x) \in R \times R^n$. There exists a sequence of points $\{t_n\}$, where we can assume $t_n > t$, such that $|x(t_n, t, x)| \nearrow V(t, x)$ as

$n \to \infty$. For each $\varepsilon > 0$ there exists an $N$ such that

$$V(t, x) \leqq |x(t_N, t, x)| + \varepsilon. \tag{7}$$

Since solutions are unique to the right, then by continuous dependence we have the existence of a $\delta(\varepsilon)$ such that

$$|x(t_N, t, x) - x(t_N, \tau, \mu)| < \varepsilon \tag{8}$$

whenever $d(\mu, x) + |\tau - t| < \delta$ and $\tau < t_N$. Combining (7) and (8) we have

$$V(t, x) \leqq |x(t_N, \tau, \mu)| + 2\varepsilon;$$

and since $|x(t_N, \tau, \mu)| \leqq V(\tau, \mu)$, we obtain

$$V(t, x) \leqq V(\tau, \mu) + 2\varepsilon$$

for all $(\tau, \mu)$ in the $\delta$-neighborhood of $(t, x)$. Hence

$$V(t, x) \leqq \liminf_{(\tau,\mu)\to(t,x)} V(\tau, \mu) + 2\varepsilon.$$

Since $\varepsilon$ is arbitrary we obtain

$$V(t, x) \leqq \liminf_{(\tau,\mu)\to(t,x)} V(\tau, \mu);$$

that is, $V$ is lower semicontinuous.

It follows easily that $V$ is nonincreasing along solutions; and using Theorem 1 we have that $\overset{*}{V}(t, x) \leqq 0$. Hence, the first part of the proof is complete.

We now prove solutions are equibounded under the added assumption that $V$ is bounded on bounded sets. Assume solutions are not equibounded. Then there exist an $\alpha > 0$, $t_1 \in R$, a sequence of points $\{x_n\} \in R^n$ such that $|x_n| \leqq \alpha$, and a sequence of points $\{t_n\}$, $t_n > t_1$, such that

$$|x(t_n, t_1, x_n)| \to \infty \quad \text{as } n \to \infty. \tag{9}$$

Using the conditions on $V$ we have

$$K(t_1, \alpha) \geqq V(t_1, x_n) \geqq V(t_n, x(t_n, t_1, x_n))$$

which contradicts (9) since $V(t_n, x(t_n, t_1, x_n)) \to \infty$.

Conversely, from the definition of equiboundedness we have, for each $\alpha > 0$ and each $t$, the existence of $\beta(t, \alpha)$ such that $V(t, x) \leqq \beta(t, \alpha)$ whenever $|x| \leqq \alpha$.

We now show that solutions are uniformly bounded when $V(t, x)$ is uniformly bounded above. If solutions are not uniformly bounded, then there exists an $\alpha_1 > 0$, two sequences of points $\{\tau_n\}$, $\{t_n\}$, where $\tau_n > t_n$, and a sequence of points $\{x_n\}$, where $|x_n| < \alpha_1$, such that

$$|x(\tau_n, t_n, x_n)| \to \infty \quad \text{as } n \to \infty. \tag{10}$$

Using the conditions on $V(t, x)$ and (10) we have

$$K_1(\alpha_1) \geqq V(t_n, x_n) \geqq V(\tau_n, x(\tau_n, t_n, x_n)) \to \infty,$$

where $K_1(\cdot)$ is the uniform upper bound on $V$. This is a contradiction.

Conversely, from the definition of uniform boundedness, it follows immediately that $V(t, x)$ is uniformly bounded above. This completes the proof of Theorem 2.

*Remarks.* We have thus characterized the various types of boundedness in terms of a single noncontinuous Lyapunov function whose behavior along solutions can be calculated without knowing the solutions. As mentioned before no characterization of boundedness has been done even when $f$ is locally Lipschitz. Moreover, the only characterization of equiboundedness (Kato and Strauss [3]) has been given under the added assumption that solutions exist in the past.

The fact that $V$ is not continuous is crucial in distinguishing between boundedness and equiboundedness since a continuous Lyapunov function is always bounded above. We also notice that when $f(t, x)$ is independent of $t$ the Lyapunov function is also independent of $t$. Moreover, when $V$ is continuous, examples can be provided in which there exists no $V$ independent of $t$ such that $\overset{*}{V}(t, x) \leqq 0$ when $f$ is independent of $t$.

Although the Lyapunov function constructed in Theorem 2 is not Lipschitz in $x$, in general, we shall show in the following example that under the conditions of Theorem 2 no Lipschitz Lyapunov function can be constructed which is radially unbounded and nonincreasing along solutions. Hence we are not able to use the usual generalized derivative of Yoshizawa [6] which requires the Lyapunov function to be Lipschitz in $x$.

*Example.* Consider the scalar equation

$$\text{(S)} \qquad \dot{x} = \begin{cases} (x - n)(n + 1 - x)^{1/2}, & n \leqq x \leqq n + 1, \\ 0, & x < 0, \end{cases}$$

where $n = 0, 1, 2, \cdots$. Solutions are unique to the right and bounded. We shall assume there exists a Lipschitz function $V(t, x)$ which is radically unbounded and nonincreasing along solutions, and we shall arrive at a contradiction. Given the point $(0, 1)$, there exists a point $(0, x_1)$, $x_1 > 1$, such that

$$|V(0, x_1) - V(0, 1)| < 1/2$$

since $V$ is Lipschitz in $x$. Letting $x_1(\cdot)$ be the solution through $(0, x_1)$, there exists a point $(t_1, x_2)$ such that $t_1 > 0$, $x_2 > 2$, $x_1(t_1) = 2$ and

$$|V(t_1, x_2) - V(t_1, x_1(t_1))| < 1/2^2.$$

Continuing this process, we obtain at the $n$th step a point $t_n > t_{n-1}$, a solution $x_n(\cdot)$ such that $x_n(t_n) = n + 1$ and a point $(t_n, x_{n+1})$, where $x_{n+1} > n + 1$, such that

$$|V(t_n, x_{n+1}) - V(t_n, x_n(t_n))| < 2^{-(n+1)}.$$

Since $V$ is nonincreasing along solutions we obtain

$$\begin{aligned} |V(t_n, x_{n+1}) - V(0, 1)| < {} & |V(t_n, x_{n+1}) - V(t_n, x_n(t_n))| \\ & + |V(t_{n-1}, x_n) - V(t_{n-1}, x_{n-1}(t_{n-1}))| + \cdots + \\ & + |V(0, x_1) - V(0, 1)| \leqq \sum_{k=1}^{n} 2^{-k} < 1. \end{aligned}$$

Therefore,

$$V(0,1) > V(t_n, x_{n+1}) - 1 \quad \text{for all } n.$$

Since $V$ is radially unbounded we have that $V(0,1)$ is not defined, thus completing our claim.

In order to characterize the existence in the future of solutions on compact sets one may use the rather complicated Okamura function (see [6, p. 8]). Similarly we may use a variation of this function in order to characterize the existence in the past of solutions on compact sets. However, the Okamura function cannot be simultaneously defined in both directions in order to obtain a characterization of the existence of solutions forever.

In the following discussion we shall first construct a Lyapunov function for solutions existing in the future and then construct one for solutions existing in the past. We shall then characterize global existence in terms of a Lyapunov function which is required to have certain properties on compact subsets of $R$.

We assume solutions of (E) are unique to the right and construct a Lyapunov function which is equivalent to the existence of solutions in the future. On the set $(-\infty, T)$, for any $T$, assuming solutions exist, define

$$V_T(t,x) = \sup_{t \leqq s \leqq T} |x(s,t,x)| \quad \text{for } (t,x) \in (-\infty, T) \times R^n.$$

Using Lemmas 1 and 2 we can show that $V_T(t,x)$ is continuous, mildly unbounded, and $\overset{*}{V}_T(t,x) \leqq 0$. These conditions are thus equivalent to the existence of solutions on $[t_0, T]$ for any initial point $(t_0, x_0) \in (-\infty, T) \times R^n$.

In previous work dealing with Lyapunov functions and existence in the future (Kato and Strauss [3], Bernfeld [1]), the construction of these functions depended on the behavior of the solutions in the past. When this behavior cannot be determined, such as in delay equations or in global semidynamical systems (see [2]) (when $f$ is autonomous, for example), then this type of construction cannot be made. We will construct a more "natural" Lyapunov function which depends upon the behavior of the solutions in the future. One consequence of this is that if $f(t,x) = f(t+w,x)$ for all $(t,x) \in R \times R^n$, then we can insure that $V(t+w,x) = V(t,x)$ for all $(t,x) \in R \times R^n$.

The following lemma for upper semicontinuous functions will be needed in order to characterize existence in the past.

LEMMA 3. *Let $V: R \times R^n \to R$ be upper semicontinuous and define*

$$\hat{V}(t,x) = \limsup_{\substack{y \to f(t,x) \\ h \to 0^+}} h^{-1}(V(t+h, x+hy) - V(t,x)). \tag{11}$$

*If solutions of* (E) *are unique to the right, then the following are equivalent*:

$$\hat{V}(t,x) \geqq 0 \tag{12}$$

*and*

$$V(t, x(t)) \text{ is a nondecreasing function of } t, \tag{13}$$

*where $x(t)$ is the solution of* (E) *satisfying $x(t) = x$.*

*Proof*. Since $V(t,x)$ is upper semicontinuous we have that $V_1(t,x) \overset{\text{def}}{\equiv} -V(t,x)$

is lower semicontinuous. In particular,

$$\hat{V}(t,x) = -\liminf_{\substack{y\to f(t,x)\\ h\to 0^+}} (V_1(t+h, x+hy) - V_1(t,x)) \tag{14}$$
$$= -\overset{*}{V}_1(t,x).$$

Assume (12) holds; then from (14) we have $\overset{*}{V}_1(t,x) \leqq 0$. Since $V_1(t,x)$ is lower semicontinuous, $V_1(t,x(t))$ is a nonincreasing function of $t$, a consequence of Theorem 1. Hence $V(t,x(t))$ is a nondecreasing function of $t$. Assume (13) holds; then $V_1(t,x)$ is lower semicontinuous, and $V_1(t,x(t))$ is a nonincreasing function of $t$. Hence from Theorem 1, $\overset{*}{V}_1(t,x) \leqq 0$. Using (14) we have $\hat{V}(t,x) \geqq 0$ proving the lemma.

Once again assume solutions of (E) are unique to the right. On the set $[S,\infty)$ for any $S$ assuming solutions exist, define

$$V_S(t,x) = \sup_{S\leqq\tau\leqq t} |x(\tau,t,x)| \quad \text{for } (t,x)\in(S,\infty)\times R^n.$$

Using Lemmas 1, 2 and 3, we can show $V_S(t,x)$ is upper semicontinuous, mildly unbounded, and $\hat{V}_S(t,x) \geqq 0$. These conditions are thus equivalent to the existence of solutions on $[S,t_0)$ for any initial point $(t_0,x_0)\in(S,\infty)\times R^n$.

We now combine these results in order to insure existence on any compact interval $[-T,T]$, $T>0$.

LEMMA 4. *Assume all solutions of* (E) *are unique to the right. Let $T>0$. Then for each point $(t_0,x_0)\in(-T,T)\times R^n$ the solutions $x(t,t_0,x_0)$ are defined for all $t\in[-T,T]$ if and only if there exists a function $V_T(t,x)$ defined on $[-T,T]\times R^n$ such that $V_T(t,x)$ is continuous and satisfies*

(a) $V_T(t,x)$ *is mildly unbounded*,

*and*

(b) $\overset{*}{V}_T(t,x) \leqq 0, \qquad \hat{V}_T(t,x) \geqq 0.$

*Sketch of proof.* The sufficiency follows by combining the previous results concerning existence in the past and in the future.

Define

$$V_T(t,x) = |x(T,t,x)|$$

for $(t,x)\in(-T,T)\times R^n$. Using Lemmas 1, 2 and 3 we can show that $V_T$ is continuous and satisfies (a) and (b).

We now present our main result concerning global existence.

THEOREM 3. *Assume all solutions of* (E) *are unique to the right. Then all solutions of* (E) *exist forever if and only if there exists a function $V(t,x)$ such that $V(t,x)$ is continuous on $D$, where $D = R\times R^n\setminus\{k\}_1^\infty\times R^n$ and*

(a′) $V(t,x)$ *is mildly unbounded on* $R\times R^n$,

*and*

(b′) $\overset{*}{V}(t,x)\leqq 0, \qquad \hat{V}(t,x)\geqq 0$ *for all* $(t,x)\in D$.

*Proof.* From (b′), Theorem 1 and Lemma 3 we have that $V$ is constant along solutions and hence from Lemma 4 solutions exist on $[-k,k]$ for all integers $k>0$. Thus solutions exist forever.

Conversely, define

$$V(t, x) = |x([t] + 1, t, x)| \quad \text{for } t \neq k, \quad t > 0,$$
$$V(k, x) = |x|,$$
$$V(t, x) = |x(0, t, x)| \quad \text{for } t < 0,$$

where $k = 0, 1, 2 \cdots$. It readily follows that for $(t, x) \in D$, $V$ is continuous due to continuity with respect to initial conditions. We also notice that for $(t, x) \in D$, $V$ is constant along solutions. Using Theorem 1 and Lemma 3 we have that (b′) is satisfied.

We now show $V$ is mildly unbounded. Assume not; then there exist a $T > 0$, an $M > 0$ and a sequence of points $\{(t_n, x_n)\}$ such that $-T \leqq t_n \leqq T, |x_n| \to \infty$, and $V(t_n, x_n) \leqq M$. We assume without loss of generality that $t_n \to t_0$ and $[t_n, t_0) \in D$. First assume $t_0$ is not a positive integer and $t_0 > 0$. We have for $n$ sufficiently large that $[t_n] = [t_0]$; hence,

$$V(t_n, x_n) = x([t_0] + 1, t_n, x_n).$$

We may assume without loss of generality that

$$y_n \overset{\text{def}}{\equiv} x([t_0] + 1, t_n, x_n) \to y_0 \quad \text{as } n \to \infty.$$

We consider the set $F_{[t_0]+1, y_0}[[t_0], [t_0] + 1]$; and since all solutions exist in the past, this set is compact. Moreover, applying Lemma 1 for sufficiently large $n$ we have

$$\rho^*(F_{[t_0]+1, y_n}[[t_0], [t_0] + 1], F_{[t_0]+1, y_0}[[t_0], [t_0] + 1]) < 1;$$

and since $x_n \in F_{[t_0]+1, y_n}[[t_0], [t_0] + 1]$, we have

$$\rho^*(x_n, F_{[t_0]+1, y_0}[[t_0], [t_0] + 1]) < 1 .$$

This leads to a contradiction since $|x_n| \to \infty$ and $F_{[t_0]+1, y_0}[[t_0], [t_0] + 1]$ is compact. If $t_0 < 0$, we can use similar techniques and arrive at a contradiction. When $t_0 = k$ for some integer $k$ we then consider the two possible cases $t_n \nearrow t_0$ and $t_n \searrow t_0$. Once again we use similar techniques as before, and for each case we arrive at a contradiction, thus completing the proof of Theorem 3.

*Remarks.* To obtain only sufficient conditions for solutions existing forever, we may replace (b′) with $\overset{*}{V}(t, x) \equiv 0$. To see this we define, for each solution $\phi(t)$,

$$V'(t, \phi(t)) \equiv \liminf_{\tau \to 0^+} [V(t + \tau, \phi(t + \tau)) - V(t, \phi(t))]\tau^{-1};$$

then since $\overset{*}{V} \leqq V'$ and $V$ is continuous, we have that $V$ is nondecreasing along solutions. Moreover, since $\overset{*}{V}(t, x) \equiv 0$ implies $V$ is nonincreasing along solutions, we have $V$ is constant along solutions. Using the same techniques used in Theorem 3 we conclude that solutions exist forever. The converse is not true; that is, we shall give an example in which a Lyapunov function is continuous, mildly unbounded, constant along solutions; and yet there exists a sequence of points $\{x_n\}$, $|x_n| \to \infty$ and $\overset{*}{V}(t, x_n) \neq 0$. Consider the scalar equation $\dot{x} = 0$ on the domain $x \geqq 0$.

Define the following Lyapunov function:

$$V(x)=\begin{cases}(3k+1)[x-3k]^{1/2}+3k[(3k+1)-x]^{1/2}, & 3k \leqq x < 3k+1,\\ \left(3k+1-\dfrac{1}{2^{3k+1}}\right)[x-(3k+1)]^{1/2}+(3k+1)[(3k+2)-x]^{1/2}, & \\ & 3k+1 \leqq x < 3k+2,\\ (3k+3)[x-(3k+2)]^{1/2}+\left((3k+1)-\dfrac{1}{2^{3k+1}}\right)[(3k+3)-x]^{1/2}, & \\ & 3k+2 \leqq x < 3k+3,\end{cases}$$

for $k = 0, 1, 2, \cdots$. We observe that $V$ is constant along all solutions of $\dot{x} = 0$, continuous, and mildly unbounded. We now show $\overset{*}{V}(3k+1) = \overset{*}{V}(3k+2) = -\infty$. From the definition of $\overset{*}{V}(x)$ we have

$$\overset{*}{V}(x) \leqq \min\left\{\frac{d^+}{dx}V(x), \frac{d^-}{dx}V(x)\right\},$$

where $(d^+/dx)V(x)$ and $(d^-/dx)V(x)$ are the ordinary right- and left-hand derivatives of $V(x)$. By direct computation, we can show that $(d^+/dx)V(3k+1) = (d^-/dx)V(3k+2) = -\infty$ and thus our claim follows. In general, we then expect that (b′) cannot be replaced by $V(x) \equiv 0$ in Theorem 3.

## REFERENCES


[1] S. BERNFELD, *Liapunov functions and global existence without uniqueness*, Proc. Amer. Math. Soc., to appear.

[2] N. BHATIA AND O. HAJEK, *Local Semi-Dynamical Systems*, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1969.

[3] J. KATO AND A. STRAUSS, *On the global existence of solutions and Liapunov functions*, Ann. Mat. Pura Appl., 77 (1967), pp. 303–316.

[4] E. ROXIN, *Stability in general control systems*, J. Differential Equations, 1 (1965), pp. 115–150.

[5] J. YORKE, *Differential inequalities and non-Lipschitz scalar functions*, Math. Systems Theory, to appear.

[6] T. YOSHIZAWA, *Stability Theory by Liapunov's Second Method*, The Mathematical Society of Japan, Tokyo, 1966.

SIAM J. MATH. ANAL.
Vol. 1, No. 4, November 1970

# ON THE RECIPROCAL MODULUS RELATION FOR ELLIPTIC INTEGRALS*

HENRY E. FETTIS†

The reciprocal modulus theorem relates the elliptic integrals with modulus $k$ to those with modulus $1/k$. For the complete integral of the first kind

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \tag{1}$$

it is given in nearly all references as[1] (see e.g., [1] through [4])

$$K(1/k) = k[K(k) + iK'(k)], \tag{2}$$

where $K'(k)$ is defined as $K(k')$, and $k' = \sqrt{1 - k^2}$ is the complementary modulus. The corresponding formula for $K'$ is

$$K'(1/k) = kK'(k). \tag{3}$$

That (2) is inconsistent is evident if it is solved for $K(k)$. This gives, after making use of (3),

$$K(k) = \frac{1}{k}[K(1/k) - iK'(1/k)]. \tag{4}$$

In fact, (2) is correct only if the additional condition

$$\operatorname{Im}(k^2) < 0$$

is imposed. For $\operatorname{Im}(k^2) > 0$ the correct relation is

$$K(1/k) = k[K(k) - iK'(k)], \tag{5}$$

a result which is now consistent with (4). On the real axis, the relation is actually ambiguous, and is different depending on whether $k^2$ is taken as $k^2 + i0$ or $k^2 - i0$. The situation is similar to that encountered with Legendre functions on the cut $(-1, 1)$. In fact, (2) and (5) can be obtained by making use of the relation between $K(k)$, $K'(k)$ and the Legendre functions of order $-\frac{1}{2}$:

$$\begin{aligned} P_{-1/2}(z) &= \frac{2}{\pi}\sqrt{\frac{2}{1+z}}\,K\left(\sqrt{\frac{z-1}{z+1}}\right), \\ Q_{-1/2}(z) &= \sqrt{\frac{2}{1+z}}\,K\left(\sqrt{\frac{2}{z+1}}\right) \end{aligned} \tag{6}$$

and using the following relation between $P_{-1/2}(z)$ and $P_{-1/2}(-z)$, valid for $\operatorname{Im}(z) > 0$:

$$P_{-1/2}(-z) = e^{i\pi/2} P_{-1/2}(z) + \frac{2}{\pi} Q_{-1/2}(z). \tag{7}$$

* Received by the editors September 10, 1969, and in final revised form April 17, 1970.

† Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio 45433.

[1] In [2] and [3] the brackets are missing. The error has been corrected in [4].

However, a more straightforward derivation of the correct sign convention can be obtained by employing one of Kummer's identities for the hypergeometric function [1, vol. 1, formula 2.9 (25)], specialized for the case where $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = 1$, $z = k^2$.[2] In the notation of the above reference we have, for $\mathrm{Im}(z) > 0$ and $|\arg z| < \pi$,

$$
\begin{aligned}
U_1 &= F(\tfrac{1}{2}, \tfrac{1}{2}, 1, k^2) = K(k), \\
U_2 &= F(\tfrac{1}{2}, \tfrac{1}{2}, 1, 1 - k^2) = K'(k), \\
U_3 &= (-z)^{-1/2} F(\tfrac{1}{2}, \tfrac{1}{2}, 1, 1/k^2) = k^{-1} e^{i\pi/2} K(1/k)
\end{aligned}
\tag{8}
$$

while relation (25) [loc. cit.] becomes

$$
e^{i\pi/2} U_2 = U_1 + e^{i\pi/2} U_3 \tag{9}
$$

which gives ultimately the desired relationship

$$
K(1/k) = k[K(k) - iK'(k)], \qquad \mathrm{Im}(k^2) > 0. \tag{10}
$$

The correctness of (5) can also be verified from the expressions for $K(k)$ and $K'(k)$ when $k$ is on the unit circle:[3]

$$
\begin{aligned}
K(e^{i\theta}) &= \tfrac{1}{2} e^{-i\theta/2} [K(\cos(\theta/2)) + iK(\sin(\theta/2))], \\
K'(e^{i\theta}) &= e^{-i\theta/2} K(\sin(\theta/2)),
\end{aligned}
\tag{11}
$$

$0 \leqq \theta < \pi/2$. Since $K(e^{-i\theta})$ and $K'(e^{-i\theta})$ are the complex conjugates of $K(e^{i\theta})$ and $K'(e^{i\theta})$ respectively, we also have

$$
\begin{aligned}
K(e^{-i\theta}) &= \tfrac{1}{2} e^{i\theta/2} [K(\cos(\theta/2)) - iK(\sin(\theta/2))] \\
&= e^{i\theta} [K(e^{i\theta}) - iK'(e^{i\theta})],
\end{aligned}
\tag{12}
$$

also in agreement with (5).

It is worthy of additional mention that the sign of $i$ in the transformed expressions for other cases of the elliptic integrals when $k$ is replaced by $1/k$, $1/k'$, $ik/k'$, and $k'/ik$ (as given, for example, in Table 4 of [1, vol. 2, p. 319]) is dependent on the sign of $\mathrm{Im}(k^2)$, and further that in the first two cases those prescribed for $E$ are actually inconsistent with those given for $K$, as can easily be verified from Legendre's relation:

$$
E(k)K'(k) + K(k)E'(k) - K(k)K'(k) = \pi/2. \tag{13}
$$

In the first case, the signs given for $K$ apply when $\mathrm{Im}(k^2) < 0$ while those for $E$ hold for $\mathrm{Im}(k^2) > 0$. In the second case, the signs given for $K$ are valid for $\mathrm{Im}(k^2) > 0$ and those for $E$ apply when $\mathrm{Im}(k^2) < 0$. In the last two cases the sign conventions for $K'$ and $E'$ are consistent and hold for $\mathrm{Im}(k^2) < 0$.

---

[2] An equivalent statement of the relationship under discussion can be found in [7, Ex. 21, p. 299] and is due originally to Barnes [8]. However, the sign convention is improperly stated in [7] and should read (in part) "opposite to the sign of $I(x)$." The author is indebted to the referee for pointing out these two references.

[3] These relations can be found from the equivalence of the elliptic integrals and Legendre function of order $-\frac{1}{2}$ (see [6]).

A final observation is that, although the discussion has heretofore referred to the imaginary part of $k^2$, similar deductions can be made which relate the sign convention to the imaginary part of $k$, provided that when $k$ lies in the second or third quadrants, its argument is defined as $\pm(\pi - \theta)$ with $0 \leqq \theta \leqq \pi/2$. The functions $K(k)$ and $K'(k)$ can then be continued analytically into these regions by the relations

$$\begin{aligned} K(ke^{i\pi}) &= K(k), \\ K'(ke^{i\pi}) &= K'(k) + 2iK(k), \end{aligned} \tag{14a}$$

when $0 \leqq \arg k \leqq \pi/2$. Values thus obtained correspond to values in the second positive Riemann sheet of the $k^2$-plane. Similarly, when $-\pi/2 \leqq \arg k \leqq 0$ the appropriate continuation is given by

$$\begin{aligned} K(ke^{i\pi}) &= K(k), \\ K'(ke^{i\pi}) &= K'(k) - 2iK(k). \end{aligned} \tag{14b}$$

With the analytic continuation defined by (14a) and (14b), equation (11) is valid in the larger interval $0 \leqq \theta \leqq \pi$.

## REFERENCES


[1] A. Erdélyi, W. Magnus, F. Oberhettinger and F. Tricomi, *Higher Transcendental Functions*, McGraw-Hill, New York, 1953.

[2] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*, Academic Press, New York, 1965.

[3] W. Magnus and F. Oberhettinger, *Formeln und Sätze fur die speziellen Functionen der mathematischen Physik*, Springer, Berlin, 1948.

[4] W. Magnus, F. Oberhettinger and R. R. Soni, *Formeln und Sältze fur die speziellen Functionen der mathematischen Physik*, Springer, Berlin, 1966.

[5] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, 3rd ed., National Bureau of Standards Applied Mathematics Series, no. 55, Washington D.C., 1965.

[6] Henry E. Fettis and James C. Caslin, *A table of the complete elliptic integral of the first kind for complex values of the modulus*, Tech. Rep. ARL 69-0172, 69-0173, Aerospace Research Laboratories, Wright-Patterson Air Force Base, Ohio, 1969.

[7] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge University Press, London, 1958.

[8] E. W. Barnes, *A new development of the theory of hypergeometric functions*, Proc. London Math. Soc. (2), 6 (1908), pp. 141–177.

SIAM J. MATH. ANAL.
Vol. 1, No. 4, November 1970

# INVERSION OF A CONVOLUTION TRANSFORM RELATED TO HEAT CONDUCTION*

HARRY POLLARD† AND D. V. WIDDER‡

**1. Introduction.** It is well known that the temperature of a semi-infinite rod which is initially at zero degrees and whose end is held at the variable temperature $\phi(t)$ as the time $t$ changes is given by the convolution

$$u(x, t) = \int_0^t h(x, t - y)\phi(y)\,dy, \qquad x > 0, \quad t > 0, \tag{1.1}$$

where

$$h(x, t) = \frac{x}{t} k(x, t), \quad k(x, t) = \frac{e^{-x^2/(4t)}}{\sqrt{4\pi t}}, \quad t > 0. \tag{1.2}$$

The rod is thought of as extending along a positive $x$-axis with its end at the origin. See, for example, D. V. Widder [1].

Let us suppose now that the temperature is known as a function of time at a single point $x$ of the rod. Is it possible from these data alone to recover the function $\phi(t)$ which is causing the flow of heat? Otherwise stated: given $u(x_0, t)$ in (1.1); find $\phi(y)$. Without restriction we set $x_0 = 1$. If $f(t) = u(1, t)$, we wish then to invert the convolution transform

$$f(t) = \int_0^t h(t - y)\phi(y)\,dy, \qquad h(t) = h(1, t). \tag{1.3}$$

H. Pollard and J. Blackman [2] have given inversion procedures for transforms of type (1.3) when the familiar Laplace transform method is not available; that is, when $f(t)$ is known only in a neighborhood of the origin or, if known for all $t$, when it has no Laplace transform, e.g., $f(t) = \exp(\exp t)$. Explicit knowledge of the kernel $h(t)$ enables us here to give an alternate solution, also having the above-mentioned advantage over the Laplace method. We shall show that the transform (1.3) is inverted by the differential operator

$$e^{\sqrt{\mathscr{D}}} = \sum_{k=0}^{\infty} \frac{\mathscr{D}^k}{(2k)!} + \frac{\mathscr{D}^k\sqrt{\mathscr{D}}}{(2k + 1)!}, \qquad \mathscr{D} = \frac{d}{dt}, \tag{1.4}$$

where $\sqrt{\mathscr{D}}$ is the classical Riemann–Liouville fractional derivative.

**2. Operational considerations.** To conjecture (1.4) by operational calculus we define, as usual,

$$e^{a\mathscr{D}}\phi(t) = \phi(t + a). \tag{2.1}$$

* Received by the editors February 20, 1970, and in revised form April 27, 1970.
† Department of Mathematics, Purdue University, Lafayette, Indiana 47907.
‡ Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138.

If $\mathscr{D}$ is regarded as a number, then from A. Erdélyi [3, (28), p. 146] we have

$$e^{-\sqrt{\mathscr{D}}} = \int_0^\infty e^{-y\mathscr{D}} h(y)\,dy. \tag{2.2}$$

Hence, by (2.1),

$$e^{-\sqrt{\mathscr{D}}}\phi(t) = \int_0^\infty \phi(t-y)h(y)\,dy = \int_0^t h(t-y)\phi(y)\,dy,$$

assuming that $\phi(y) = 0$ when $y < 0$. Thus

$$e^{-\sqrt{\mathscr{D}}}\phi(t) = f(t), \qquad \phi(t) = e^{\sqrt{\mathscr{D}}}f(t).$$

Since $e^{\sqrt{\mathscr{D}}}$ is not itself a Laplace transform no definition of $e^{\sqrt{\mathscr{D}}}$ similar to (2.2) is available. We proceed differently in the following section.

**3. Definition of the inversion operator.** To obtain an effective interpretation of the operator $e^{\sqrt{\mathscr{D}}}$ we expand in infinite series.

DEFINITION 3.1.

$$e^{r\sqrt{\mathscr{D}}}f(t) = \cosh r\sqrt{\mathscr{D}}f(t) + \sinh r\sqrt{\mathscr{D}}f(t),$$

$$\cosh r\sqrt{\mathscr{D}}f(t) = \sum_{n=0}^\infty \frac{r^{2n}}{(2n)!} f^{(n)}(t),$$

$$\sinh r\sqrt{\mathscr{D}}f(t) = \sum_{n=1}^\infty \frac{r^{2n-1}}{(2n-1)!} g^{(n)}(t),$$

$$g(t) = \frac{1}{\sqrt{\mathscr{D}}} f(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{f(y)}{\sqrt{t-y}}\,dy. \tag{3.1}$$

Here we have used in (3.1) the classical Riemann–Liouville definition of the fractional integral $\mathscr{D}^{-1/2}$. For the operation of the definition to be applicable the function $f(t)$ must be such that the integral (3.1) exists and the two series converge.

Let us apply Definition 3.1 to the function $h(t)$ of § 1. In this case,

$$g(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{h(y)}{\sqrt{t-y}}\,dy = 2k(1, t). \tag{3.2}$$

See, for example, D. V. Widder [4, p. 289]. Thus

$$g^{(n)}(t) = 2\frac{\partial^n}{\partial t^n} k(1, t) = 2\frac{\partial^{2n}}{\partial x^{2n}} k(x, t)|_{x=1}.$$

Here we have used the fact that $k(x, t)$ satisfies the heat equation

$$\frac{\partial^2 k}{\partial x^2} = \frac{\partial k}{\partial t}, \qquad t > 0.$$

Obviously,

$$h(x, t) = -2\frac{\partial}{\partial x} k(x, t), \tag{3.3}$$

so that

$$g^{(n)}(t) = -\frac{\partial^{2n-1}}{\partial x^{2n-1}} h(x, t)|_{x=1}.$$

Since $h(x, t)$ also satisfies the heat equation we have

$$e^{r\sqrt{\mathscr{D}}}h(t) = \sum_{n=0}^{\infty} \frac{(-r)^n}{n!} \frac{\partial^n h(x,t)}{\partial x^n}\bigg|_{x=1}.$$

But this is the Maclaurin series for $h(1 - r, t)$, an entire function of $r$ for fixed $t > 0$. We have thus proved the following result, basic for the inversion of the transform (1.3).

THEOREM 3.1. *If* $-\infty < r < \infty$, $0 < t < \infty$, *then*

$$e^{r\sqrt{\mathscr{D}}}h(1,t) = h(1 - r, t).$$

**4. Properties of the kernel $h(x, t)$.** The following facts about $h(x, t)$ are either self-evident or easily proved.

| | | |
|---|---|---|
| (a) | $h(x,t) > 0,$ | $x, t > 0.$ |
| (b) | $h(x, 0+) = h(x, \infty) = 0,$ | $x > 0.$ |
| (c) | $h(x_0, t) \in \uparrow,$ | $t < x_0^2/6, \quad x_0 > 0,$ |
| | $h(x_0, t) \in \downarrow,$ | $t > x_0^2/6, \quad x_0 > 0.$ |
| (d) | $\int_0^\infty h(x,t)\,dt = 1,$ | $x > 0.$ |
| (e) | $\lim_{x\to 0+} \int_0^c h(x,t)\,dt = 1,$ | $c > 0,$ |
| | $\lim_{x\to 0+} \int_c^\infty h(x,t)\,dt = 0,$ | $c > 0.$ |

We now prove two further results.

LEMMA 4.1. *If* $c > 0$,

$$\lim_{x\to 0+} \int_c^\infty |h_y(x,y)|\,dy = 0.$$

For, by property (c) above, if $x < \sqrt{6c}$,

$$\int_c^\infty |h_y(x,y)| dy = -\int_c^\infty h_y(x,y)\,dy = h(x,c).$$

The result now follows by (b).

LEMMA 4.2. *If* $0 < x < \infty$,

$$\int_0^\infty y|h_y(x,y)|\,dy < 2.$$

For, by (c),

$$\begin{aligned}\int_0^\infty |h_y(x,y)|y\,dy &= \int_0^{x^2/6} h_y(x,y)y\,dy - \int_{x^2/6}^\infty h_y(x,y)y\,dy \\ &= 2h(x, x^2/6)(x^2/6) - \int_0^{x^2/6} h(x,y)\,dy + \int_{x^2/6}^\infty h(x,y)\,dy \\ &< (6/\pi)^{1/2}e^{-3/2} + 1 < 2.\end{aligned}$$

Here we have integrated by parts and used (a) and (d).

**5. The Lebesgue integral convolution with** $h(x, t)$**.** If $\phi(y)$ is Lebesgue integrable on an interval $(0, c)$, the convolution (1.1) is well-defined for $0 < x < \infty$, $0 < t < c$. It is a familiar fact (H. S. Carslaw and J. C. Jaeger [5]), as we indicated in § 1, that the function $\varphi(y)$ of (1.1) may be obtained from $u(x, t)$ simply by allowing $t$ to approach zero (assuming $u(x, t)$ known for all $x$ and not just for $x = 1$). Since it seems impossible to quote a reference for this fact, in the desired generality, we prove the basic facts here.

THEOREM 5.1. *If*

(i) $\phi(y) \in L, \quad 0 \leqq y \leqq c,$

(ii) $\alpha(x) = \int_0^x [\phi(t_0 - y) - \phi(t_0)]\, dy = o(x), x \to 0+, 0 < t_0 < c,$

(iii) $F(x) = \int_0^{t_0} h(x, y)\phi(t_0 - y)\, dy,$

*then*

$$F(0+) = \phi(t_0).$$

The result is true when $\phi$ is constant by (e). Hence we need only show that

$$\begin{aligned} \int_0^{t_0} h(x, y)\, d\alpha(y) &= \int_0^{t_0} h(x, y)[\phi(t_0 - y) - \phi(t_0)]\, dy \\ &= \alpha(t_0)h(x, t_0) - \int_0^{t_0} h_y(x, y)\alpha(y)\, dy = o(1), \qquad x \to 0+. \end{aligned} \tag{5.1}$$

By (b), $h(0+, t_0) = 0$. Let $0 < \delta < t_0$. Then by Lemma 4.2,

$$\left| \int_0^{\delta} h_y(x, y)\alpha(y)\, dy \right| \leqq 2 \max_{0 \leqq y \leqq \delta} \frac{|\alpha(y)|}{y}.$$

Hence by Lemma 4.1,

$$\left| \int_{\delta}^{t_0} h_y(x, y)\alpha(y)\, dy \right| \leqq \max_{0 \leqq y \leqq t_0} |\alpha(y)| \int_{\delta}^{t_0} |h_y(x, y)|\, dy = o(1), \qquad x \to 0+,$$

$$\limsup_{x \to 0+} \left| \int_0^{t_0} h_y(x, y)\alpha(y)\, dy \right| \leqq 2 \max_{0 \leqq y \leqq \delta} |\alpha(y)|/y.$$

This inequality, with hypothesis (ii), shows that the integral (5.1) tends to zero with $x$, as desired.

COROLLARY 5.1.

$$F(0+) = \phi(t_0) \quad \textit{for almost all } t_0 \textit{ in } (0, c),$$

$$F(0+) = \phi(t_0-) \quad \textit{when } \phi(t_0-) \textit{ exists.}$$

For, hypothesis (ii) holds in points $t_0$ of the Lebesgue set for $\phi$ or when $\phi(t_0-)$ exists and equals $\phi(t_0)$.

**6. The Stieltjes integral convolution with** $h(x, t)$**.** If the integral (1.1) is replaced by a Stieltjes integral, a modification of the inversion formula is necessary.

THEOREM 6.1. *If*

(i) $\alpha(y)$ *is of bounded variation on* $0 \leqq y \leqq c$,

(ii) $F(x, t) = \int_0^t h(x, t-y)\, d\alpha(y)$,

*then*

$$\lim_{x \to 0+} \int_0^{t_0} F(x, y)\, dy = \alpha(t_0-) - \alpha(0), \qquad 0 < t_0 \leqq c. \tag{6.1}$$

Obvious calculations give

$$F(x, t) = -\alpha(0)h(x, t) + \int_0^t h_y(x, t-y)\alpha(y)\, dy,$$

$$\begin{aligned}\int_0^{t_0} F(x, r)\, dr &= -\alpha(0) \int_0^{t_0} h(x, r)\, dr + \int_0^{t_0} dr \int_0^r h_y(x, r-y)\alpha(y)\, dy \\ &= -\alpha(0) \int_0^{t_0} h(x, r)\, dr + \int_0^{t_0} \alpha(y)\, dy \int_y^{t_0} h_y(x, r-y)\, dr \\ &= -\alpha(0) \int_0^{t_0} h(x, r)\, dr + \int_0^{t_0} h(x, t_0 - y)\alpha(y)\, dy.\end{aligned}$$

Now by (e) and Corollary 5.1, equation (6.1) follows at once.

**7. The inversion theorems.** We can now solve the problem originally posed.

THEOREM 7.1. *If*

(i) $\phi(y) \in L$, $0 \leqq y \leqq c$,

(ii) $f(t) = \int_0^t h(1, t-y)\phi(y)\, dy$,

*then*

$$\lim_{r \to 1-} e^{r\sqrt{\mathscr{D}}} f(t) = \phi(t)$$

*for almost all t in* $(0, c)$ *or when* $\phi(t) = \phi(t-)$.

If it is permissible to apply the operator $e^{r\sqrt{\mathscr{D}}}$ under the integral sign, we have, by Theorem 3.1, that

$$e^{r\sqrt{\mathscr{D}}} f(t) = F(1-r) = \int_0^t h(1-r, t-y)\phi(y)\, dy. \tag{7.1}$$

The conclusion of the theorem then follows by Corollary 5.1.

To establish (7.1) we show that $F$ can be expanded in Maclaurin's series. Setting $h(t) = h(1, t)$, $k(t) = k(1, t)$, we have

$$f^{(n)}(t) = \int_0^t h^{(n)}(t-y)\phi(y)\, dy, \qquad n = 0, 1, 2, \cdots,$$

since $h^{(n)}(0+) = 0$. By (3.2),

$$g(t) = \frac{1}{\sqrt{\mathscr{D}}} f(t) = 2\int_0^t k(t-y)\phi(y)\,dy,$$

$$g^{(n)}(t) = 2\int_0^t k^{(n)}(t-y)\phi(y)\,dy, \qquad n = 0, 1, 2, \cdots.$$

Using the heat equation as in § 3, we have

$$F^{(2n)}(1) = \int_0^t h^{(n)}(t-y)\phi(y)\,dy = f^{(n)}(t).$$

By (3.3),

$$F(r) = -2\int_0^t \frac{\partial}{\partial r} k(r, t-y)\phi(y)\,dy,$$

$$F^{(2n-1)}(1) = -2\int_0^t k^{(n)}(t-y)\phi(y)\,dy = -g^{(n)}(t).$$

Thus, if the Maclaurin expansion is valid,

$$(7.2)\quad F(1-r) = \sum_{n=0}^{\infty} \frac{(-r)^n}{n!} F^{(n)}(1) = \sum_{n=0}^{\infty} \frac{r^{2n}}{(2n)!} f^{(n)}(t) - \sum_{n=1}^{\infty} \frac{r^{2n-1}}{(2n-1)!} g^{(n)}(t).$$

But a change of variable gives

$$F(r) = \frac{r}{\pi}\int_{1/(4t)}^{\infty} e^{-r^2 z} \frac{\phi(1/(4z))}{\sqrt{z}}\,dz, \qquad z = \frac{1}{4t}.$$

This integral clearly converges for $r > 0$. Hence by a familiar property of the Laplace transform, $F(r)$ is analytic for $\operatorname{Re} r > 0$, so that (7.2) is valid for $|r| < 1$. But the right-hand side of (7.2) is $e^{r\sqrt{\mathscr{D}}} f(t)$ by Definition 3.1. This completes the proof.

In a similar manner, an appeal to Theorem 6.1 gives the following result.

THEOREM 7.2. *If*

(i) $\alpha(y)$ *is of bounded variation on* $0 \leqq y \leqq c$,

(ii) $f(t) = \int_0^t h(1, t-y)\,d\alpha(y)$,

*then*

$$\lim_{r\to 1-} \int_0^{t_0} e^{r\sqrt{\mathscr{D}}} f(t)\,dt = \alpha(t_0-) - \alpha(0), \qquad 0 < t_0 \leqq c.$$

## REFERENCES

[1] D. V. WIDDER, *Positive temperatures on a semi-infinite rod*, Trans. Amer. Math. Soc., 75 (1953), pp. 510–525.

[2] J. BLACKMAN AND H. POLLARD, *The finite convolution transform*, Ibid., 91 (1959), pp. 399–409.

[3] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER AND F. TRICOMI, *Tables of Integral Transforms*, vol. 1, McGraw-Hill, New York, 1954.

[4] D. V. WIDDER, *Integral transforms related to heat conduction*, Ann. Mat. Pura Appl., 42 (1956), pp. 279–305.

[5] H. S. CARSLAW AND J. C. JAEGER, *Conduction of Heat in Solids*, Oxford University Press, London, 1948.

SIAM J. MATH. ANAL.
Vol. 1, No. 4, November 1970

# A PARADOX IN ASYMPTOTICS*

F. W. J. OLVER†

**Abstract.** An example is given of a convergent series expansion which has twice itself as its own asymptotic expansion.

Can a convergent series possess a sum which differs from its asymptotic sum? For ordinary Poincaré expansions in powers of the asymptotic variable the answer is always no. In the case of generalized asymptotic expansions [1], however, the answer can be affirmative. This phenomenon was pointed out by van der Corput in 1962 [2]. The principal examples in this reference are somewhat artificial, however, in the sense that the terms of the series are discontinuous. The purpose of the present note is to draw attention to an example which occurs naturally in special function theory and has been discussed erroneously in the literature.

Consider the Legendre polynomial $P_n(\cos\theta)$ for large $n$ and fixed $\theta$, with $0 < \theta < \pi$. We have

$$P_n(\cos\theta) \sim \left(\frac{2}{\sin\theta}\right)^{1/2} \sum_{\nu=0}^{\infty} \binom{-\frac{1}{2}}{\nu}\binom{\nu-\frac{1}{2}}{n}\frac{\cos\theta_{n,\nu}}{(2\sin\theta)^{\nu}}, \qquad n\to\infty, \tag{1}$$

where

$$\theta_{n,\nu} = (n-\nu+\tfrac{1}{2})\theta + (n-\tfrac{1}{2}\nu-\tfrac{1}{4})\pi,$$

in the sense that the difference between $P_n(\cos\theta)$ and the $p$th partial sum of the series is $O\left\{\binom{p-\frac{1}{2}}{n}\right\}$ which is equivalent to $O(n^{-p-(1/2)})$. This result is derivable by the method of Darboux;[1] the leading term is, in effect, Laplace's well-known approximation.

When $2\sin\theta > 1$, that is, when $\frac{1}{6}\pi < \theta < \frac{5}{6}\pi$, the series on the right of (1) converges. But contrary to our natural expectations, and also to a statement in Szegö's comprehensive treatise,[2] the sum is not $P_n(\cos\theta)$ but $2P_n(\cos\theta)$. This may be verified as follows. By expansion in Taylor series at the point $t = e^{-i\theta}$, we have

$$\frac{1}{(1-2t\cos\theta+t^2)^{1/2}} = \frac{e^{-\pi i/4}}{(2\sin\theta)^{1/2}}\sum_{\nu=0}^{\infty}\binom{-\frac{1}{2}}{\nu}\frac{(e^{-i\theta}-t)^{\nu-(1/2)}}{(2i\sin\theta)^{\nu}}.$$

This series converges uniformly in the disc $|e^{-i\theta}-t| \leqq 2\sin\theta-\delta$, where $\delta$ is an arbitrary small positive number. If $2\sin\theta > 1$, then $t = 0$ is includable in the region of uniform convergence. Differentiating $n$ times, setting $t = 0$, and equating

* Received by the editors April 3, 1970.

† Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland 20742. This research was supported by the U.S. Army Research Office, Durham, under Contract 67-C-0062.

[1] See [3, § 8.4]. The statement of the result in [4, p. 534] contains a misprint in the cosine term.

[2] [3, § 8.4, (3)].

real parts, we obtain

$$P_n(\cos\theta) = \frac{1}{(2\sin\theta)^{1/2}} \sum_{\nu=0}^{\infty} \binom{-\frac{1}{2}}{\nu} \binom{\nu - \frac{1}{2}}{n} \frac{\cos\theta_{n,\nu}}{(2\sin\theta)^{\nu}}, \qquad \tfrac{1}{6}\pi < \theta < \tfrac{5}{6}\pi, \tag{2}$$

as asserted. The result (2) can be easily checked in the case $n = 0$, for example.

The explanation of the paradox is that the tail of the series (2), that is, the sum from $\nu = p$ (fixed) to $\nu = \infty$, is not of the same $O$-order as the first neglected term (as it always is with Poincaré expansions). For example, apart from the oscillatory factor $\cos\theta_{n,\nu}$ the term for which $\nu = 2n$ contributes

$$\binom{-\frac{1}{2}}{\nu} \binom{\nu - \frac{1}{2}}{n} \frac{1}{(2\sin\theta)^{\nu+(1/2)}} = \frac{\Gamma(2n + \frac{1}{2})\Gamma(n + \frac{1}{4})\Gamma(n + \frac{3}{4})}{2\pi\Gamma(2n + 1)\Gamma(n + \frac{1}{2})\Gamma(n + 1)(\sin\theta)^{2n+(1/2)}}$$
$$\sim \frac{1}{2^{3/2}\pi n(\sin\theta)^{2n+(1/2)}},$$

as $n \to \infty$. This is infinitely large compared with the $(p + 1)$th term, whatever the value of $p$.

## REFERENCES

[1] A. Erdélyi and M. Wyman, *The asymptotic evaluation of certain integrals*, Arch. Rational Mech. Anal., 14 (1963), pp. 217–260.

[2] J. G. van der Corput, *Can a series possess a sum 1 and an asymptotic sum 0*? Studies in Mathematical Analysis and Related Topics, Stanford University Press, Stanford, 1962, pp. 410–418.

[3] G. Szegö, *Orthogonal Polynomials*, Colloquium Publications, vol. 23, 3rd ed., American Mathematical Society, New York, 1967.

[4] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. 1, Interscience, New York, 1953.

SIAM J. MATH. ANAL.
Vol. 1, No. 4, November 1970

# THE NONOSCILLATION OF A SOLUTION OF A THIRD ORDER EQUATION*

W. R. UTZ†

The differential equation

$$y''' + t^2 y' + 3ty = 0 \tag{1}$$

considered by D. E. Amos in a problem posed [1] and solved by Amos, and others, has three linearly independent oscillatory solutions and, at first glance, would seem to have all solutions oscillatory. In this note we observe that this equation, and more generally (2), below, belongs to the class of linear third order equations with three linearly independent oscillatory solutions but for which there are (nontrivial) nonoscillatory solutions. Such equations have recently been identified and studied in [3].

According to Amos, (1) arises in a problem of describing the motion of a particle in a magnetic field.

The solution of (1) given by Sidney Spital [1, p. 387] reveals the three linearly independent solutions

$$y_1 = \cos\frac{t^2}{2}, \qquad y_2 = \sin\frac{t^2}{2}$$

and

$$y_3 = C\left(\frac{t}{\sqrt{\pi}}\right)\cos\frac{t^2}{2} + S\left(\frac{t}{\sqrt{\pi}}\right)\sin\frac{t^2}{2},$$

where $C(t/\sqrt{\pi})$ and $S(t/\sqrt{\pi})$ are the Fresnel integrals

$$C\left(\frac{t}{\sqrt{\pi}}\right) = \frac{1}{\sqrt{\pi}}\int_0^t \cos\frac{x^2}{2}\,dx, \qquad S\left(\frac{t}{\sqrt{\pi}}\right) = \frac{1}{\sqrt{\pi}}\int_0^t \sin\frac{x^2}{2}\,dx.$$

One may write

$$y_3 = M(t)\sin(t^2/2 + \phi(t)),$$

where

$$M(t) = \sqrt{C^2 + S^2}, \qquad \phi(t) = \arctan C/S.$$

Thus, $y_3$ is oscillatory and (1) has three linearly independent oscillatory solutions. Some nontrivial solution does not oscillate according to a theorem of Lazer [2, Theorem 3.3]. (The inequality in the statement of this theorem is reversed.)

Equation (1) can be generalized to preserve the property of having three linearly independent oscillatory solutions by examining the solution of H. E. Fettis [1, p. 388] for the Amos problem.

* Received by the editors April 2, 1970, and in revised form June 3, 1970.

† Department of Mathematics, University of Missouri—Columbia, Columbia, Missouri 65201.

THEOREM. *Let $m > 0$ be an integer. The equation*

$$ty''' + (2 - m)y'' + t^{2m-1}y' + (2m - 1)t^{2m-2}y = 0, \tag{2}$$

$t > 0$, *has three linearly independent solutions such that each of these solutions oscillates, yet some nontrivial solution of* (2) *is nonoscillatory.*

*Proof.* Consider

$$y'' + y = ax^{-s}, \tag{3}$$

wherein $a$ is any real number and $s > 0$ is real. Let $x = t^m/m$ in (3) to secure

$$ty'' + (1 - m)y' + t^{2m-1}y = am^s t^{-sm+2m-1}. \tag{4}$$

In order to eventually secure an equation without a forcing term, set

$$-sm + 2m - 1 = 0.$$

Now, let $m$ be any positive integer and select

$$s = (2m - 1)/m$$

(the Amos case corresponds to $m = 2$). Then (4) becomes

$$ty'' + (1 - m)y' + t^{2m-1}y = \text{const.}$$

Differentiate this equation to secure (2).

Solutions of (3) are all functions of the form

$$y = A\cos x + B\sin x + aK(x), \qquad x > 0,$$

where $A$ and $B$ are arbitrary and

$$K(x) = \sin x \int_1^x u^{-s}\cos u\, du + \cos x \int_1^x u^{-s}\sin u\, du.$$

Thus all solutions of (2) are given by

$$y = A\cos(t^m/m) + B\sin(t^m/m) + aK(t^m/m), \qquad t > 0,$$

where, now, there are three arbitrary constants $A$, $B$, $a$.

That $K(t^m/m)$ is linearly independent of $\cos(t^m/m)$ and $\sin(t^m/m)$ is easily seen from an examination of the form of $K(t^m/m)$ (this examination will also reveal that it is oscillatory), but it is also obvious from the fact that $K(x)$ must satisfy (3) whereas $\sin x$ and $\cos x$ satisfy $y'' + y = 0$ (and so would any linear combination of them).

In (2), let

$$y = ut^{(m-2)/3}$$

to secure an equation free of the second derivative and to which one can apply the nonoscillation test of Lazer [2, Theorem 3.3]. This transformation, which does not change oscillation, yields

$$u''' + p(t)u' + q(t)u = 0,$$

where

$$p(t) = \frac{-1}{3}(m - 2)(m + 1)t^{-2} + t^{2m-2}$$

and

$$q(t) = \frac{-2}{27}(m-2)(m-5)(m+1)t^{-3} + \frac{1}{3}(7m-5)t^{2m-3}.$$

Hence,

$$2q - p' = t^{-3}\left[\frac{4}{3}(2m-1)t^{2m} - \frac{2}{27}(2m-1)(m-2)(m+1)\right].$$

Thus, $2q - p' > 0$ for any positive integer $m$ if $t$ is large. This completes the proof of the theorem.

## REFERENCES

[1] DONALD E. AMOS ET AL., *A third order differential equation*, SIAM Rev., 8 (1966), pp. 386–388.

[2] A. C. LAZER, *The behavior of solutions of the differential equation* $y''' + p(x)y' + q(x)y = 0$, Pacific J. Math., 17 (1966), pp. 435–466.

[3] W. R. UTZ, *Oscillating solutions of third order differential equations*, Proc. Amer. Math. Soc., to appear.

SIAM J. MATH. ANAL.
Vol. 1, No. 4, November 1970

# ON THE EVALUATION OF CERTAIN SUMS INVOLVING THE NATURAL NUMBERS RAISED TO AN ARBITRARY POWER*

KEITH B. OLDHAM†

**1. Introduction.** Among the mathematical requirements which arose recently in this laboratory in connection with electrochemical investigations (see, for example, [1]) were those for:

(a) A value of the $l \to \infty$ limit of the sum $[1 - \sqrt{2} + \sqrt{3} - \sqrt{4} + \cdots \mp \sqrt{l-2} \pm \sqrt{l-1} \mp \frac{1}{2}\sqrt{l}]$.

(b) and (c) Asymptotic representations for the sums $[1 - \sqrt{3} + \sqrt{5} - \cdots \mp \sqrt{L}]$ and $[1 - 2^{7/6} - 3^{7/6} + 4^{7/6} + 5^{7/6} - \cdots \pm l^{7/6}]$ valid for large odd $L$ and large $l$. When standard mathematical references [2], [3] and monographs [4], [5] failed to provide answers to these problems, the present study was undertaken.

These problems and others were generalized to a study of the sums

$$g(1) + \frac{g(2)}{2^r} + \frac{g(3)}{3^r} + \cdots + \frac{g(l)}{l^r} = \sum_{k=1}^{l} \frac{g(k)}{k^r}, \tag{1}$$

where $r$ is any real number (positive or negative), $l$ is any positive integer and each $g(k)$ takes values according to some repetitive sequence of $v$ elements, i.e.,

$$g(\mu) = g(v + \mu) = g(2v + \mu) = \cdots$$

for $1 \leqq \mu \leqq v$. Note that any $g(k)$, including $g(l)$, is permitted to be zero, so that the term in $l^{-r}$ may be absent from sum (1). These sums have, of course, been intensively studied for certain values of $r$, $l$ and $g(k)$, especially for $r$ integer [3, Chap. 0], [6, p. 67]. A number of texts [7, § 3.5], [8, p. 25 and Chap. XIII],[1] [9, Chap. XIV] make mention of nonintegral $r$ instances, but none with a generality sufficient to embrace the present problems.

The sequence $g(1), g(2), \cdots, g(l)$ is specified if either the first $v$ members

$$G_o \equiv g(1), g(2), \cdots, g(v)$$

or the last $v$ members

$$G_o \equiv g(l - v + 1), g(l - v + 2), \cdots, g(l)$$

are specified, where we term $G_o$ and $G_c$ the opening and closing sequences. Thus, for example, the sum

$$1 - \frac{1}{3^r} + \frac{1}{4^r} - \frac{1}{6^r} + \cdots + \frac{1}{(l-1)^r} \tag{2}$$

has an opening sequence $G_o = \{+1, 0, -1\}$ and a closing sequence $G_c = \{-1, +1, 0\}$. Though $G_o$ and $G_c$ necessarily contain the same elements, they are identical if and only if $v$ is a divisor of $l$.

* Received by the editors March 11, 1969, and in revised form January 5, 1970.

† Science Center, North American Rockwell Corporation, Thousand Oaks, California. Now at Department of Chemistry, Trent University, Peterborough, Ontario, Canada.

[1] The definition of the Bernoulli numbers used in this text differs from the more usual one used here.

We seek a diminishing power series in $l$ to express sum (1). It will transpire that the series sought are all of the form

$$(3)\quad \sum_{k=1}^{l} \frac{g(k)}{k^r} = T(r, G_o) + \sum_{j=0}^{2m} \frac{\Gamma(r+j-1)C_{j-1}(G_c)}{\Gamma(r) l^{r+j-1}} + O([l-v+1]^{-r-2m-1}),$$

where $T(r, G_o)$ depends upon $r$ and upon the opening sequence but not on $l$ or $G_c$ while $G_{j-1}(G_c)$ depends on the closing sequence but not $G_o$, $r$ or $l$. Many of the $C$ coefficients are often zero, as will be shown later.

The object of this article is to establish (3) and to determine $T(r, G_o)$ and $C_{j-1}(G_c)$. First, however, two lemmas are needed.

LEMMA 1. *Let $p$ be any positive rational number and $q$ be a positive rational number in the range $0 < q \leqq 1$, such that $p - q$ is an integer. Then*

$$(4)\quad \sum_{k=0}^{p-q} (k+q)^{-r} = \zeta(r, q) - \sum_{j=0}^{2m} \frac{\Gamma(r+j-1)}{j!\Gamma(r)} p^{1-r-j} B_j + O(p^{-r-2m-1}),$$

*where $m$ is any integer exceeding $-(r+1)/2$ and $B_j$ is the $j$-th Bernoulli number.*

Result (4) follows by asymptotic expansion of the definition (see [10, §§ 1.10, 1.18]) of the bivariate zeta function $\zeta(r, q)$, known as the Hurwitz function or the generalized zeta function. It is valid for all $r$ except $r = 1$.

LEMMA 2. *Here we establish the identity*

$$(5)\quad \sum_{j=0}^{2m} u^j \frac{\Gamma(r+j-1)}{j!} (l+h)^{1-r-j} B_j \equiv - \sum_{j=0}^{2m} (-u)^j \frac{\Gamma(r+j-1)}{j!} l^{1-r-j} B_j\left(\frac{u+h}{u}\right),$$

*where $B_j(x)$ denotes the Bernoulli polynomial of order $j$ and argument $x$, $u$ and $h$ are real numbers, and $m$ is any positive integer.*

Let $m$, $J$ and $j$ be positive integers such that $2m \geqq J \geqq j$; then the binomial theorem may be used to prove that the coefficient of $l^{1-r-J}$ in the expansion of

$$u^j \frac{\Gamma(r+j-1)}{j!} (l+h)^{1-r-j}$$

may be written

$$u^J \frac{\Gamma(r+J-1)}{J!} \binom{J}{j} \left[\frac{-h}{u}\right]^{J-j}.$$

Hence, the coefficient of $l^{1-r-J}$ in the summation on the left-hand side of (5) is

$$u^J \frac{\Gamma(r+J-1)}{J!} \sum_{j=0}^{J} \binom{J}{j} \left[\frac{-h}{u}\right]^{J-j} B_j$$

which is known to equal

$$u^J \frac{\Gamma(r+J-1)}{J!} B_J\left(\frac{-h}{u}\right).$$

Accordingly,

$$\sum_{j=0}^{2m} u^j \frac{\Gamma(r+j-1)}{j!} (l+h)^{1-r-j} B_j \equiv \sum_{j=0}^{2m} u^j \frac{\Gamma(r+j-1)}{j!} l^{1-r-j} B_j\left(\frac{-h}{u}\right),$$

from which (5) follows on application of the identity $B_j(1-x) \equiv (-)^j B_j(x)$.

**2. Unit sums.** Instances of (1) in which all but one of the coefficients $g(1)$, $g(2), \cdots, g(v)$ are zero, the remaining coefficient equaling unity, play a special role in the derivation of relationship (3). Such a sum will be designated a "unit sum" and may be denoted by $S(r, l, v, \mu, \lambda)$. This notation indicates that it is the $\mu$th element of $G_o$ which is nonzero and the $\lambda$th element of $G_c$ which is nonzero. The numbers $\mu$ and $\lambda$ are related by the congruence[2] $\lambda \equiv 1 + [\mu - l - 1] \pmod v$.

For the special case $\mu = \lambda = v$, the summation is easily accomplished with the help of Lemma 1. Thus

$$S(r, l, v, v, v) = \frac{1}{v^r} + \frac{1}{(2v)^r} + \frac{1}{(3v)^r} + \cdots + \frac{1}{l^r}$$

$$(6) \qquad = v^{-r} \sum_{k=1}^{l/v} k^{-r}$$

$$= v^{-r}\zeta(r, 1) - \sum_{j=0}^{2m} v^{j-1} \frac{\Gamma(r + j - 1)}{j!\Gamma(r)} l^{1-r-j} B_j + O(l^{-r-2m-1}).$$

In (6), as in all succeeding relations, it is assumed that $r$ does not equal unity.

If $\lambda = v \neq \mu$, $G_o$ and $G_c$ are "out of phase." Result (4) may again be used to achieve the summation

$$S(r, l, v, \mu, v) = \frac{1}{\mu^r} + \frac{1}{(\mu + v)^r} + \frac{1}{(\mu + 2v)^r} + \cdots + \frac{1}{l^r}$$

$$(7) \qquad = v^{-r} \sum_{k=0}^{K} \left(k + \frac{\mu}{v}\right)^{-r} \quad \text{with } K = \frac{l}{v} - \frac{\mu}{v}$$

$$= v^{-r}\zeta\left(r, \frac{\mu}{v}\right) - \sum_{j=0}^{2m} v^{j-1} \frac{\Gamma(r + j - 1)}{j!\Gamma(r)} l^{1-r-j} B_j + O(l^{-r-2m-1}).$$

Now we consider $\lambda$ and $\mu$ unrestricted. The derivation of the unit sum follows a similar pattern to that of (7) initially, but Lemma 2 is employed to execute the final step:

$$S(r, l, v, \mu, \lambda) = \frac{1}{v^r} + \frac{1}{(\mu + v)^r} + \frac{1}{(\mu + 2v)^r} + \cdots + \frac{1}{(l - v + \lambda)^r}$$

$$= v^{-r} \sum_{k=0}^{K} \left(k + \frac{\mu}{v}\right)^{-r} \left(\text{with } K = \frac{l - v + \lambda - \mu}{v}\right)$$

$$(8) \qquad = v^{-r}\zeta\left(r, \frac{\mu}{v}\right) - \sum_{j=0}^{2m} v^{j-1} \frac{\Gamma(r + j - 1)}{j!\Gamma(r)} (l + \lambda - v)^{1-r-j} B_j$$

$$+ O([l + \lambda - v]^{-r-2m-1})$$

$$= v^{-r}\zeta\left(r, \frac{\mu}{v}\right) + \sum_{j=0}^{2m} (-v)^{j-1} \frac{\Gamma(r + j - 1)}{j!\Gamma(r)} l^{1-r-j} B_j\left(\frac{\lambda}{v}\right)$$

$$+ O([l + \lambda - v]^{-r-2m-1}).$$

Since $B_j(0) = (-)^j B_j(1) = B_j$, formulas (6) and (7) are seen to be special cases of (8). Notice that the first term on the right-hand side of (8) depends on $\mu$, $v$ and $r$, but

[2] The notation is that of Hardy and Wright [11].

not on $l$ or $\lambda$; that the second right-hand term does not explicitly involve $\mu$; and that the remainder term may be made small by a suitable choice of $m$.

**3. Sums other than unit sums.** It is evident that any summation (1) may be decomposed into $v$ unit sums, the $g$ coefficients serving as weighting factors:

$$\sum_{k=1}^{l} \frac{g(k)}{k^r} = \sum_{\mu=1}^{v} g(\mu) S(r, l, v, \mu, \lambda). \tag{9}$$

For example, sum (2) may be written as $S(r, l, 3, 1, 2) - S(r, l, 3, 3, 1)$ since in this case $g(1) = 1$, $g(2) = 0$, and $g(3) = -1$. In writing expression (9) weighting factors have been selected from the opening sequence of the sum. With equal validity, the closing sequence could have been chosen, leading to

$$\sum_{k=1}^{l} \frac{g(k)}{k^r} = \sum_{\lambda=1}^{v} g(l - v + \lambda) S(r, l, v, \mu, \lambda). \tag{10}$$

Recalling that $S(r, l, v, \mu, \lambda)$ is given by (8), we may compose a nonunit sum via either of expressions (9) or (10). For the present purpose it is best to sum the $l$-independent terms over $\mu$, as indicated in (9), and the $l$-dependent terms over $\lambda$, as in (10). Thereby the general formula for sum (1) is

$$\begin{aligned} v^{-r} \sum_{\mu=1}^{v} g(\mu) \zeta\left(r, \frac{\mu}{v}\right) &+ \sum_{j=0}^{2m} (-v)^{j-1} \frac{\Gamma(r+j-1)}{j!\Gamma(r)} l^{1-r-j} \sum_{\lambda=1}^{v} g(l - v + \lambda) B_j\left(\frac{\lambda}{v}\right) \\ &+ O([l + 1 - v]^{-r-2m-1}). \end{aligned} \tag{11}$$

This result does accord with (3), with

$$T = v^{-r} \sum_{\mu=1}^{v} g(\mu) \zeta\left(r, \frac{\mu}{v}\right)$$

and

$$C_{j-1} = \frac{(-v)^{j-1}}{j!} \sum_{\lambda=1}^{v} g(l - v + \lambda) B_j\left(\frac{\lambda}{v}\right);$$

and we see, as asserted earlier, that $T$ is a function only of $r$ and $G_o$, whereas $C_{j-1}$ depends only on $G_c$.

These results suggest that the evaluation of a sum with a repetition factor $v$ requires that as many as $v$ numerical values of the bivariate zeta function and as many as $(2m + 1)v$ values of the Bernoulli polynomials be available. Certain fortunate properties of these functions, however, greatly reduce the data needed to make use of formula (11), especially for small values of $v$.

Consider first the trivial $v = 1$ case. Formula (11) then degenerates to

$$\zeta(r, 1) - \sum_{j=0}^{2m} \frac{\Gamma(r+j-1)}{j!\Gamma(r)} l^{1-r-j} B_j - O(l^{-r-2m-1}). \tag{12}$$

Moreover, since $B_3 = B_5 = B_7 = \cdots = 0$, almost half the terms within the $j$-summation are zero.

With $v = 2$, formula (11) reduces to

$$2^{-r}\{g(1)\zeta(r,\tfrac{1}{2}) + g(2)\zeta(r,1)\}$$
$$(13) \qquad + \sum_{j=0}^{2m} (-2)^{j-1}\frac{\Gamma(r+j-1)}{\Gamma(r)j!} l^{1-r-j} \cdot \{g(l-1)B_j(\tfrac{1}{2}) + g(l)B_j(1)\}$$
$$+ O([l-1]^{-r-2m-1}).$$

However, the relationship $\zeta(r,\frac{1}{2}) = [2^r - 1]\zeta(r,1)$, which is a special case of

$$(14) \qquad \sum_{i=1}^{n} \zeta\left(r, \frac{i}{n}\right) = n^r\zeta(r,1),$$

enables the $l$-independent term to be condensed. Similarly the relationship $B_j(\frac{1}{2}) = [2^{1-j} - 1]B_j$, which is a special case of

$$(15) \qquad \sum_{i=1}^{n} B_j\left(\frac{i}{n}\right) = [n^{1-j} - 1 + (-1)^j]B_j,$$

may be used to replace the Bernoulli polynomials in (13) by Bernoulli numbers. The result of these replacements is

$$\{[1 - 2^{-r}]g(1) + 2^{-r}g(2)\}\zeta(r,1)$$
$$(16) \qquad + \sum_{j=0}^{2m} \frac{\Gamma(r+j-1)}{j!\Gamma(r)} l^{1-r-j} \cdot \{[2^{j-1} - 1]g(l-1) - 2^{j-1}g(l)\}B_j$$
$$+ O([l-1]^{-r-2m-1}).$$

Like (12), formula (16) converges rapidly since almost half of the terms within the $j$-summation are zero.

Though condensation to the extent possible with $v = 2$ no longer generally occurs, formula (11) can be made considerably more arithmetically tractable for $v = 3$ or 4. Thus, making use of the identities $\zeta(r,\frac{2}{3}) = [3^r - 1]\zeta(r,1) - \zeta(r,\frac{1}{3})$, $\zeta(r,\frac{3}{4}) = [4^r - 2^r]\zeta(r,1) - \zeta(r,\frac{1}{4})$, $B_j(\frac{2}{3}) = [3^{1-j} - 1]B_j - B_j(\frac{1}{3})$ and $B_j(\frac{3}{4}) = [4^{1-j} - 2^{1-j}]B_j - B_j(\frac{1}{4})$, which follow from (14) and (15), we can convert formula (11) into

$$3^{-r}\{[g(1) - g(2)]\zeta(r,\tfrac{1}{3}) + [g(3) + (3^r - 1)g(2)]\zeta(r,1)\}$$
$$(17) \qquad + \sum_{j=0}^{2m} \frac{\Gamma(r+j-1)}{\Gamma(r)j!} l^{1-r-j}\{[-3]^{j-1}[g(l-2) - g(l-1)]B_j(\tfrac{1}{3})$$
$$+ [(3^{j-1} - 1)g(l-1) - 3^{j-1}g(l)]B_j\} + O([l-2]^{-r-2m-1})$$

and

$$4^{-r}\{[g(1) - g(3)]\zeta(r,\tfrac{1}{4}) + [g(4) + (4^r - 2^r)g(3) + (2^r - 1)g(2)]\zeta(r,1)\}$$
$$(18) \qquad + \sum_{j=0}^{2m} \frac{\Gamma(r+j-1)}{\Gamma(r)j!} l^{1-r-j}\{[-4]^{j-1}[g(l-3) - g(l-1)]B_j(\tfrac{1}{4})$$
$$+ [(4^{j-1} - 2^{j-1})g(l-2) + (2^{j-1} - 1)g(l-1) - 4^{j-1}g(l)]B_j\}$$
$$+ O([l-3]^{-r-2m-1})$$

for $v = 3$ and 4 respectively. When $g(l-2) = g(l-1)$ for $v = 3$ and when $g(l-3) = g(l-1)$ for $v = 4$, the Bernoulli polynomials vanish from the above formulas, leaving the rapidly converging type of asymptotic expansion which, as we have seen, is general for $v = 1$ and 2.

For $v = 5$, terms involving $\zeta(r, \frac{1}{5})$, $\zeta(r, \frac{2}{5})$, $\zeta(r, 1)$, $B_j(\frac{1}{5})$, $B_j(\frac{2}{5})$ and $B_j$ will generally be present. We shall not further pursue sums having five or more elements within a repeating sequence.

**4. Data required.** In this section we consider the numerical values which are needed to make practical use of formulas (12), (16), (17) and (18).

As far as the $l$-independent terms are concerned, it is values of $\zeta(r, 1)$, $\zeta(r, \frac{1}{3})$ and $\zeta(r, \frac{1}{4})$ which are needed for the $r$ value of interest. The $\zeta(r, 1)$ is the Riemann zeta function $\zeta(r)$ and is extensively tabulated, but tabulations of the bivariate function $\zeta(r, q)$ are few [6, p. 521] and do not include $q = \frac{1}{3}$ or $q = \frac{1}{4}$. To evaluate $\zeta(r, q)$ requires only the rearrangement of the $p \to \infty$ limit of (4). Thence we obtain

$$\zeta(r, q) = \lim_{p\to\infty} \left\{ \sum_{k=0}^{p-q} (k+q)^{-r} + \sum_{j=0}^{2m} \frac{\Gamma(r+j-1)}{\Gamma(r)j!} p^{1-r-j} B_j \right\},$$

where $m$ exceeds $-(r+1)/2$. For $r > 1$ no terms are needed in the second summation, though the inclusion of a few will hasten convergence. A computer program was written to exploit the formula above and a selection of the output is reproduced as Table 1. Existing tabulations have been used to check Table 1 at integral arguments. Since $\zeta(r, q)$ displays a singularity at $r = +1$, computation in the vicinity of this value is inaccurate. On the other hand, the function $W(r, q) = q/(1-r) + q^r\zeta(r, q)$ is well-behaved there. Moreover, as Fig. 1 shows, $W(r, q)$ is an excellent vehicle for interpolation in both $r$ and $q$.

Required to evaluate the $l$-dependent terms are values of $B_j/j!$, $[-3]^{j-1}B_j(\frac{1}{3})/j!$ and $[-4]^{j-1}B_j(\frac{1}{4})/j!$ for small values of $j$. Table 2 presents these data for $j$ values up to 5. The identity $z^jB_j(1/z) \equiv \sum_0^j \binom{j}{i} z^iB_i$ was used in constructing the final and penultimate columns of this tabulation.

**5. Examples.** To exemplify the utility of the foregoing treatment, it will suffice to consider the three problems which motivated this study.

When $r = -\frac{1}{2}$, $g(1) = +1$, $g(2) = -1$, $g(l-1) = \pm 1$, and $g(l) = \mp 1$ are inserted into formula (16), there results

$$\begin{aligned} 1 - 2^{1/2} + 3^{1/2} - \cdots \pm (l-1)^{1/2} \mp l^{1/2} \\ = (1 - 2^{3/2})\zeta(-\tfrac{1}{2}, 1) \pm \{-l^{1/2}/2 + O(l^{-1/2})\}. \end{aligned}$$

This equation provides the key to problem (a) since thence

$$\begin{aligned} \lim_{l\to\infty} \{1 - 2^{1/2} + 3^{1/2} - \cdots \pm (l-1)^{1/2} \mp \tfrac{1}{2}l^{1/2}\} &= (1 - 2^{3/2})\zeta(-\tfrac{1}{2}, 1) \\ &= 0.3801048. \end{aligned}$$

Because problem (b) involves a sum with $v = 4$, formula (18) is applicable. Inserting into it the values $r = -\frac{1}{2}$, $g(1) = +1$, $g(2) = g(4) = 0$, $g(3) = -1$,

TABLE 1

| $r$ | $\zeta(r, 1)$ | $\zeta(r, \frac{1}{3})$ | $\zeta(r, \frac{1}{4})$ |
|---|---|---|---|
| $-4$ | 0 | $+0.00411527$ | $+0.00488281$ |
| $-3\frac{1}{2}$ | $+0.00444098$ | $+0.00131732$ | $+0.00400423$ |
| $-3$ | $+0.00833333$ | $-0.00401235$ | $-0.00045573$ |
| $-2\frac{1}{2}$ | $+0.00851692$ | $-0.00998427$ | $-0.00803810$ |
| $-2$ | 0 | $-0.01234568$ | $-0.01562500$ |
| $-1\frac{1}{2}$ | $-0.02548520$ | $-0.00353146$ | $-0.01510930$ |
| $-1\frac{1}{3}$ | $-0.04006133$ | $+0.00376158$ | $-0.01055877$ |
| $-1\frac{1}{6}$ | $-0.05896522$ | $+0.01403623$ | $-0.00241808$ |
| $-1$ | $-0.08333333$ | $+0.02777778$ | $+0.01041667$ |
| $-\frac{5}{6}$ | $-0.11469997$ | $+0.04539704$ | $+0.02922294$ |
| $-\frac{2}{3}$ | $-0.15519690$ | $+0.06704321$ | $+0.05539997$ |
| $-\frac{1}{2}$ | $-0.20788622$ | $+0.09244628$ | $+0.09032226$ |
| $-\frac{1}{3}$ | $-0.27734305$ | $+0.12032492$ | $+0.13499200$ |
| $-\frac{1}{6}$ | $-0.37073766$ | $+0.14746313$ | $+0.18925307$ |
| 0 | $-0.50000000$ | $+0.16666667$ | $+0.25000000$ |
| $+\frac{1}{6}$ | $-0.68658158$ | $+0.16215837$ | $+0.30688655$ |
| $+\frac{1}{3}$ | $-0.97336025$ | $+0.09786236$ | $+0.33101390$ |
| $+\frac{1}{2}$ | $-1.46035451$ | $-0.11808333$ | $+0.23996352$ |
| $+\frac{2}{3}$ | $-2.44758074$ | $-0.77591123$ | $-0.24411861$ |
| $+\frac{5}{6}$ | $-5.43504324$ | $-3.36389701$ | $-2.59333588$ |
| $+1$ | $\infty$ | $\infty$ | $\infty$ |
| $+1\frac{1}{6}$ | $+6.58921554$ | $+9.72877813$ | $+11.26251968$ |
| $+1\frac{1}{3}$ | $+3.60093775$ | $+7.44662281$ | $+9.56763334$ |
| $+1\frac{1}{2}$ | $+2.61237535$ | $+7.30992572$ | $+10.21305536$ |
| $+2$ | $+1.64493407$ | $+10.09559713$ | $+17.19732915$ |
| $+2\frac{1}{2}$ | $+1.34148726$ | $+16.33304416$ | $+32.84745195$ |
| $+3$ | $+1.20205690$ | $+27.56106120$ | $+64.66386997$ |
| $+3\frac{1}{2}$ | $+1.12673387$ | $+47.21062129$ | $+128.546959$ |
| $+4$ | $+1.08232323$ | $+81.36396942$ | $+256.463691$ |

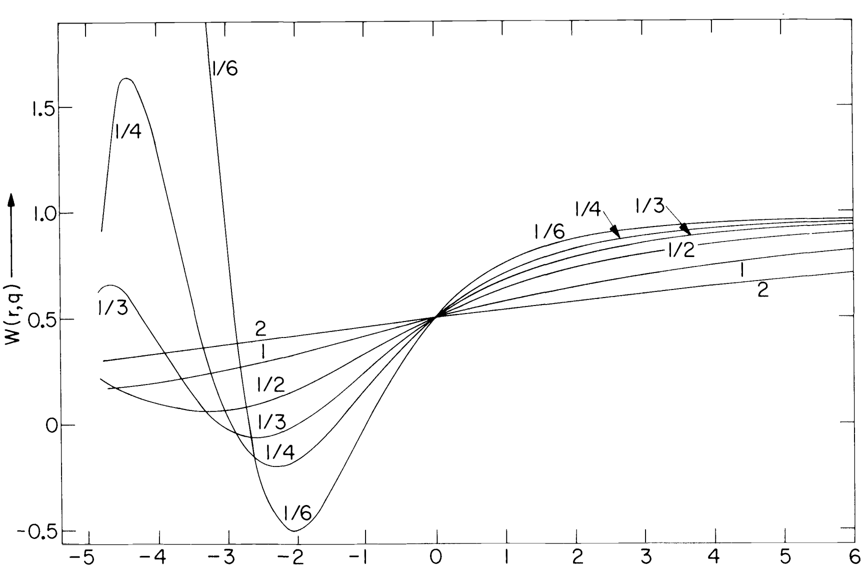


FIG. 1. *Values of the function* $W(r, q)$ *for* $-5 < r \leqq 6$ *for* $q = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$, 1 *and* 2

TABLE 2

| $j$ | $B_j/j!$ | $[-3]^{j-1}B_j(\frac{1}{3})/j!$ | $[-4]^{j-1}B_j(\frac{1}{4})/j!$ |
|---|---|---|---|
| 0 | 1 | $-\frac{1}{3}$ | $-\frac{1}{4}$ |
| 1 | $-\frac{1}{2}$ | $-\frac{1}{6}$ | $-\frac{1}{4}$ |
| 2 | $\frac{1}{12}$ | $\frac{1}{12}$ | $\frac{1}{24}$ |
| 3 | 0 | $\frac{1}{18}$ | $\frac{1}{8}$ |
| 4 | $-\frac{1}{720}$ | $-\frac{13}{720}$ | $-\frac{7}{1440}$ |
| 5 | 0 | $-\frac{1}{72}$ | $-\frac{5}{96}$ |

$g(L-3) = g(L-1) = 0$, $g(L-2) = \mp 1$, and $g(L) = \pm 1$, we obtain the simple result

$$1 - 3^{1/2} + 5^{1/2} - \cdots \mp [L-2]^{1/2} \pm L^{1/2} = 4\zeta(-\tfrac{1}{2}, \tfrac{1}{4})$$
$$+ [\sqrt{2} - 1]\zeta\left(-\frac{1}{2}, 1\right) \pm \left\{\frac{L^{1/2}}{2} + \frac{L^{-1/2}}{4} - \frac{L^{-5/2}}{16} + O(L^{-9/2})\right\}.$$

Table 2 gives the $L$-independent term as 0.2751797.

Problem (c) involves a more complex application of formula (18). Four different asymptotic representations apply, according as $l \pmod 4$ is 0, 1, 2 or 3. The same $l$-independent term applies to all four and is found straightforwardly to be

$$T = 4^{5/3}\zeta(-\tfrac{7}{6}, \tfrac{1}{4}) + [4^{5/3} - 1]\zeta(-\tfrac{7}{6}, 1) = -0.55974.$$

By noting that in problem (c), $g(l-3)$ and $g(l-1)$ are necessarily of opposite sign, as are $g(l-2)$ and $g(l)$, we may reduce formula (18), with $r = \frac{7}{6}$, to

$$T + [g(l-1) + g(l)]\frac{l^{7/6}}{2} + g(l)\frac{7l^{1/6}}{12} - g(l-1)\frac{7l^{-5/6}}{144} + g(l)\frac{35l^{-11/6}}{1296} + \cdots.$$

The four representations now result by giving $g(l)$ and $g(l-1)$ each the values $\pm 1$.

**Acknowledgment.** It is a pleasure to record my indebtedness to Dr. Jerome Spanier for his advice and encouragement.

REFERENCES


[1] K. B. OLDHAM AND L. E. TOPOL, J. Phys. Chem., 73 (1969), pp. 1462–1469.

[2] M. ABRAMOWITZ AND I. STEGUN, eds., *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, National Bureau of Standards Appl. Math. Series, No. 55, U.S. Government Printing Office, Washington, D.C., 1964.

[3] I. S. GRADSTEYN AND I. M. RYZHIK, *Table of Integrals, Series and Products*, Academic Press, New York, 1965.

[4] L. B. W. JOLLEY, *Summation of Series*, Dover, New York, 1961.

[5] A. D. WHEELON, *A Short Table of Summable Series*, Rep. SM-14642, Douglas Aircraft Co., Inc., Santa Monica, Calif., 1953.

[6] A. FLETCHER, J. C. P. MILLER, L. ROSENHEAD AND L. J. COMRIE, *An Index of Mathematical Tables*, Blackwell, Oxford, 1962.

[7] N. G. DEBRUYN, *Asymptotic Methods in Analysis*, North-Holland, Amsterdam, 1961.

[8] G. H. HARDY, *Divergent Series*, Oxford University Press, London, 1949.

[9] K. KNOPP, *Theory and Application of Infinite Series*, Hafner, New York, 1951.

[10] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, vol. I, McGraw-Hill, New York, 1953.

[11] G. H. Hardy and E. M. Wright, *Introduction to the Theory of Numbers*, Oxford University Press, London, 1938.

SIAM J. MATH. ANAL.
Vol. 1, No. 4, November 1970

# FRACTIONAL INTEGRALS OF DISTRIBUTIONS*

A. ERDÉLYI AND A. C. MCBRIDE†

**Abstract.** Certain operators of fractional integration arising in connection with singular differential operators, Hankel transforms, and dual integral equations involve integration of fractional order with respect to $r^2$ and multiplication of functions by fractional powers of the independent variable. Such operations are not meaningful for distributions. In this paper a class of generalized functions is introduced on which such operations can meaningfully be performed. The operations are defined as adjoints of corresponding operations on a suitably selected space of testing functions. Relations to spherically symmetric $n$-dimensional distributions and to the singular differential operator

$$\frac{d^2}{dr^2} + \frac{2\nu + 1}{r}\frac{d}{dr}$$

are discussed.

**1.** The integral of order $\alpha$ of a locally integrable function $f$ can, for $\operatorname{Re}\alpha > 0$, be defined by the formula

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1} f(y)\,dy. \tag{1.1}$$

One has

$$I^\alpha f(x) = \frac{d^n}{dx^n} I^{\alpha+n} f(x), \qquad \operatorname{Re}\alpha > 0, \quad n = 0, 1, 2, \cdots, \tag{1.2}$$

and this formula can be used to extend the definition of $I^\alpha$ to $\operatorname{Re}\alpha > -n$, for instance, if $f$ is $n-1$ times differentiable, $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$, and $f^{(n-1)}$ is locally absolutely continuous.

Let us set $f(x) = 0$ for $x < 0$ and

$$p_\alpha(x) = \begin{cases} \dfrac{x^{\alpha-1}}{\Gamma(\alpha)}, & x > 0, \\ 0, & x \leqq 0. \end{cases} \tag{1.3}$$

Then (1.1) can be written as a convolution,

$$I^\alpha f(x) = \int_{-\infty}^{\infty} p_\alpha(x-y) f(y)\,dy,$$

or, more briefly,

$$I^\alpha f = p_\alpha * f; \tag{1.4}$$

and this form can be used to define fractional integrals of distributions whose support is in $[0, \infty[$. Indeed, for $\operatorname{Re}\alpha > 0$, the locally integrable function (1.3) defines a (regular) distribution whose support is $[0, \infty[$, i.e., an element of $\mathscr{D}'_+$. This distribution is an analytic function of $\alpha$ and can be continued analytically to the

* Received by the editors December 23, 1969.
† Mathematical Institute, University of Edinburgh, Edinburgh EH1 1HZ, Scotland.

entire $\alpha$-plane. From now on, $p_\alpha$ will represent the entire function of $\alpha$ with values in $\mathscr{D}'_+$ obtained by this process. For each $\alpha \in C$ and each $f \in \mathscr{D}'_+$, the convolution $p_\alpha * f$ exists and defines $I^\alpha f$. Important basic relations, e.g.,

$$I^\alpha I^\beta f = I^{\alpha+\beta} f, \qquad I^{-n} f = f^{(n)}, \qquad n = 0, 1, 2, \cdots \tag{1.5}$$

($f^{(n)}$ denotes the distribution derivative of order $n$ of $f$), follow from the theory of convolution of distributions in conjunction with

$$p_{-n} = \delta^{(n)}, \qquad n = 0, 1, 2, \cdots.$$

This theory is well known; see, for instance, [2, Chap. 1, § 5].

For certain purposes, for instance, in connection with Hankel transforms, certain partial differential equations, and dual integral equations, certain modifications and extensions of $I^\alpha$ must be considered, in particular, the operator $I^{\eta;\alpha}_{r^m}$ defined by

$$I^{\eta;\alpha}_{r^m} f(r) = \frac{m}{\Gamma(\alpha)} r^{-m\alpha - m\eta} \int_0^r (r^m - u^m)^{\alpha-1} u^{m\eta+m-1} f(u)\, du \tag{1.6}$$

when $\operatorname{Re} \alpha > 0$, $m > 0$, $|f|^p$ is locally integrable, $1 \leqq p \leqq \infty$, and $m \operatorname{Re} \eta + m > 1/p$. Here one integrates with respect to $r^m$ rather than $r$, and in addition multiplies both $f$ and the integral by appropriate powers of the variable. The case $m = 2$, when $r$ is the distance in a Euclidean space, is especially important. The operators were known to Poisson [8]; their theory was developed and applied to Hankel transforms by Kober, partly in collaboration with one of us [4], [5]; applications to abstract differential equations (including partial differential equations) were most thoroughly investigated by Lions [6], see also [7, Chap. 12]; while applications to dual integral equations were given by Sneddon and Erdélyi [9] and others.

In spite of the close connection between (1.6) and (1.1), it does not appear to be possible to base a theory of $I^{\eta;\alpha}_{r^m}$ on convolution in $\mathscr{D}'_+$. Even when $m = 1$ and the integral in (1.6) is a convolution integral, a difficulty arises since (pointwise) multiplication of a distribution by an arbitrary power of the variable lacks meaning; and when $m \neq 1$ there is the added difficulty that, in general, $x = r^m$ is not a permissible change of variable in $\mathscr{D}'_+$. For this reason we propose an alternative approach to (1.1) and will show how this alternative approach can be extended to (1.6).

Along with $I^\alpha$ we consider the adjoint operator $K^\alpha$ defined, for $\operatorname{Re} \alpha > 0$ and a function $\phi$ with suitable integrability properties, by

$$K^\alpha \phi(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (y - x)^{\alpha-1} \phi(y)\, dy. \tag{1.7}$$

One has

$$K^\alpha \phi(x) = \left( -\frac{d}{dx} \right)^n K^{\alpha+n} \phi(x), \tag{1.8}$$

and if $\phi$ is a testing function, i.e., an infinitely differentiable function with compact support, (1.8) can be used to define $K^\alpha \phi$ for all complex numbers $\alpha$. With this extended definition,

$$K^\alpha K^\beta \phi = K^{\alpha+\beta} \phi, \qquad K^{-n} \phi(x) = (-1)^n \phi^{(n)}(x). \tag{1.9}$$

Now, for a locally integrable $f$, $\phi \in \mathscr{D}$, and $\operatorname{Re}\alpha > 0$,

$$\int_0^\infty I^\alpha f(x)\phi(x)\,dx = \int_0^\infty f(x)K^\alpha\phi(x)\,dx$$

follows by Fubini's theorem and suggests the definition, for $f \in \mathscr{D}'_+$ and all $\alpha \in C$, of $I^\alpha f \in \mathscr{D}'_+$ as the distribution given by

$$(1.10) \qquad (I^\alpha f, \phi) = (f, K^\alpha\phi).$$

Although $K^\alpha\phi \notin \mathscr{D}$, the right-hand side of (1.10) is well-defined for $f \in \mathscr{D}'_+$; it is easy to show that $I^\alpha f \in \mathscr{D}'_+$ and, for instance, (1.5) is a simple consequence of (1.9).

We shall not carry out the details of this program but present an analogous approach to the theory of $I^{\eta,\alpha}_{r^m}$. The latter operator will be applied to generalized functions that are continuous linear functionals on certain testing function spaces. Our testing functions are infinitely differentiable functions on $]0, \infty[$ and have compact support in $[0, \infty[$.

In §§ 2, 3 we introduce this testing function space and investigate its properties, including its behavior under the operation of $K^{\eta,\alpha}_{r^m}$. In §§ 4, 5 we introduce the dual space, define $I^{\eta,\alpha}_{r^m}$ on it, and investigate the connection between our generalized functions and spherically symmetric $n$-dimensional distributions in the sense of Schwartz. In the final section, § 6, we prove

$$(1.11) \qquad I^{\nu,\alpha}_{r^2}\left(\frac{d^2}{dr^2} + \frac{2\nu+1}{r}\frac{d}{dr}\right)r^2 f = \left(\frac{d^2}{dr^2} + \frac{2\nu+2\alpha+1}{r}\frac{d}{dr}\right)I^{\nu,\alpha}_{r^2} r^2 f$$

for our generalized functions $f$.

It is known that (1.11) is the key to the application of integrals of fractional order to differential equations (including partial differential equations and abstract differential equations) and to Hankel transforms. We shall not follow this up here, but hope to develop on a future occasion such applications and also the theory of fractional integration in $\mathscr{D}'_{L_p}$ which can be approached in a similar manner.

**2.** For each $l > 0$, let $\mathscr{I}_l$ be the collection of all those complex-valued infinitely differentiable functions $\phi$ of a positive variable $r$ which vanish outside the interval $[0, l]$ and for which

$$(2.1) \qquad \gamma_k(\phi) = \sup\left\{r^k\left|\frac{d^k\phi(r)}{dr^k}\right| : r > 0\right\}$$

is finite for each nonnegative integer $k$. Clearly $\mathscr{I}_l$, with the usual pointwise operations of addition and multiplication by a complex number, is a vector space over the complex numbers, the $\gamma_k$ are seminorms on $\mathscr{I}_l$, and, in particular, $\gamma_0$ is a norm. The collection $M = \{\gamma_k : k = 0, 1, 2, \cdots\}$ is a countable multinorm [11, p. 8] on $\mathscr{I}_l$ and with the topology generated by this multinorm $\mathscr{I}_l$ is a countably multinormed space [11, § 1.6]. As in the case of $\mathscr{D}_K$ (see [11, Example 1.6.1]) it is easily seen that this space is complete and hence a Fréchet space. It is equally easily seen that

$$(2.2) \qquad \mathscr{I} = \bigcup_{l=1}^{\infty} \mathscr{I}_l$$

is a complete strict countable union space.

There is some resemblance between $\gamma_k$ and the seminorms used by Zemanian [11, § 4.2] in connection with the Mellin transformation, but our condition on the behavior of the test functions at $\infty$ is much more drastic. In fact, $\mathscr{I} \subset \mathscr{M}_{1,b}$ for any $b$.

**3.** We next consider some operators on $\mathscr{I}$.

For any fixed complex $\mu$ with $\operatorname{Re}\mu \geqq 0$, we define the operator $r^\mu$ by

$$(r^\mu\phi)(r) = r^\mu\phi(r), \qquad r > 0.$$

No confusion will arise from using the same symbol for the function $r^\mu$ and multiplication by this function. Clearly $r^\mu$ is a linear operator, and to prove its continuity, it is sufficient to establish continuity at the origin. If $\phi \in \mathscr{I}$, then $\phi \in \mathscr{I}_l$ for some $l$, and

$$\frac{d^k(r^\mu\phi(r))}{dr^k} = \sum_{h=0}^{k}\binom{k}{h}\frac{\Gamma(\mu+1)}{\Gamma(\mu-h+1)}r^{\mu-h}\frac{d^{k-h}\phi(r)}{dr^{k-h}}.$$

Then

$$\gamma_k(r^\mu\phi) \leqq l^{\operatorname{Re}\mu}\sum_{h=0}^{k}\binom{k}{h}\left|\frac{\Gamma(\mu+1)}{\Gamma(\mu-h+1)}\right|\gamma_{k-h}(\phi),$$

so that $r^\mu$ maps $\mathscr{I}$ into itself continuously. Except when $\operatorname{Re}\mu = 0$, the map is not onto, but it is one-to-one.

Next we define the operator

$$\delta = r\frac{d}{dr}$$

by

$$(\delta\phi)(r) = r\frac{d\phi(r)}{dr}.$$

Since

$$\frac{d^k}{dr^k}\left(r\frac{d\phi}{dr}\right) = r\frac{d^{k+1}\phi}{dr^{k+1}} + k\frac{d^k\phi}{dr^k},$$

we have

$$\gamma_k(\delta\phi) \leqq \gamma_{k+1}(\phi) + k\gamma_k(\phi)$$

so that $\delta$ is a continuous linear mapping of $\mathscr{I}$ into itself. $\delta$ is clearly one-to-one, since $\delta\phi = 0$ only for constant functions and the only constant function in $\mathscr{I}$ is 0. $\delta$ is not onto. Consider, for example,

$$\phi_1(r) = \begin{cases} \exp\left(-\dfrac{1}{1-r}\right) & \text{if } 0 < r < 1, \\ 0 & \text{if } r \geqq 1. \end{cases}$$

Clearly $\phi_1 \in \mathscr{I}$. If $\delta\phi = \phi_1$, then $\phi = 0$ for $r \geqq 1$ and for $0 < r < 1$,

$$\phi(r) = -\int_r^1 \exp\left(-\frac{1}{1-u}\right)\frac{du}{u}.$$

But then $\gamma_0(\phi) = +\infty$ and $\phi \notin \mathscr{I}$.

Similarly the operator $\delta'$ defined by

$$\delta'\phi(r) = \frac{d}{dr} r\phi(r)$$

is a continuous one-to-one linear mapping of $\mathscr{I}$ into itself. Clearly,

$$\delta'\phi - \delta\phi = \phi.$$

For $m > 0$, complex numbers $\alpha$ and $\eta$ with $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\eta > 0$, and $\phi \in \mathscr{I}$, we define the "homogeneous" operator $K_{r^m}^{\eta,\alpha}$ of fractional integration of order $\alpha$ with respect to $r^m$ by

$$K_{r^m}^{\eta,\alpha}\phi(r) = \frac{mr^{m\eta}}{\Gamma(\alpha)} \int_r^\infty (u^m - r^m)^{\alpha-1}\phi(u)u^{-\alpha m - \eta m + m - 1}\, du. \tag{3.1}$$

The integral is a finite integral, and

$$K_{r^m}^{\eta,\alpha}\phi(r) = \frac{m}{\Gamma(\alpha)} \int_1^\infty (t^m - 1)^{\alpha-1}\phi(rt)t^{-\alpha m - \eta m + m - 1}\, dt \tag{3.2}$$

is infinitely differentiable if $r > 0$.

$$\frac{d^k}{dr^k}(K_{r^m}^{\eta,\alpha}\phi(r)) = \frac{m}{\Gamma(\alpha)} \int_1^\infty (t^m - 1)^{\alpha-1}\phi^{(k)}(rt)t^{k-\alpha m - \eta m + m - 1}\, dt,$$

and so

$$\gamma_k(K_{r^m}^{\eta,\alpha}\phi) \leqq \frac{m}{|\Gamma(\alpha)|} \int_1^\infty |(t^m - 1)^{\alpha-1}t^{-\alpha m - \eta m + m - 1}|\, dt\gamma_k(\phi).$$

Since the infinite integral is convergent, $K_{r^m}^{\eta,\alpha}$ is a continuous linear mapping of $\mathscr{I}$ into itself.

For $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$, $\operatorname{Re}\eta > 0$, we have

$$K_{r^m}^{\eta,\alpha}K_{r^m}^{\eta+\alpha,\beta}\phi = K_{r^m}^{\eta,\alpha+\beta}\phi \tag{3.3}$$

by an interchange in the order of integrations on the left-hand side; and

$$\begin{aligned} r^m \frac{d}{dr^m} K_{r^m}^{\eta,\alpha+1}\phi &= \frac{1}{m}\delta K_{r^m}^{\eta,\alpha+1}\phi = \frac{1}{m}K_{r^m}^{\eta,\alpha+1}\delta\phi \\ &= \eta K_{r^m}^{\eta,\alpha+1}\phi - K_{r^m}^{\eta+1,\alpha}\phi \end{aligned} \tag{3.4}$$

by straightforward differentiation of (3.2) and (3.1). A counterpart to (3.4) is

$$\begin{aligned} K_{r^m}^{\eta,\alpha+1}\left(r^m \frac{d}{dr^m}\phi(r)\right) &= \frac{1}{m}K_{r^m}^{\eta,\alpha+1}\delta\phi = \frac{1}{m}\delta K_{r^m}^{\eta,\alpha+1}\phi \\ &= (\eta + \alpha)K_{r^m}^{\eta,\alpha+1}\phi - K_{r^m}^{\eta,\alpha}\phi \end{aligned} \tag{3.5}$$

and can be established by integration by parts.

Equation (3.5) can be used to extend the definition of $K_{r^m}^{\eta,\alpha}$ in the first place to $\operatorname{Re}\alpha > -1$ and then, by repeated application, to the entire complex $\alpha$-plane. The extended operator is an analytic function of $\alpha$, and (3.3) to (3.5) are valid for all $\alpha$, $\beta$

and $\operatorname{Re}\eta > 0$, by analytic continuation, provided that in the case of (3.3) also $\operatorname{Re}(\eta + \alpha) > 0$.

From (3.5),

$$K_{rm}^{\eta,0}\phi(r) = \eta K_{rm}^{\eta,1}\phi(r) - \frac{1}{m}K_{rm}^{\eta,1}\delta\phi(r)$$

$$= -r^{\eta m}\int_r^\infty \frac{d}{du}(\phi(u)u^{-\eta m})\,du$$

and so

$$K_{rm}^{\eta,0}\phi = \phi. \tag{3.6}$$

As a consequence,

$$K_{rm}^{\eta,\alpha}K_{rm}^{\eta+\alpha,-\alpha}\phi = \phi \tag{3.7}$$

if $\operatorname{Re}\eta > 0$, $\operatorname{Re}(\eta + \alpha) > 0$.

From (3.4) and (3.5) we also have

$$\delta' K_{rm}^{\eta,\alpha+1}\phi = K_{rm}^{\eta,\alpha+1}\delta'\phi = (m\eta + 1)K_{rm}^{\eta,\alpha+1}\phi - mK_{rm}^{\eta+1,\alpha}\phi \tag{3.8}$$

and

$$K_{rm}^{\eta,\alpha+1}\delta'\phi = \delta' K_{rm}^{\eta,\alpha+1}\phi = (m\eta + m\alpha + 1)K_{rm}^{\eta,\alpha+1}\phi - mK_{rm}^{\eta,\alpha}\phi. \tag{3.9}$$

**4.** A locally integrable function $f$ on $[0, \infty[$ defines a continuous linear functional on $\mathscr{I}$ by means of the formula

$$(f, \phi) = \int_0^\infty f(r)\phi(r)\,dr.$$

We shall consider all continuous linear functionals on $\mathscr{I}$ as generalized functions. For these we can use the theory set out in §§ 1.8 and 1.9 of [11]. The dual of $\mathscr{I}$ will as usual be denoted by $\mathscr{I}'$. It is complete since $\mathscr{I}$ is a countable union of complete countably multinormed spaces.

The operators formerly defined on $\mathscr{I}$ will give rise to adjoint operators on the dual space [11, § 1.6].

The operator $r^\mu$ on $\mathscr{I}'$ is defined by

$$(r^\mu f, \phi) = (f, r^\mu \phi)$$

and the operators $\delta$ and $\delta'$ by

$$(\delta f, \phi) = -(f, \delta'\phi),$$

$$(\delta' f, \phi) = -(f, \delta\phi).$$

All three are continuous linear mappings of $\mathscr{I}'$ into itself [11, Theorem 1.10.1], and the notation has been so chosen as to be consistent in case $f$ is an element generated by a differentiable function.

For $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\eta > m^{-1} - 1$, and a function $f$ that is locally integrable on $[0, \infty[$, the "homogeneous" operator $I_{rm}^{\eta,\alpha}$ of fractional integration of order $\alpha$ with respect to $r^m$ is defined by

$$I_{r^m}^{\eta,\alpha}f(r) = \frac{m}{\Gamma(\alpha)} r^{-m\eta-m\alpha} \int_0^r (r^m - u^m)^{\alpha-1} f(u) u^{m\eta+m-1}\, du$$
$$= \frac{m}{\Gamma(\alpha)} \int_0^1 (1 - t^m)^{\alpha-1} f(rt) t^{m\eta+m-1}\, dt.$$

$I_{r^m}^{\eta,\alpha}f$ itself is locally integrable and for any $\phi \in \mathscr{I}$,

$$(I_{r^m}^{\eta,\alpha}f, \phi) = \frac{m}{\Gamma(\alpha)} \int_0^\infty \phi(r) r^{-m\eta-m\alpha} \left( \int_0^r (r^m - u^m)^{\alpha-1} f(u) u^{m\eta+m-1}\, du \right) dr.$$

Here the order of integrations may be interchanged and one has

$$(4.1) \qquad (I_{r^m}^{\eta,\alpha}f, \phi) = (f, K_{r^m}^{\eta',\alpha}\phi),$$

where $\eta' = \eta + 1 - m^{-1}$.

As $\phi$ ranges over $\mathscr{I}$, the right-hand side of (4.1) defines a functional on $\mathscr{I}$ for any $f \in \mathscr{I}'$ and complex number $\alpha$ provided $\operatorname{Re}\eta > m^{-1} - 1$, and we take this functional as the definition of $I_{r^m}^{\eta,\alpha}f$. With this definition, $I_{r^m}^{\eta,\alpha}$ is a continuous linear operator on $\mathscr{I}'$ and the notation is consistent in the case of elements of $\mathscr{I}'$ generated by a locally integrable function.

The relations

$$(4.2) \qquad I_{r^m}^{\eta,0}f = f,$$

$$(4.3) \qquad I_{r^m}^{\eta+\alpha,\beta} I_{r^m}^{\eta,\alpha}f = I_{r^m}^{\eta,\alpha+\beta}f$$

provided

$$\operatorname{Re}\eta > m^{-1} - 1, \qquad \operatorname{Re}(\eta + \alpha) > m^{-1} - 1,$$

$$(4.4) \qquad \delta I_{r^m}^{\eta,\alpha+1}f = I_{r^m}^{\eta,\alpha+1}\delta f = m I_{r^m}^{\eta,\alpha}f - (m\eta + m\alpha + m) I_{r^m}^{\eta,\alpha+1}f,$$

$$(4.5) \qquad I_{r^m}^{\eta,\alpha+1}\delta f = \delta I_{r^m}^{\eta,\alpha+1}f = m I_{r^m}^{\eta+1,\alpha}f - (m\eta + m) I_{r^m}^{\eta,\alpha+1}f,$$

$$(4.6) \qquad \delta' I_{r^m}^{\eta,\alpha+1}f = I_{r^m}^{\eta,\alpha+1}\delta' f = m I_{r^m}^{\eta,\alpha}f - (m\eta + m\alpha + m - 1) I_{r^m}^{\eta,\alpha+1}f,$$

$$(4.7) \qquad I_{r^m}^{\eta,\alpha+1}\delta' f = \delta' I_{r^m}^{\eta,\alpha+1}f = m I_{r^m}^{\eta+1,\alpha}f - (m\eta + m - 1) I_{r^m}^{\eta,\alpha+1}f,$$

$$(4.8) \qquad I_{r^m}^{\eta+\alpha,-\alpha} I_{r^m}^{\eta,\alpha}f = f$$

provided

$$\operatorname{Re}\eta > m^{-1} - 1 \quad \text{and} \quad \operatorname{Re}(\eta + \alpha) > m^{-1} - 1$$

are consequences of (3.3) to (3.9).

**5.** There is a connection between certain subspaces of $\mathscr{I}'$ and $n$-dimensional spherically symmetric distributions. We shall assume familiarity with the elements of distribution theory [11, Chap. 2] and shall use the customary notations.

$$x = (x_1, \cdots, x_n)$$

will denote a typical point of $\mathbb{R}^n$, and we set

$$r = |x| = (x_1^2 + \cdots + x_n^2)^{1/2}.$$

A function $f$ on $\mathbb{R}^n$ is said to be spherically symmetric if it depends only on $|x|$, i.e., if for any orthogonal transformation $S$ of $\mathbb{R}^n$, $f(Sx) = f(x)$. A distribution $f \in \mathscr{D}'_n$ is said to be spherically symmetric if for any orthogonal transformation $S$ of $\mathbb{R}^n$,

$$\langle f(Sx), \chi(x)\rangle = \langle f(x), \chi(S^{-1}x)\rangle = \langle f(x), \chi(x)\rangle$$

for each $\chi \in \mathscr{D}_n$.

Let $f(x)$ be a spherically symmetric function that is locally integrable on $\mathbb{R}^n$. Then

$$\langle f, \chi\rangle = \int_{R^n} f(x)\chi(x)\,dx.$$

If we introduce polar coordinates in $\mathbb{R}^n$, setting $x = r\xi$ where $\xi$, a unit vector, ranges over the $n$-dimensional unit sphere $\Omega$ with surface element $d\omega$, then $dx = r^{n-1}\,dr\,d\omega$ and

$$\langle f, \chi\rangle = \int_0^\infty g(r)\phi(r)\,dr = (g, \phi), \tag{5.1}$$

where $f(x) = g(r)$ and

$$\phi(r) = r^{n-1}\int_\Omega \chi(r\xi)\,d\omega = r^{n-1}\psi(r), \tag{5.2}$$

say. Since $\chi$ is infinitely differentiable, so is $\psi$, and since $\chi$ vanishes for sufficiently large $|x|$, $\phi(r) = 0$ for $r > l$ for some $l$. It follows from the formula

$$\frac{d^k\psi(r)}{dr^k} = \int_\Omega \sum_{|v|=k} (D^v\chi)(r\xi)\xi^v\,d\omega$$

in which $v = (v_1, \cdots, v_n)$ is a multi-index of length $|v| = v_1 + \cdots + v_n$,

$$D_i = \frac{\partial}{\partial x_i}, \quad \xi^v = \xi_1^{v_1}\cdots\xi_n^{v_n}, \quad D^v = D_1^{v_1}\cdots D_n^{v_n},$$

that $\phi = T\chi$ defines a continuous linear mapping $T$ of $\mathscr{D}_n$ into $\mathscr{I}$. Clearly the map is many-to-one, and it is not onto. For instance, let $\rho(r)$ be infinitely differentiable, $= 1$ for $0 \leqq r \leqq 1$, and $= 0$ for $r \geqq 2$. Then $r^{1/2}\rho(r)$ is an element of $\mathscr{I}$ but not in $T\mathscr{D}_n$.

We shall now show by induction that

$$\begin{aligned}\left(\frac{d}{dr^2}\right)^k\psi(r) &= \left(\frac{d}{dr^2}\right)^k\int_\Omega \chi(r\xi)\,d\omega\\ &= \frac{1}{2^{2k-1}(k-1)!}\int_0^1 (1-u^2)^{k-1}\left(\int_\Omega \sum_{|v|=2k}(D^v\chi)(ru\xi)\xi^v\,d\omega\right)du.\end{aligned} \tag{5.3}$$

Indeed,

$$\frac{d\psi(r)}{dr^2} = \frac{1}{2r}\int_\Omega \sum_{i=1}^n \xi_i(D_i\chi)(r\xi)\,d\omega,$$

the integral vanishes if $r = 0$ since the integrand is then skew symmetric on $\Omega$, and consequently,

$$\int_\Omega \sum \xi_i D_i \chi(r\xi)\, d\omega = \int_0^r \frac{d}{d\rho}\left(\int_\Omega \sum \xi_i D_i \chi(\rho\xi)\, d\omega\right) d\rho.$$

Now,

$$\frac{d}{d\rho} g(\rho\xi) = \sum_{j=1}^n \xi_j (D_j g)(\rho\xi).$$

Using this and setting $\rho = ru$, we have (5.3) for $k = 1$.

Suppose (5.3) holds for $k$. Then

$$\left(\frac{d}{dr^2}\right)^{k+1} \psi(r) = \frac{1}{2^{2k}(k-1)!\,r} \int_0^1 (1-u^2)^{k-1} \left(\int_\Omega \sum_{|\nu|=2k+1} \xi^\nu (D^\nu \chi)(ru\xi)\, d\omega\right) u\, du.$$

As before, the integrand is skew symmetric, and the integral over $\Omega$ vanishes if $r = 0$, so that

$$\left(\frac{d}{dr^2}\right)^{k+1} \psi(r) = \frac{1}{2^{2k}(k-1)!} \int_0^1 \int_0^1 u^2(1-u^2)^{k-1} \cdot \left(\int_\Omega \sum_{|\nu|=2k+2} \xi^\nu (D^\nu \chi)(ruv\,\xi)\, d\omega\right) du\, dv.$$

Here we change the variables of integration from $u, v$ to $u$ and $w = uv$ and integrate with respect to $u$ to obtain (5.3) for $k + 1$.

It follows from (5.3) that $\psi$ is infinitely differentiable with respect to $r^2$ for $r \geqq 0$. Let $\mathscr{I}^\lambda$ denote that subspace of $\mathscr{I}$ consisting of all functions in $\mathscr{I}$ of the form $\phi(r) = r^{\lambda-1}\psi(r)$, where $\psi(r)$ is infinitely differentiable with respect to $r^2$ for $r \geqq 0$. There is a many-to-one correspondence, defined by (5.2), between $n$-dimensional test functions and elements of $\mathscr{I}^n$. To show that $T\mathscr{D}_n = \mathscr{I}_n$, we need only to note that corresponding to each $\phi(r) = r^{n-1}\psi(r)$ in $\mathscr{I}^n$ we may take

$$\chi(x) = \psi(|x|) \Big/ \int_\Omega d\omega$$

and then have $T\chi = \phi$.

Alternatively, $T$ establishes a one-to-one correspondence between spherically symmetric $n$-dimensional testing functions and elements of $\mathscr{I}^n$. $T$ is continuous but its inverse is not. However, if we equip $\mathscr{I}^n$ with the topology generated by the seminorms

$$\gamma_k^0(\phi) = \sup\left\{\left|\left(\frac{d}{dr^2}\right)^k \psi(r)\right| : r \geqq 0\right\}$$

and call the resulting space $\mathscr{I}^{n,0}$, then it can be shown that $T$ is an isomorphism between the subspace of spherically symmetric elements of $\mathscr{D}_n$ and $\mathscr{I}^{n,0}$. Unfortunately for our approach to fractional integrals, $\mathscr{I}^n$ is not an invariant subspace of $\mathscr{I}$ for the operator $K_{r^2}^{\eta,\alpha}$.

We can now show that each $f \in \mathscr{I}'$, indeed each $f \in \mathscr{I}^{n\prime}$, corresponds to a spherically symmetric distribution $T'f$ on $\mathbb{R}^n$ defined by

$$\langle T'f, \chi\rangle = (f, T\chi), \qquad \chi \in \mathscr{D}_n. \tag{5.4}$$

By [11, Theorem 1.10.1], $T'$ is a continuous linear mapping of $\mathscr{I}^{n'}$ into $\mathscr{D}'_n$, and so $T'f$ is a distribution. To show that $T'f$ is spherically symmetric, it is sufficient to remark that for each orthogonal transformation $S$ of $\mathbb{R}^n$, $T\chi(Sx) = T\chi(x)$.

Not all spherically symmetric distributions correspond to elements of $\mathscr{I}^{n'}$. For instance, the function $f(x) = |x|^{-3/2}$ is locally integrable in $\mathbb{R}^n$ for $n \geqq 2$ and generates a regular spherically symmetric distribution, but $g(r) = r^{-3/2}$ is not locally integrable and does not appear to correspond to an element of $\mathscr{I}^{n'}$ for $n \geqq 2$. Equation (5.4) can also be used to extend the mapping $T'$ from $\mathscr{I}^{n'}$ to the larger space $\mathscr{I}^{n,0'}$ of generalized functions, and as $T$ is an isomorphism from spherically symmetric $n$-dimensional testing functions to $\mathscr{I}^{n,0}$, $T'$ defines an isomorphism from $\mathscr{I}^{n,0'}$ to spherically symmetric $n$-dimensional distributions.

**6.** When Laplace's operator is applied to spherically symmetric functions on $\mathbb{R}^n$, the singular differential operator

$$\frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr}$$

makes its appearance. Accordingly, this operator turns up in connection with spherically symmetric waves in $n$ spatial dimensions, axially symmetric potentials in $n + 1$ dimensions and other problems. Moreover, it makes sense to consider the differential operator

$$L_\nu = \frac{d^2}{dr^2} + \frac{2\nu+1}{r}\frac{d}{dr} \tag{6.1}$$

for other than integer values of $2\nu$. See [10] or [3, Chap. 4] for the importance of this operator and for further references.

Now, $L_\nu$ is not an operator on $\mathscr{I}$, and for this reason we shall consider the modified operators

$$\begin{aligned} r^2L_\nu &= \delta^2 + 2\nu\delta, \\ rL_\nu r &= \delta'^2 + 2\nu\delta', \\ L_\nu r^2 &= \delta\delta' + (2\nu+3)\delta + 4(\nu+1). \end{aligned} \tag{6.2}$$

These are continuous linear operators on $\mathscr{I}$, since $\delta$ and $\delta'$ have this property, and as $\delta$ and $-\delta'$ are adjoint operators, so are $r^2L_\nu$ and $rL_{-\nu}r$.

There is a connection between the differential operator $L_\nu$ and fractional integration with respect to $r^2$. This was utilized already by Poisson, and among more recent authors, Lions [6], [7] gave the most precise and thorough discussion of this connection.

One of the forms in which the connection between $L_\nu$ and fractional integration can be expressed is [1]

$$I_{r^2}^{\nu,\alpha}L_\nu = L_{\nu+\alpha}I_{r^2}^{\nu,\alpha}.$$

We write this in the form

$$r^2I_{r^2}^{\nu,\alpha}L_\nu = r^2L_{\nu+\alpha}I_{r^2}^{\nu,\alpha}$$

which is the adjoint of the relation

$$rL_{-\nu}rK_{r^2}^{\nu-1/2,\alpha} = K_{r^2}^{\nu+1/2,\alpha}rL_{-\nu-\alpha}r, \tag{6.3}$$

and we will establish this last relation for testing functions.

Indeed, by (6.2) and (3.8),

$$rL_{-\nu}rK_{r^2}^{\nu-1/2,\alpha}\phi = \delta'(\delta' - 2\nu)K_{r^2}^{\nu-1/2,\alpha}\phi$$
$$= -2\delta' K_{r^2}^{\nu+1/2,\alpha-1}\phi;$$

and by (6.2) and (3.9),

$$K_{r^2}^{\nu+1/2,\alpha}rL_{-\nu-\alpha}r\phi = K_{r^2}^{\nu+1/2,\alpha}(\delta' - 2\nu - 2\alpha)\delta'\phi$$
$$= -2K_{r^2}^{\nu+1/2,\alpha-1}\delta'\phi.$$

Since $\delta'$ and $K$ commute, (6.3) is true for $\phi \in \mathscr{I}$, at least for $\operatorname{Re}\nu > \frac{1}{2}$ when $\operatorname{Re}(\nu \pm \frac{1}{2}) > 0$.

Since $L_\nu$ does not map $\mathscr{I}'$ into $\mathscr{I}'$, we write

$$r^2 I_{r^2}^{\nu,\alpha} L_\nu r^2 f = r^2 L_{\nu+\alpha} I_{r^2}^{\nu,\alpha} r^2 f, \qquad f \in \mathscr{I}'. \tag{6.4}$$

Now, $r^2, L_\nu r^2, r^2 L_{\nu+\alpha}, I_{r^2}^{\nu,\alpha}$ are all continuous operators on $\mathscr{I}'$ provided $\operatorname{Re}\nu > -\frac{1}{2}$. Both sides of (6.4) are analytic functions of $\nu$ for $\operatorname{Re}\nu > -\frac{1}{2}$; the two sides are equal by (6.3) if $\operatorname{Re}\nu > \frac{1}{2}$; and by analytic continuation, (6.4) holds for $\operatorname{Re}\nu > -\frac{1}{2}$.

## REFERENCES


[1] A. ERDÉLYI, *An application of fractional integrals*, J. Analyse Math., 14 (1965), pp. 113–126.

[2] I. M. GEL'FAND AND G. E. SHILOV, *Generalized Functions*, vol. 1, Academic Press, New York, 1964.

[3] R. P. GILBERT, *Function Theoretic Methods in Partial Differential Equations*, Academic Press, New York, 1969.

[4] H. KOBER, *On fractional integrals and derivatives*, Quart. J. Math. Oxford, 11 (1940), pp. 193–211.

[5] H. KOBER AND A. ERDÉLYI, *Some remarks on Hankel transforms*, Ibid., 11 (1940), pp. 212–221.

[6] J. L. LIONS, *Opérateurs de Delsarte et problèmes mixtes*, Bull. Soc. Math. France, 84 (1956), pp. 9–95.

[7] ———, *Équations différentielles opérationelles et problèmes aux limites*, Springer-Verlag, Berlin, 1961.

[8] S. D. POISSON, *Mémoire sur l'intégration des équations linéaires aux différences partielles*, J. École Polytechnique, 12 (1823), pp. 215–248.

[9] I. N. SNEDDON AND A. ERDÉLYI, *Fractional integration and dual integral equations*, Canad. J. Math., 14 (1962), pp. 685–693.

[10] A. WEINSTEIN, *On a singular differential operator*, Ann. Mat. Pura Appl. (4), 49 (1960), pp. 359–366.

[11] A. H. ZEMANIAN, *Generalized Integral Transformations*, Interscience, New York, 1968.